A NOTE ON THE BEHAVIOR OF VERY LONG WAVES IN SIMPLE BAROCLINIC MODELS

A. Wiin-Nielsen

Air Weather Service

(Manuscript received 25 May 1960)

ABSTRACT

The behavior of very long waves in a two-parameter model with no divergence in the mean flow has been investigated. It is found that the temperature field and the pressure field move almost independently of each other. The pressure field will retrograde with a speed comparable to the Rossby speed for non-divergent waves, while the temperature field progresses slowly. As a result, the latter field will precede the pressure field after a while, verifying an earlier observation in these forecasts. Introducing a divergence in the vertical mean flow not only greatly reduces the retrogression, but a stronger coupling then exists between the temperature and pressure fields.

1. Introduction

Numerical forecasts with a two-parameter, frictionless and adiabatic model have a number of characteristic errors which have been recently summarized by Thompson (1959). It is, for instance, observed that a pressure system which initially shows the usual westward slope with height will at times develop in such a way that the axis of the system becomes more vertical during the forecast and even may develop the opposite slope as the forecast progresses in time. The characteristic development in time of short stable and longer unstable baroclinic waves has been investigated by Ogura (1957). He showed that the shorter stable waves will have fluctuations in their amplitude and a monotonic increase of the phase angle, while the longer unstable waves will develop toward a state which has a westward slope. Ogura's solution of the equations for the two-parameter model are finite-amplitude wave solutions. It follows from his results that the short stable waves initially will increase their slope with height. Therefore, the only waves which can possibly behave as described by Thompson are the long stable waves.

It is the purpose of the present note to investigate the behavior of very long waves in a two-parameter model. The investigation will be restricted to solutions of the two-parameter equations consisting of a single finite-amplitude wave, independent of the north-south direction, superimposed on a zonal current with no lateral shear. For ease of reference, the model equations and the solution will be reproduced in the next section.

2. Solution of the model equation

The atmosphere will be represented by the simplest possible baroclinic model. The information levels will be the 40- and 80-cb levels. The atmosphere is assumed to be frictionless and adiabatic, and the vertical boundary conditions will be $\omega = 0$ for $p = 20$ cb and $p = 100$ cb.

The model equations for this model may be written (Wiin-Nielsen, 1959) as

$$\frac{\partial \zeta^*}{\partial t} + V^* \cdot \nabla(\zeta^* + f) + V' \cdot \nabla \zeta' = r^2 \frac{\partial \psi^*}{\partial t} \quad (2.1)$$

and

$$\frac{\partial \zeta'}{\partial t} + V^* \cdot \nabla \zeta' + V' \cdot \nabla(\zeta^* + f) = \lambda^2 \left(\frac{\partial \psi'}{\partial t} + V^* \cdot \nabla \psi' \right). \quad (2.2)$$

In (2.1) and (2.2), $V = k \times \nabla\psi$ denotes a horizontal wind, $\zeta = \nabla^2\psi$ the relative vorticity, and f the Coriolis parameter, while ψ is a stream function. Quantities with a superscript "star" refer to the arithmetic mean of the corresponding quantities at the two information levels, while a superscript "prime" refers to one half of the difference between quantities at the same levels. The parameter $\lambda^2 = 2f_0^2/\sigma P^2$ is assumed to be constant. f_0 is a standard value of the Coriolis parameter, $P = 40$ cb, and $\sigma = -\alpha \partial \ln \theta / \partial p$ is a measure of static stability. In the following, we are going to refer to the ψ'-field as the temperature field. ψ^* will be called the pressure field. The term $r^2 \partial\psi^*/\partial t$ represents an estimate to the divergence in the mean flow. For $r^2 = 0$, (2.1) and (2.2) reduce to the usual equations for the two-parameter model. The importance and interpretation of this term has been discussed earlier by the author (1959).

