The Rate of Change of the Kinetic Energy Spectrum of Flow in a Compressible Fluid

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ABSTRACT

The variance spectrum of velocities in a non-homogeneous, compressible fluid does not represent the wave-number distribution of kinetic energy, as it does in incompressible, homogeneous (constant density) fluids. Use of a truncated Fourier transform and the assumption that the flow occurs in a finite area show that the kinetic energy spectrum in the former case is the co-spectrum between the velocity and the momentum. The Navier-Stokes equations are used to study the time rates of change of the kinetic energy spectrum produced by the various physical effects contained in those equations. Introduction of the assumption of homogeneity and incompressibility in the equations derived here gives the same qualitative results as Batchelor's (1953) study of the time rate of change of the spectrum of turbulent flow. Kinetic energy in a compressible, non-homogeneous fluid can draw on internal and potential energy, but these energy sources are not available to flow in incompressible, homogeneous fluids. It is shown that compressibility effects are not important in the action of the inertial or viscous effects on the total kinetic energy.

1. Introduction

Theoretical and observational studies of fluid motion have made increasing use of the wave-number distribution of kinetic energy provided by velocity variance spectra. The time rates of change of the spectrum caused by the various physical effects represented by the separate terms of the hydrodynamic equations are of special interest.

Theoretical spectral studies have emphasized, primarily for mathematical convenience, motion or turbulence in incompressible fluids with density constant in space. In this case, the variance spectrum multiplied by the density is the kinetic energy spectrum.

Perhaps the best-known modern study of this type is Batchelor's (1953), in which he used a stochastic Fourier-Stieltjes transform to determine the rate of change of the spectrum. His work deals with turbulence occurring in an infinite space and, as pointed out by Birkhoff and Kampé de Fériet (1958), the integral upon which the theory is based [equation 2.5.3 of Batchelor (1953)] is valid only for a stochastic process, not for an individual velocity field selected from the ensemble.

The purpose of this paper is to study the time rate of change of the true kinetic energy spectrum (for the meteorologically important case) of flow in a compressible, viscous fluid with density changes in space, and to use a method which is valid for an individual velocity field. It will appear that the introduction of the incompressibility assumption (by which we shall mean both that the fluid is incompressible and that its density is constant in space) produces the same qualitative results which Batchelor obtained. Furthermore, differences between compressible and incompressible flow are thus demonstrated.

The physically reasonable assumption that the area in which the flow occurs is finite allows use of a truncated Fourier transform, and all reference to correlation functions is avoided. The validity of this technique when the motion is viewed as an infinite stochastic process is dubious (Parzen, 1961), but it is valid when the physical variables are assumed to have a well-defined value at any point of a finite space at any given time. Furthermore, difficult questions of the existence of integrals and limits, such as encountered by Batchelor, are completely avoided. The fundamental theorem for this investigation relates the total variance to the integral of the spectrum; its corollary, the convolution theorem, provides a method of handling the nonlinearities of the Navier-Stokes equations. The Navier-Stokes equations and the continuity equation allow investigation of the true kinetic energy spectrum rather than the variance spectrum of the flow velocity.

The model. The flow to be studied here occurs in a finite, three dimensional volume in a baroclinic fluid subject to gravitational and Coriolis forces. It is assumed that all physical variables are zero outside the

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area of interest. The fluid may, for example, be confined by solid walls. Mathematically, we shall say that the variables are functions with compact support; that is, they have non-zero values only on a compact domain, where compact means closed and bounded. We assume that the compact support is constant in time, and that the velocity component perpendicular to the boundary vanishes at the boundary and hence there is no advection through the boundaries.

It is assumed that there is no mean motion and that the dynamic viscosity is constant in time and space. The use of a compressible fluid with constant viscosity seems to be reasonable for atmospheric studies where the temperature range may be presumed relatively small; the analysis is not intended to cover high Mach number flows or such phenomena as shock waves. It is, however, applicable to atmospheric turbulence confined by non-turbulent areas.

The physical assumptions about the topology of the problem are convenient ones to assure the existence of the truncated Fourier transforms; it is the author's opinion that this form of the transform is the simplest mathematical tool for studying the kinetic energy spectrum.

2. Notation and fundamental theorems

The vector (see Table of Symbols) $\mathbf{x}=\mathbf{i}_1x_1+\mathbf{i}_2x_2+\mathbf{i}_3x_3$ denotes a point in E_3 (three-dimensional Euclidean space). Similarly, the notation

$$\mathbf{u}(\mathbf{x})=\sum_{i=1}^{3}\mathbf{i}_iu_i(\mathbf{x})=\mathbf{i}_iu_i(\mathbf{x})$$

denotes a vector-valued function with domain in E_3, where the usual tensor convention of summing on double indices is valid.