The solutions to (2.1) and (2.2) will be assumed in the form

$$\left.\begin{aligned} \psi^*(x, y, t) &= -U^*y + \hat{\psi}^* e^{ik(x-ct)} \\ \psi'(x, y, t) &= -U'y + \hat{\psi}' e^{ik(x-ct)}. \end{aligned}\right\} \quad (2.3)$$

$\hat{\psi}^*$ and $\hat{\psi}'$ denote the (complex) amplitudes, $k=2\pi/L$ the wave-number, and c the phase-speed, while U^* and U' are the constant zonal wind speeds. (2.3) will be solutions to (2.1) and (2.2) if the following two homogeneous linear equations are satisfied:

$$[(1+r^2/k^2)c-U^*+\beta/k^2]\hat{\psi}^*-U'\hat{\psi}'=0 \quad (2.4)$$

$$-U'(1-\lambda^2/k^2)\hat{\psi}^* + [(1+\lambda^2/k^2)(c-U^*+\beta/k^2)-\beta\lambda^2/k^4]\hat{\psi}'=0. \quad (2.5)$$

The condition that the determinant has to be zero in order to obtain non-trivial solutions results in an equation from which we can determine the phase speed c. We find

$$c=\frac{2+r^2/k^2}{2(1+r^2/k^2)}U^*-\frac{(2+r^2/k^2+\lambda^2/k^2)\beta/k^2}{2(1+r^2/k^2)(1+\lambda^2/k^2)} \pm\frac{D^{\frac{1}{2}}}{2(1+r^2/k^2)(1+\lambda^2/k^2)} \quad (2.6)$$

where

$$D=[(1+\lambda^2/k^2)(r^2/k^2)U^*+(\lambda^2/k^2-r^2/k^2)\beta/k^2]^2 +4(1+r^2/k^2)(1-\lambda^2/k^2)U'^2. \quad (2.7)$$

The two solutions (2.6) will be denoted c_+ and c_-. The complete solutions to the model equations may now be written as

$$\psi^*(x,t)=\hat{\psi}_+^*e^{ik(x-c_+t)}+\hat{\psi}_-^*e^{ik(x-c_-t)} \quad (2.8)$$

$$\psi'(x,t)=\hat{\psi}_+'e^{ik(x-c_+t)}+\hat{\psi}_-'e^{ik(x-c_-t)}, \quad (2.9)$$

since superposition of solutions is possible.

The four amplitudes appearing in (2.8) and (2.9) are determined by the initial conditions for the fields, ψ^* and ψ'. Let these be given by

$$\left.\begin{aligned}\psi^*_{t=0}&=A_0^*e^{ikx}\\ \psi'_{t=0}&=A_0'e^{i(kx+\alpha_0')}\end{aligned}\right\} \quad (2.10)$$

where A_0^* and A_0' are the initial amplitudes and α_0' is the initial difference in phase. If $\alpha_0'>0$, the temperature field will be lagging behind the pressure field. By equating (2.8) and (2.9) for $t=0$ to the expressions (2.10), we get

$$\hat{\psi}_+^*+\hat{\psi}_-^*=A_0^* \quad (2.11)$$

$$\hat{\psi}_+'+\hat{\psi}_-'=A_0'e^{i\alpha_0'}. \quad (2.12)$$

The two remaining equations are obtained from either (2.4) or (2.5), which are equivalent since the determinant of the equations is zero. The simplest forms are obtained from (2.4). By introducing the notations

$$\left.\begin{aligned}x_+&=(1+r^2/k^2)c_+-c_R\\ x_-&=(1+r^2/k^2)c_--c_R\end{aligned},\quad c_R=U^*-\beta/k^2\right\} \quad (2.13)$$

we obtain

$$x_+\hat{\psi}_+^*-U'\hat{\psi}_+'=0 \quad (2.14)$$

$$x_-\hat{\psi}_-^*-U'\hat{\psi}_-'=0. \quad (2.15)$$

The solution to the system (2.11, 12, 14, 15) may be written

$$\left.\begin{aligned}\hat{\psi}_+^*&=\frac{U'A_0'e^{i\alpha_0'}-x_-A_0^*}{x_+-x_-}, &\hat{\psi}_+'&=\hat{\psi}_+^*\frac{x_+}{U'}\\ \hat{\psi}_-^*&=\frac{x_+A_0^*-U'A_0'e^{i\alpha_0'}}{x_+-x_-}, &\hat{\psi}_-'&=\hat{\psi}_-^*\frac{x_-}{U'}\end{aligned}\right\} \quad (2.16)$$

3. On the behavior of very long waves

The solutions obtained in section 2 apply to all scales. Since Ogura (1957) has given a discussion of the short stable and the unstable waves, we shall here restrict ourselves to a discussion of very long waves. It is well known that the very long waves are never unstable, according to the instability criterion for a two-parameter model using observed values of the vertical shear, U'. We may, therefore, in the investigation of these waves safely assume that both c_+ and c_- are real roots. x_+ and x_- are therefore also real numbers. In order to keep the investigation relatively simple from the mathematical point of view, we shall investigate what happens as $k\to 0$, although it is realized that there is an upper limit for the wavelength in the atmosphere.