Integrations must be performed with respect to the volume element $d\mathbf{x}=dx_1dx_2dx_3$, and we thus define

$$\int_{E_3}\mathbf{u}(\mathbf{x})d\mathbf{x}=\mathbf{i}_i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_i(x_1,x_2,x_3)dx_1dx_2dx_3. \quad (2.1)$$

Hence the integral of a vector-valued function $\mathbf{u}(\mathbf{x})$ is again a vector whose scalar magnitudes are the integrals of the scalar magnitudes of $\mathbf{u}(\mathbf{x})$. In case $\mathbf{u}(\mathbf{x})$ has compact support, $\mathbf{X}\subset E_3$, we write

$$\int_{E_3}\mathbf{u}(\mathbf{x})d\mathbf{x}=\int_{\mathbf{X}}\mathbf{u}(\mathbf{x})d\mathbf{x}. \quad (2.2)$$

Formal use of the infinite integral on the left will frequently simplify computations.

The Fourier transform is defined here for a measurable scalar function of one variable, x, as

$$\hat{u}(\kappa)=\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}u(x)e^{-i\kappa x}dx \quad (2.3)$$

and when $u(x)$ has compact support $[-X,X]$, the identity

$$\hat{u}(\kappa)=\pi^{-\frac{1}{2}}\int_{-X}^{X}u(x)e^{-i\kappa x}dx \quad (2.4)$$

holds. These transforms always exist if $u(x)$ is a bounded function with compact support.

When $u=u(\mathbf{x})=u(x_1,x_2,x_3)$, the appropriate exponent is the scalar product $-i\boldsymbol{\kappa}\cdot\mathbf{x}$ and the vector-valued Fourier transform of a vector-valued function $\mathbf{u}(\mathbf{x})$ is

$$\hat{\mathbf{u}}(\boldsymbol{\kappa})=\mathbf{i}_i\hat{u}_i(\boldsymbol{\kappa})$$
$$=\mathbf{i}_i\pi^{-\frac{3}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_i(x_1,x_2,x_3)e^{-i\kappa_jx_j}dx_1dx_2dx_3. \quad (2.5)$$

This transform has three components at each point of the three-dimensional wave-number space.

The square modulus of the transform (2.5),

$$|\hat{\mathbf{u}}(\boldsymbol{\kappa})|^2=\bar{\hat{\mathbf{u}}}(\boldsymbol{\kappa})\cdot\hat{\mathbf{u}}(\boldsymbol{\kappa})=|\hat{u}_i(\boldsymbol{\kappa})|^2 \quad (2.6)$$

(note the summation of the double index i) gives a scalar magnitude at each point of wave-number space. When $u_i(\mathbf{x})$ is a component of the velocity, this square modulus is the sum of the specific kinetic energy of all three velocity components as a function of wave-number. It should be noted that the spectrum depends on the entire velocity field; it has no general validity as a local concept.

It is sufficient, in view of the Navier-Stokes equations and the fact that Lebesgue integration is used, for atmospheric variables to be twice partially differentiable almost everywhere. This condition, however, is unnecessarily severe. Only one differentiation in space is required for all variables except flow velocities, which require two spatial differentiations. Since functions of bounded variation on compact sets are bounded on such sets and possess finite derivatives almost everywhere on such sets, the sufficient conditions for the validity of all of the following theorems are:

1) All physical variables have a compact support constant in time and are measurable on that support;
2) All physical variables have uniformly bounded partial derivatives with respect to time;
3) Flow velocities possess bounded derivatives in space which are of bounded variation (note that the velocities themselves are thus continuous functions on compact sets and therefore are bounded themselves);
4) All other physical variables are of bounded variation in space.

Preliminary theorems. Interchanges in the order of integration can be justified by the theorems of Tonelli and Fubini, which combine to yield the result that if $f(x,y)$ is a measurable function of (x,y) and if either

$$\int_c^d dy \int_a^b |f(x,y)|dx < \infty$$

or

$$\int_a^b dx \int_c^d |f(x,y)|dy < \infty$$

then

$$\int_c^d \int_a^b f(x,y)dxdy = \int_c^d dy \int_a^b f(x,y)dx = \int_a^b dx \int_c^d f(x,y)dy. \quad (2.7)$$

We shall call this result the general Fubini theorem.

With this theorem and the multiplicative property of the exponential function, (2.5) may be written as

$$\hat{\mathbf{u}}(\boldsymbol{\kappa}) = \mathbf{i}_i\pi^{-\frac{3}{2}} \int_{-X_3}^{X_3} dx_3 e^{-i\kappa_3 x_3} \int_{-X_2}^{X_2} dx_2 e^{-i\kappa_2 x_2} \times \int_{-X_1}^{X_1} u_i(x_1,x_2,x_3)e^{-i\kappa_1 x_1}dx_1. \quad (2.8)$$

It is thus sufficient to prove theorems about Fourier transforms in the pair of transform variables (x,κ) and apply them to the pair $(\mathbf{x},\boldsymbol{\kappa})$ considered as vectors and integrating with respect to $d\mathbf{x}$.

The fundamental theorem. The theorem upon which this study is based relates the mean square of a function to the integral of the square modulus of its truncated Fourier transform. It can be shown (Wiener, 1930) that if $f(x)$ is a bounded, measurable function with compact support $[-X,X]$ then

$$\int_{-X}^{X} |f(x)|^2 dx = \frac{1}{2}\int_{-\infty}^{\infty} |\hat{f}(\kappa)|^2 d\kappa. \quad (2.9)$$

When $f(x)$ is a velocity, the quantity on the left is twice the total specific kinetic energy. Thus the integrand on the right gives the wave-number distribution of specific kinetic energy in wave-number space. For other variables, we obtain the wave-number distribution of the mean square, and the integrand is known as a variance spectral density function.

In view of the result (2.8), the theorem is easily extended to the case of three dimensions. The proper constant, as computation of an example shows, is 2^{-3} to correspond with $\pi^{-\frac{3}{2}}$. Hence for functions with compact support $\mathbf{X}$,

$$\int_{\mathbf{X}} |f(\mathbf{x})|^2 d\mathbf{x} = \frac{1}{8}\int_{E_3} |\hat{f}(\boldsymbol{\kappa})|^2 d\boldsymbol{\kappa}. \quad (2.10)$$

The convolution theorem. Since Fourier transforms are additive, we may write (following Wiener, 1930)

$$\begin{matrix} A \\ B \end{matrix} \int_{-\infty}^{\infty} |f(x) \pm g(x)|^2 dx = \frac{1}{2}\int_{-\infty}^{\infty} |\hat{f}(\lambda) \pm \hat{g}(\lambda)|^2 d\lambda,$$
$$\begin{matrix} C \\ D \end{matrix} \int_{-\infty}^{\infty} |f(x) \pm ig(x)|^2 dx = \frac{1}{2}\int_{-\infty}^{\infty} |\hat{f}(\lambda) \pm i\hat{g}(\lambda)|^2 d\lambda, \quad (2.11)$$

where the top letter goes with the plus sign. The operation $(A-B+iC-iD)/4$ produces

$$\int_{-\infty}^{\infty} f(x)\bar{g}(x)dx = \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}(\lambda)\bar{\hat{g}}(\lambda)d\lambda. \quad (2.12)$$

If we put $g(x) = \bar{h}(x)e^{i\kappa x}$, then

$$\bar{\hat{g}}(\lambda) = \pi^{-\frac{1}{2}} \int_{-\infty}^{\infty} h(x)e^{-i(\kappa-\lambda)x}dx = \hat{h}(\kappa-\lambda) \quad (2.13)$$

so that substitution for g and $\hat{g}$ in (2.12) yields

$$\int_{-\infty}^{\infty} f(x)h(x)e^{-i\kappa x}dx = \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}(\lambda)\hat{h}(\kappa-\lambda)d\lambda = \frac{1}{2}\int_{-\infty}^{\infty} \hat{f}(\kappa-\lambda)\hat{h}(\lambda)d\lambda. \quad (2.14)$$

The second integral on the right is obtained from the other by the formal substitution $\lambda = \kappa - \lambda$. Thus the transform of a product is the convolution of the transforms of the factors.

It will be necessary to interchange the order of convolution. To do so, we note that $\bar{\hat{g}}(\kappa) = \hat{\bar{g}}(-\kappa)$, and obtain the formula

$$\int_\kappa \hat{h}(\kappa)\left\{\int_\lambda \bar{\hat{f}}(\lambda)\bar{\hat{g}}(\kappa-\lambda)d\lambda\right\}d\kappa = \int_\lambda \bar{\hat{f}}(\lambda)\left\{\int_\kappa \hat{h}(\kappa)\hat{\bar{g}}(\lambda-\kappa)d\kappa\right\}d\lambda, \quad (2.15)$$

in which the symbols below the integrals indicate that the integration should be performed over all possible values of that variable [in this study $(-\infty,\infty)$]. Use of (2.14) on the inner integrals of both sides and subsequent use of (2.12) proves the equality. The general Fubini theorem is inconclusive here because integrability of the modulus of the transforms is not assured.

Additional results. The commutativity of spatial integration and time differentiation is the crux of the operational method. We may put

$$\frac{d}{dt}\int_{-X}^{X} f(x,t)dx = \lim_{\epsilon\to 0}\int_{-X}^{X} \frac{f(x,t+\epsilon)-f(x,t)}{\epsilon}dx. \quad (2.16)$$

An intuitively obvious extension of Lebesgue's dominated convergence theorem to functions of two variables rather than sequences of functions allows interchange of the limit and the integration if $\partial f(x,t)/\partial t$ exists in the whole domain (x,t), (that is, for $-X \leq x \leq X$, all t) and if $|\partial f/\partial t| \leq K$ (K a constant). Thus a sufficient condition for the relation

$$\frac{d}{dt}\int_{-X}^{X} f(x,t)dx = \int_{-X}^{X} \frac{\partial f(x,t)}{\partial t}dx \qquad (2.17)$$

to be valid is the existence of a uniformly bounded derivative in time and X constant in time, which we have assumed. Details of the theorem are given by Hobson (1926). This theorem shows that the rate of change of the spatial transform of a physical variable is the transform of the rate of change with respect to time.