Case A: No divergence in the mean flow.

The model equations in this case are (2.1) and (2.2) with $r^2=0$. In this case, the model is similar to most two-parameter models and for practical purposes identical to the model applied in the computations summarized by Thompson.

The definitions (2.13) reduce now to

$$\left.\begin{aligned}x_+&=c_+-c_R\\ x_-&=c_--c_R.\end{aligned}\right\} \quad (3.1)$$

From (2.6) with $r^2=0$, it can be shown that

$$c_R\to-\infty,\quad x_+\to+\infty,\quad x_-\to 0\quad \text{for}\quad k\to 0. \quad (3.2)$$

The relations (3.2) are also illustrated in fig. 5a–8a of Wiin-Nielsen (1959). By using the relations (3.2), it is now easily seen from (2.16) that

$$\left.\begin{aligned}\frac{\hat{\psi}_+^*}{A_0^*}&\to 0, &\frac{\hat{\psi}_+'}{A_0'e^{i\alpha_0'}}&\to 1\\ \frac{\hat{\psi}_-^*}{A_0^*}&\to 1, &\frac{\hat{\psi}_-'}{A_0'e^{i\alpha_0'}}&\to 0\end{aligned}\right\} \quad (3.3)$$

when $k\to 0$.

The results (3.3) show that the stream function ψ^* for very large values of the wave length will move predominantly with the speed c_-, which is close to the Rossby speed and therefore numerically large and negative for very long waves. The amplitude $\hat{\psi}_+^*$ in the part of the solution moving with speed c_+ will be small. (3.3) shows also that the major portion of the thermal wave $\hat{\psi}'$ will move with the speed c_+, which for small values of k is positive but small. The pressure wave ψ^* and the thermal wave ψ' move therefore in opposite directions and with speeds which are very different. Consider as an example the case where the two waves, ψ^* and ψ', are initially in phase ($\alpha_0' = 0$). According to (3.3), we will find the ψ^*-wave retrograding with a large speed, while the ψ'-wave will move slowly towards the east. As a consequence, we will find the temperature wave preceding the pressure wave in the forecast. Although our results are obtained for infinitely long waves in the so-called β-plane, it is likely that similar but perhaps slightly modified results will apply to the planetary waves on the sphere. The waves in which it is found that the temperature wave gradually will precede the pressure wave in forecasts with the two-parameter model treated in this section are therefore sufficiently long stable waves.

Case B: Divergence in the mean flow.

The term $r^2\partial\psi^*/\partial t$ now appearing in (2.1) may be interpreted as part of the divergence in the mean flow, ψ^*. By identifying this term with the term $-f_0\nabla\cdot V^*$ in the vorticity equation, we find that we have semi-empirically introduced an amount of divergence equal to

$$\nabla\cdot V^* = -\frac{r^2}{f_0}\frac{\partial\psi^*}{\partial t}. \quad (3.4)$$

By assuming for a moment that

$$\psi^* = A\sin k(x-ct), \quad (3.5)$$

we find

$$\nabla\cdot V^* = \frac{r^2}{f_0}ckA\cos k(x-ct) = \frac{r^2}{f_0}cv^*. \quad (3.6)$$

According to (3.6), we have, for the very long waves where c is negative, introduced a divergence where $v^* < 0$ and a convergence where $v^* > 0$. Instantaneously, the divergence patterns are therefore going to give an additional velocity component in the phase-speed towards the east and will consequently act to decrease the rapid retrogression of the very long waves.

It is interesting to notice that a distribution of divergence of this type in the mid-troposphere will be part of the divergence field in a model with a finer vertical resolution. We may therefore justify the presence of the term $r^2\partial\psi^*/\partial t$ by saying that we have introduced a divergence field which cannot be represented directly by two parameters, because it depends on the deviation from a linear vertical wind profile, but which is present in the real atmosphere due to the vertical structure. With this further justification, it becomes interesting to investigate whether the model treated in this section will modify the results obtained in case A.