The last theorem relates the transform of a derivative of a function to the transform of the function itself. Formal inversion of the transform $\hat{f}(\kappa)$ produces

$$f(x) = \frac{(\pi)^{-\frac{1}{2}}}{2}\int_{-\infty}^{\infty} \hat{f}(\kappa)e^{i\kappa x}d\kappa. \qquad (2.18)$$

Differentiation of both sides yields

$$\frac{\partial f(x)}{\partial x} = \frac{1}{2(\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} i\kappa\hat{f}(\kappa)e^{i\kappa x}d\kappa,$$

so that by the formal reciprocity of the transforms, we obtain the relations

$$\widehat{\frac{\partial f}{\partial x}}(\kappa) = i\kappa\hat{f}(\kappa) \qquad (2.19)$$

and

$$\overline{\widehat{\frac{\partial f}{\partial x}}}(\kappa) = -i\kappa\bar{\hat{f}}(\kappa). \qquad (2.20)$$

A proof which does not depend on the invertibility of the transform is given by Bochner and Chandrasekharan (1949).

3. The time rate of change of the kinetic energy spectrum

In addition to the mathematical assumptions listed in the previous section, the necessary or convenient physical assumptions about the fluid and its motion are:

1) The motion obeys the Navier-Stokes equations and the equation of continuity;
2) The fluid obeys the equation of state (perfect gas law) and the first law of thermodynamics;
3) There is no net advection through the boundary of the area;
4) The fluid has no mean motion with respect to the Cartesian coordinate system being used.

The Navier-Stokes equations, in tensor notation, for a Cartesian coordinate system, are

$$\frac{\partial u_i(\mathbf{x},t)}{\partial t} = F_i(\mathbf{x},t)$$

$$= -u_k\frac{\partial u_i}{\partial x_k} - \alpha\frac{\partial p}{\partial x_i} - 2\epsilon_{ijk}\omega_j u_k - g\delta_{i3} + \mu\alpha\frac{\partial^2 u_i}{\partial x_j^2}. \qquad (3.1)$$

Reference to space or time dependency of the terms on the right will generally be omitted.

Combination of (3.1) and the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0 \qquad (3.2)$$

produces the momentum equation

$$\frac{\partial}{\partial t}(\rho u_i) = G_i(\mathbf{x},t)$$

$$= -\frac{\partial}{\partial x_k}(\rho u_i u_k) - \frac{\partial p}{\partial x_i}$$

$$-2\rho\epsilon_{ijk}\omega_j u_k - \rho g\delta_{i3} + \mu\frac{\partial^2 u_i}{\partial x_j^2}. \qquad (3.3)$$

Use of (2.12) in three dimensional form similar to (2.10) gives the relation

$$\int_{\mathbf{x}} \rho u_i^2 d\mathbf{x} = \frac{1}{8}\int_{\boldsymbol{\kappa}} \widehat{\rho u_i}(\boldsymbol{\kappa})\bar{\hat{u}}_i(\boldsymbol{\kappa})d\boldsymbol{\kappa}. \qquad (3.4)$$

The imaginary parts of the transform are odd functions of the wave number, so that

$$\frac{1}{8}\int_{\boldsymbol{\kappa}} \widehat{\rho u_i}\bar{\hat{u}}_i d\boldsymbol{\kappa} = \frac{1}{8}\int_{\boldsymbol{\kappa}} Re\{\widehat{\rho u_i}\bar{\hat{u}}_i\}d\boldsymbol{\kappa}, \qquad (3.5)$$

in which

$$Re\{\widehat{\rho u_i}\hat{u}_i\} = Re(\widehat{\rho u_i})Re(\hat{u}_i) + Im(\widehat{\rho u_i})Im(\hat{u}_i). \qquad (3.6)$$

Thus $Re\{\widehat{\rho u_i}\bar{\hat{u}}_i\}$, the co-spectrum between velocity and momentum, is the kinetic energy spectral density function.

The time rate of change of the kinetic energy spectrum is therefore obtained with the aid of (3.1) and (3.3), and the theorems of the previous section and is given by

$$\frac{\partial}{\partial t}\Phi_{ii}(\boldsymbol{\kappa},t) = \frac{\partial}{\partial t}Re\{\widehat{\rho u_i}\bar{\hat{u}}_i\} = Re\{\widehat{\rho u_i}\bar{\hat{F}}_i + \bar{\hat{u}}_i\hat{G}\}. \qquad (3.7)$$

Separation of (3.7) into the sums of the corresponding terms of the Navier-Stokes and momentum equations representing the same physical effects will facilitate the analysis. Hence the time derivative will be written as equal to each effect separately. The notation

$$i\kappa_k\hat{u}_k(\boldsymbol{\kappa})=\widehat{\nabla\cdot\mathbf{V}}(\boldsymbol{\kappa}) \qquad (3.8)$$

will be used to emphasize the effects of divergence. Also, since u is real, we have

$$\bar{\hat{u}}(\boldsymbol{\kappa})=\hat{u}(-\boldsymbol{\kappa}).$$

Inertial effects. The time rate of change of the kinetic energy spectrum due to inertial effects is given, according to (3.1), (3.3), and (3.7) by

$$\frac{\partial}{\partial t}\Phi_{ii}=-Re\left\{\bar{\hat{u}}_i\frac{\partial}{\partial t}(\widehat{\rho u_i})+\widehat{\rho u_i}\frac{\partial\bar{\hat{u}}_i}{\partial t}\right\}$$

$$=-Re\left\{\left[\widehat{\frac{\partial}{\partial x_k}(\rho u_i u_k)}\right]\bar{\hat{u}}_i+\widehat{\rho u_i}\left[\overline{\widehat{u_k\frac{\partial u_i}{\partial x_k}}}\right]\right\}. \qquad (3.9)$$

Use of (2.14), (2.19), and (2.20) yields

$$\frac{\partial}{\partial t}\Phi_{ii}=-\frac{Re}{8}\left\{i\kappa_k\bar{\hat{u}}_i(\boldsymbol{\kappa})\int_{\boldsymbol{\lambda}}\bar{\hat{u}}_k(\boldsymbol{\kappa}-\boldsymbol{\lambda})\widehat{\rho u_i}(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right.$$

$$\left.-\widehat{\rho u_i}(\boldsymbol{\kappa})\int_{\boldsymbol{\lambda}}\bar{\hat{u}}_k(\boldsymbol{\kappa}-\boldsymbol{\lambda})i\lambda_k\bar{\hat{u}}_i(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right\}. \qquad (3.10)$$

This equation shows that the effect of the inertial terms is complicated and the effect at one wave-number depends on the entire velocity and momentum transforms. Furthermore, the effect on the three-dimensional spectrum of each component depends on all three components of the motion.

Integration over all wave-numbers produces an expression for the change in the total kinetic energy. Reversal of the order of convolution [see (2.15)] shows that

$$\frac{\partial}{\partial t}\int_{\boldsymbol{\kappa}}\Phi_{ii}d\boldsymbol{\kappa}=-\frac{Re}{8}\left\{\int_{\boldsymbol{\lambda}}d\boldsymbol{\lambda}\widehat{\rho u_i}(\boldsymbol{\lambda})\int_{\boldsymbol{\kappa}}\bar{\hat{u}}_k(\boldsymbol{\lambda}-\boldsymbol{\kappa})i\kappa_k\bar{\hat{u}}_i(\boldsymbol{\kappa})d\boldsymbol{\kappa}\right.$$

$$\left.-\int_{\boldsymbol{\kappa}}d\boldsymbol{\kappa}\widehat{\rho u_i}(\boldsymbol{\kappa})\int_{\boldsymbol{\lambda}}\bar{\hat{u}}_k(\boldsymbol{\kappa}-\boldsymbol{\lambda})i\lambda_k\bar{\hat{u}}_i(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right\}=0, \qquad (3.11)$$

since the two integrals have only the formal difference in the variable of integration. Thus inertial effects can cause no change in the total kinetic energy and hence must merely advect kinetic energy from one part of wave-number space to another. Indeed, the sum in (3.11) vanishes separately for each component of the motion, so that inertial effects can cause no change in the total energy associated with individual components. This agrees with Batchelor's result for an incompressible fluid, so that compressibility is not important in the action of the inertia terms on the total kinetic energy.

Pressure gradient effects. The rate of change of the kinetic energy spectrum given by the pressure gradient term is

$$\frac{\partial}{\partial t}\Phi_{ii}=-Re\left\{\widehat{\frac{\partial p}{\partial x_i}}\bar{\hat{u}}_i+\widehat{\rho u_i}\overline{\widehat{\alpha\frac{\partial p}{\partial x_i}}}\right\}. \qquad (3.12)$$

For incompressibility, $\rho=\alpha^{-1}=$constant, so that

$$\frac{\partial}{\partial t}\Phi_{ii}=-Re\{i\kappa_i\hat{p}\bar{\hat{u}}_i-i\kappa_i\bar{\hat{p}}\hat{u}_i\}$$

$$=Re\{\hat{p}\overline{\widehat{\nabla\cdot\mathbf{V}}}+\bar{\hat{p}}\widehat{\nabla\cdot\mathbf{V}}\}=0, \qquad (3.13)$$

since $\nabla\cdot\mathbf{V}=0$ by assumption.

In the general case, the equation of state and (3.12) yield the expression[2]

$$\frac{\partial}{\partial t}\Phi_{ii}=Re\left\{-i\kappa_i\bar{\hat{u}}_i\hat{p}+\widehat{\rho u_i}\left[\overline{i\kappa_i\hat{T}R+\widehat{p\frac{\partial\alpha}{\partial x_i}}}\right]\right\}$$

$$=Re\left\{\hat{p}\overline{\widehat{\nabla\cdot\mathbf{V}}}+R\bar{\hat{T}}\widehat{\nabla\cdot\rho\mathbf{V}}-\frac{\widehat{\rho u_i}(\boldsymbol{\kappa})}{8}\right.$$

$$\left.\times\int_{\boldsymbol{\lambda}}\bar{\hat{p}}(\boldsymbol{\kappa}-\boldsymbol{\lambda})i\lambda_i\bar{\hat{\alpha}}(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right\}. \qquad (3.14)$$