From the analysis in an earlier paper (Wiin-Nielsen 1959), it can be shown that the two speeds, c_+ and c_-, are numerically smaller than in the case $r^2 = 0$. In the limit, we find

$$c_+ = U^* - \frac{\beta}{\lambda^2}, \qquad c_- = -\frac{\beta}{r^2} \qquad \text{for} \qquad k\to 0. \quad (3.7)$$

By assuming that $U^* = 20$ msec^{-1}, $\lambda^2 = 4\times 10^{-12}$ m^{-2}, $r^2 = 2\times 10^{-12}$ m^{-2} and $\beta = 16\times 10^{-12}$ m^{-1} sec^{-1}, we get $c_+ = 16$ msec^{-1} and $c_- = -8$ msec^{-1}. Even if the results (3.3) obtained for the model with no mean divergence should hold here, we find, due to the smaller speeds, that the tendency to let the temperature field precede the pressure field would be greatly reduced. The main point is, however, that it is possible to show that we will obtain a coupling between the pressure and temperature fields in the present case.

It is immediately obvious that since $c_R\to -\infty$, while $c_+\to U^* - \beta/\lambda^2 > 0$, we have $x_+\to +\infty$ for $k\to 0$. Due to the compensatory effect between the two terms in c_- in (2.13), it is somewhat more laborious to find the limiting value of c_- for $k\to 0$. However, by applying l'Hospitals rule twice, it is possible to show that

$$x_- \to \frac{U'^2}{U^* + (r^{-2}-\lambda^{-2})\beta} \qquad \text{for} \qquad k\to 0. \quad (3.8)$$

By using the results derived above, it is a straightforward matter to determine the amplitude ratio corresponding to those reproduced in (3.3). We obtain now for $k\to 0$

$$\begin{aligned}
&\frac{\hat{\psi}_+^*}{A_0^*}\to 0\\
&\frac{\hat{\psi}_-^*}{A_0^*}\to 1\\
&\frac{\hat{\psi}_+'}{A_0'e^{i\alpha_0'}}\to 1 - \frac{U'}{U_+^*(r^{-2}-\lambda^{-2})\beta}\cdot\frac{A_0^*}{A_0'e^{i\alpha_0'}}\\
&\frac{\hat{\psi}_-'}{A_0'e^{i\alpha_0'}}\to \frac{U'}{U_+^*(r^{-2}-\lambda^{-2})\beta}\cdot\frac{A_0^*}{A_0'e^{i\alpha_0'}}.
\end{aligned} \quad (3.9)$$

It is thus seen that the behavior of the pressure wave ψ^* is unchanged in the limit; *i.e.*, the major portion of the wave will retrograde with the speed c_- which, however, is now much smaller. The behavior of the thermal wave ψ' is changed to the extent that a finite portion of the initial amplitude will now go into that part of the wave which moves with the speed c_-. In order to obtain an estimate of the fraction of the initial amplitude in the ψ'-wave traveling with the speed c_-, we may assume that

$$\left|\frac{A_0{}^*}{A_0{}'e^{i\alpha_0{}'}}\right| \sim 2, \tag{3.10}$$

assuming a linear increase of the initial amplitude with $-p$.

By using again $U^* = 20$ msec^{-1} and $U' = 10$ msec^{-1}, we find

$$\frac{U'}{U_+{}^*(r^{-2} - \lambda^{-2})\beta} \sim 0.4 \tag{3.11}$$

and, consequently,

$$\left|\frac{\psi_-{}'}{A_0{}'e^{i\alpha_0{}'}}\right| \sim 0.8. \tag{3.12}$$

We find, therefore, that a major portion of the thermal wave will retrograde in this case, and we may conclude that the mid-tropospheric divergence introduces a coupling between the temperature and pressure fields.

4. Conclusions

The analysis of very long waves in a two-parameter model with no divergence in the mean flow shows that the pressure field and temperature field move almost independently of each other, the first field retrograding with a speed comparable to the Rossby speed for non-divergent waves, the latter progressing slowly toward the east. The analysis is based on finite-amplitude linear solutions of the prognostic equations for the model.

The introduction of a semi-empirical divergence in the mean flow causes a stronger coupling between the two fields. The retrogression of the pressure wave is at the same time greatly reduced.

It is possible that the semi-empirical nature of the divergence can be avoided if a model with a finer vertical resolution can be applied.

REFERENCES

Ogura, Y., 1957: Wave solutions of the vorticity equations for the 2½-dimensional model. *J. Meteor.*, **14**, 60–64.

Thompson, P. D., 1959: Some statistical aspects of the dynamic processes of growth and occlusion in simple baroclinic models, in *The Rossby Memorial Volume*. New York, Oxford Univ. Press in assoc. with Rockefeller Inst. Press, pp. 350–358.

Wiin-Nielsen, A., 1959: On barotropic and baroclinic models with special emphasis on ultra-long waves. *Mon. Wea. Rev.*, **87**, 171–183.