Integration of (3.12) over all wave-numbers with the aid of (2.12) expressed for three dimensions [as is (2.10)] gives

$$\frac{\partial}{\partial t}\int_{\boldsymbol{\kappa}}\Phi_{ii}d\boldsymbol{\kappa}=8\frac{\partial}{\partial t}\int_{\mathbf{x}}\rho u_i^2d\mathbf{x}=-16\int_{\mathbf{x}}u_i\frac{\partial p}{\partial x_i}d\mathbf{x}$$

$$=-16\int_{\mathbf{x}}\mathbf{V}\cdot\nabla p d\mathbf{x}. \qquad (3.15)$$

This, followed by use of the first law of thermodynamics and the equation of continuity combined in the form

$$\left(\frac{\partial u_i}{\partial x_k}\right)\Delta_{ik}(\mu)+\rho\frac{dq}{dt}=\rho C_V\frac{dT}{dt}+p\nabla\cdot\mathbf{V}=\rho T\frac{dS}{dt}, \qquad (3.16)$$

[2] Equation (3.14) is written in this form to show explicitly the dependence on temperature and the momentum divergence; it can be written directly from (3.12) as the first term of (3.14) plus the product of the momentum transform and a convolution of the transforms of specific volume and the derivative of pressure.

along with Gauss's theorem, produces

$$\frac{\partial}{\partial t}\int_{\mathbf{X}}\frac{\rho u_i^2}{2}d\mathbf{x}=-\int_{\mathbf{X}}\left\{C_V\frac{dT}{dt}-T\frac{dS}{dt}\right\}\rho d\mathbf{x}. \qquad (3.17)$$

This is a form of the well-known meteorological energy equation, with the gravitational potential energy term and an advection term (see assumption 3) missing.

Thus the three terms of (3.14) show how internal energy is converted into the kinetic energy of fluid motion. Depending on the signs and magnitudes of all three terms, there may be a conversion of internal energy into kinetic energy at one wave-number, and the opposite conversion at another.

Returning to (3.14), we note that the first two terms are co-spectra, so that they imply a conversion of internal to kinetic energy at a given wave-number if the pressure and divergence and if the temperature and momentum divergence are in phase; the opposite if they are 180 deg out of phase, and no conversion if they are 90 deg out of phase. The effect of these two terms at a given wave-number depends only on their values at that wave-number; the third term of (3.14) expresses the interdependency of effects at various wave-numbers.

For incompressible fluids, (3.16) shows that the right side of (3.17) vanishes and so changes in total kinetic energy due to pressure gradient effects are not possible.

Vertical motion and gravity. The role of gravitational effects will be examined after determining the transform of g, assuming that it is a constant. We find

$$\hat{g}(\boldsymbol{\kappa})=\frac{g}{\pi^{\frac{3}{2}}}\int_{\mathbf{X}}e^{-i\kappa_i x_i}d\mathbf{x}=g\left(\frac{2}{\pi^{\frac{1}{2}}}\right)^3\prod_{j=1}^{3}\frac{\sin\kappa_j X_j}{\kappa_j} \qquad (3.18)$$

(no sum on j).

The rate of change of the cross spectrum between velocity components is

$$\frac{\partial}{\partial t}\hat{u}_i\bar{\hat{u}}_j=-(\hat{u}_i\bar{\hat{g}}\delta_{j3}+\bar{\hat{u}}_j\hat{g}\delta_{i3}), \qquad (3.19)$$

but since both i and j cannot be 3, we take $i=3$ and obtain

$$\frac{\partial}{\partial t}\hat{u}_3\bar{\hat{u}}_j=-g\left(\frac{2}{\pi^{\frac{1}{2}}}\right)^3\bar{\hat{u}}_j(\boldsymbol{\kappa})\prod_{j=1}^{3}\frac{\sin\kappa_j X_j}{\kappa_j}. \qquad (3.20)$$

Thus gravitational effects cause no changes in the cross spectrum between the horizontal components of the wind. The action in transferring variance to or from the vertical component to or from a horizontal component depends on the interaction between the transform $\bar{\hat{u}}_j$ and the product of the sine functions.

The rate of change of the energy spectrum becomes

$$\frac{\partial}{\partial t}\Phi_{ii}=-Re\{\widehat{\rho u_i}\bar{\hat{g}}+\widehat{g\rho}\bar{\hat{u}}_i\}\delta_{i3}, \qquad (3.21)$$

which reduces to

$$\frac{\partial}{\partial t}\Phi_{ii}=-Re\{\widehat{\rho u_3}\hat{g}+g\hat{\rho}\bar{\hat{u}}_3\}. \qquad (3.22)$$

The change in the total kinetic energy is

$$\frac{\partial}{\partial t}\int_{\mathbf{X}}\frac{\rho u_i^2}{2}d\mathbf{x}=-\int_{\mathbf{X}}g\rho u_3 d\mathbf{x}=-\frac{d}{dt}\int_{\mathbf{X}}g\rho z d\mathbf{x}, \qquad (3.23)$$

in which the last integral on the right is obtained from its predecessor with the aid of (2.17), the continuity equation (3.2), and the assumption that the net advection through the boundary is zero. This is the familiar integral for the rate of change of potential energy, and thus (3.22) shows how the transfer of potential energy to kinetic energy is accomplished as a function of wave number. When the density is constant, (3.23) must vanish by the assumption of no mean motion.

Coriolis force effects. The Coriolis force changes the kinetic energy spectrum according to the relation (assuming that the region is small enough that the latitude may be taken as constant)

$$\frac{\partial}{\partial t}\Phi_{ii}=-2\epsilon_{ijk}\omega_j Re\{\widehat{\rho u_i}\bar{\hat{u}}_k+\widehat{\rho u_k}\bar{\hat{u}}_i\}. \qquad (3.24)$$

The changes are thus dependent on both amplitude and phase relations between the various velocity components.

Integration over all wave-numbers shows that

$$\frac{\partial}{\partial t}\int_{\mathbf{X}}\frac{\rho u_i^2}{2}dx=-\epsilon_{ijk}\omega_j\int_{\mathbf{X}}\rho u_i u_k d\mathbf{x}=0, \qquad (3.25)$$

since expansion of this sum produces complete cancellation. The Coriolis effects do not affect the total kinetic energy.

Viscous effects. Two applications of the theorem (2.19) on the transform of a derivative show that the rate of change of the kinetic energy spectrum due to viscous effects (as specified by the Navier-Stokes equations) is

$$\frac{\partial}{\partial t}\Phi_{ii}=\mu Re\left\{\bar{\hat{u}}_i\widehat{\frac{\partial^2 u_i}{\partial x_j^2}}+\widehat{\rho u_i}\overline{\widehat{\alpha\frac{\partial^2 u_i}{\partial x_j^2}}}\right\}$$

$$=-\mu\left\{|\boldsymbol{\kappa}|^2|\hat{u}_i(\boldsymbol{\kappa})|^2+Re\left(\frac{\widehat{\rho u_i}(\boldsymbol{\kappa})}{8}\int_{\lambda}\bar{\hat{\alpha}}(\boldsymbol{\kappa}-\boldsymbol{\lambda})|\boldsymbol{\lambda}|^2\bar{\hat{u}}_i(\boldsymbol{\lambda})d\boldsymbol{\lambda}\right)\right\}. \qquad (3.26)$$

The first term in brackets is clearly always positive, and is exactly half the rate of change of the spectrum for an incompressible fluid. Hence, the compressibility effects are given by the second term. For an incompressible fluid, the first term of (3.26) would be the conjugate and of the second and hence (3.26) reduces to twice the first term, so that the result will agree with Batchelor's.

The first term of (3.26) shows that the viscous effects increase with the square of the wave number. For an incompressible fluid with only viscous forces present, the energy spectrum as a function of time is given by

$$\Phi_{ii}(\boldsymbol{\kappa},t)=\Phi_{ii}(\boldsymbol{\kappa},t_0)e^{-2\mu|\boldsymbol{\kappa}|^2(t-t_0)}, \qquad (3.27)$$

in which t_0 is the time when the other forces vanish.

The change in the total energy is found by integration over all wave-numbers. Reversing the order of integration on the second term of (3.27) produces the result

$$Re\int_{\boldsymbol{\lambda}}\bar{\hat{u}}_i(\boldsymbol{\lambda})|\boldsymbol{\lambda}|^2\left\{\frac{1}{8}\int_{\boldsymbol{\kappa}}\hat{\alpha}(\boldsymbol{\lambda}-\boldsymbol{\kappa})\widehat{\rho u}_i(\boldsymbol{\kappa})d\boldsymbol{\kappa}\right\}d\boldsymbol{\lambda}$$

$$=\int_{\boldsymbol{\lambda}}|\boldsymbol{\lambda}|^2|\hat{u}_i(\boldsymbol{\lambda})|^2d\boldsymbol{\lambda}, \qquad (3.28)$$

since, by virtue of (2.14) the inner integral is the transform of the product

$$\alpha\cdot\rho u_i=u_i.$$

Hence the change in the total energy is given by

$$\frac{\partial}{\partial t}\int_{\mathbf{X}}\frac{\rho u_i^2}{2}d\mathbf{X}=-\frac{\mu}{8}\int_{\boldsymbol{\kappa}}|\boldsymbol{\kappa}|^2|\hat{u}_i(\boldsymbol{\kappa})|^2d\boldsymbol{\kappa}$$

$$=-\mu\int_{|\boldsymbol{\kappa}|\geq 0}|\boldsymbol{\kappa}|^2|\hat{u}_i(\boldsymbol{\kappa})|^2d\boldsymbol{\kappa}.$$

Therefore the viscous forces always dissipate kinetic energy, and the change in the total energy is the same whether the fluid is compressible or not.

Energy equation. If we had used the complete dissipation tensor for the viscous effects (rather than the Navier-Stokes assumption that $\nabla^2\mathbf{V}$ is the dominant term), we would find that the viscous contribution to the rate of change of kinetic energy is

$$\frac{\partial}{\partial t}\int_{\mathbf{X}}\frac{\rho u_i^2}{2}d\mathbf{X}=\int_{\mathbf{X}}u_i\frac{\partial}{\partial x_k}(\Delta_{ik}(\mu))\rho d\mathbf{X}. \qquad (3.29)$$

Combining this result with (3.17) and (3.23), we find

$$\frac{\partial}{\partial t}\int_{\mathbf{X}}\frac{\rho u_i^2}{2}d\mathbf{X}=-\int_{\mathbf{X}}\left\{C_v\frac{dT}{dt}-\frac{dq}{dt}+gu_3\right\}\rho dx$$

$$+\iint u_i\Delta_{ik}(\mu)d\sigma, \qquad (3.30)$$

where Gauss's theorem has been used to convert the last term to a surface integral on the boundary σ. Defining the total energy to be the sum of kinetic, internal, and gravitational potential energies, we see that (3.30) implies that the total energy can be changed only by external heat addition and boundary effects. The energy equation results are well-known and do not depend on the Fourier analysis, but it is worth noting that some of the terms which do not contribute to changes in total energy (inertial or Coriolis effects, for example) may cause changes in the shape of the spectrum.

4. Conclusions

The results here show that the time rate of change of the true kinetic energy spectrum of motion in a compressible baroclinic fluid can be examined by relatively straightforward use of the Fourier transform, given the physically reasonable assumption that the area of interest is finite. It was demonstrated that the introduction of the assumptions of incompressibility and homogeneity give results which are qualitatively the same as Batchelor's for turbulence.

The kinetic energy of flow in compressible fluids can increase by drawing on the reservoir of internal energy through pressure gradient effects and on potential energy through vertical momentum. Kinetic energy is dissipated by viscous effects, and the total dissipation is the same as in an incompressible fluid.

In an incompressible fluid with constant density, however, there is no energy source if there is neither mean motion nor advection through the boundary. We must conclude that small scale motion in homogeneous incompressible fluids actually draws upon the kinetic energy of the mean motion.

It was shown that for flow in compressible fluids, the time rate of change of the kinetic energy spectrum at a given wave number due to inertial, pressure gradient, and viscous effects depends at least in part on the entire transform (and hence the entire field) of the appropriate variables.

The explicit calculation of any of the terms of the expressions given here can be accomplished, given an adequate set of data, by an appropriate reduction of the Fourier transforms to Fourier series and the use of theorems analogous to the ones given.

TABLE OF SYMBOLS

Physical variables and constants

$\mathbf{V}, \mathbf{u}$	velocity vectors
u_i	i^{th} component of velocity
p	pressure
ρ	density
α	specific volume: $\alpha=\rho^{-1}$
T	absolute temperature
C_V	specific heat at constant volume
R	gas constant

dq/dt rate of external heat addition (mechanical units)
S entropy (mechanical units)
μ dynamic viscosity
g the acceleration of gravity
ω_j the j^{th} component of the Coriolis parameter
F_i defined in (3.1)
G_i defined in (3.2)

$\Delta_{ik}(\mu)$ viscous tensor: $\Delta_{ik}(\mu)=\mu\left\{\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}-\frac{2\partial u_j}{3\partial x_j}\right\}$

Independent variables and coordinates

$\mathbf{x}$ three dimensional position vector $\mathbf{x}=\mathbf{i}_1x_1+\mathbf{i}_2x_2+\mathbf{i}_3x_3$
$\mathbf{i}_i$ unit vector in the i^{th} direction
x_i scalar magnitude of i^{th} component
E_3 infinite, three-dimensional Euclidean space
$\boldsymbol{\kappa}$ three dimensional wave-number vector $\boldsymbol{\kappa}=\mathbf{i}_1\kappa_1+\mathbf{i}_2\kappa_2+\mathbf{i}_3\kappa_3$
$\mathbf{X}$ compact support of physical variables in E_3 $\mathbf{X}=\{\mathbf{x}\,|\,-X_1\leq x_1\leq X_1,\ -X_2\leq x_2\leq X_2,\ -X_3\leq x_3\leq X_3\}$

Mathematical symbols

$\int\{\ \}d$ Lebesgue integral: range of integration indicated by differential and subscript on integral: E_3, X, $\mathbf{X}$, κ, λ, $\boldsymbol{\kappa}$, $\boldsymbol{\lambda}$
f, g, h arbitrary functions satisfying mathematical assumptions
d total derivative
∂ partial derivative
∇ Hamiltonian operator
$a\cdot b$ scalar product
e exponential
π 3.1415...
$\hat{f}$ Fourier transform of f: defined in (2.3)
Φ_{ii} kinetic energy spectral density function: defined in (3.7)
i $(-1)^{\frac{1}{2}}$
$\prod$ product operator
$\sum$ summation operator
$\|$ modulus
δ_{ij} Kronecker delta: $\delta_{ij}=1$ for $i=j$, 0 otherwise
ϵ_{ijk} alternating unit tensor:

$$\epsilon_{ijk}=\begin{cases}0 \text{ unless } i\neq j\neq k\\ 1 \text{ if subscripts are an even permutation of } 1, 2, 3\\ -1 \text{ if subscripts are an odd permutation of } 1, 2, 3\end{cases}$$

$\subset$ set inclusion
$\bar{\ }$ complex conjugate
Re, Im real and imaginary parts: $z=Re(z)+iIm(z)$.

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