Stability of Jets in a Divergent Barotropic Fluid

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ABSTRACT

The stability of a two-layer incompressible fluid system on a rotating earth is investigated. The upper layer has infinite depth and is inert; the lower layer has finite depth and a basic west to east zonal velocity of form sec^y. The linearized potential vorticity equation is used for the stability investigation. It is found that both the beta effect due to the curvature of the earth and the divergence tend to stabilize the jet if the winds are from west to east everywhere. However, if there are easterly winds away from the center of the jet, the divergence may not be stabilizing.

This stability theory is applied to a jet at 45 deg latitude in the atmosphere. The maximum wind is 60 m sec^-1 and the half-width of the jet is 1000 km. For the case of no divergence the most unstable wavelength is 5500 km and this disturbance has an e-fold amplification in 1.8 days. If we include divergence, the most unstable wavelength is again 5500 km but the e-fold amplification time is 14 days.

The theory can also be applied to the Gulf Stream. For a current with a maximum velocity of 1.5 m sec^-1, a half-width of 31 km and a depth of 350 m, the most unstable wavelength is 180 km and the e-fold amplification time is 4 days.

1. Introduction

The purpose of this investigation is to study the effect of horizontal mass divergence upon the stability of barotropic flow. There are several areas of geophysical fluid dynamics in which the above problem is relevant. Rossby et al. (1939) were the first to consider a one-layer atmospheric model in which divergence played an important part in determining the phase speed of waves travelling in the direction of the basic flow. Bolin (1955) extended this idea by considering a two-layer atmosphere in which the upper layer was inert and infinite in depth. The lower layer was finite and corresponded to the troposphere. This model predicted the motion of the long waves much better than the nondivergent model and is basically the model widely used for numerical weather prediction.

Another problem where the theory presented below is relevant is that of the stability of the Gulf Stream, considered recently, for example, by Stern (1961). A third area of interest is some laboratory experiments by Faller (1960). In these experiments barotropic jets are set up by means of sources and sinks in a rotating fluid. In addition to the usual effect of divergence caused by the free surface, there is a systematic effect (variation of potential vorticity) due to the free-surface slope induced by the rotation. These experiments, however, are different from the theory presented below in that friction is important at the bottom boundary.

The barotropic stability problem without divergence has previously been considered by Kuo (1949) and Lipps (1962). Recently Wiin-Nielsen (1961) has made an analysis of the barotropic stability problem with divergence. He found that for a channel of finite width the effect of divergence was to stabilize the flow. Thus, both the "beta" effect (due to curvature of the earth) and the divergence tend to stabilize the flow. The present analysis extends that of Wiin-Nielsen in two ways. First, a sufficient condition for stability is established, namely that the gradient of potential vorticity does not change sign anywhere in the fluid. This result is in agreement with the work of Charney and Stern who considered the stability problem for fluid systems with continuous density gradients. Second, we consider the stability of a basic velocity profile in an infinite fluid without the artifice of walls at given latitude circles.

In the next part of the paper we explain the physical model. Then the assumptions used are stated and the perturbation potential vorticity equation is given. In this investigation we consider a fluid made up of two homogeneous, incompressible layers in which the upper layer is infinite and inert. We assume that the flow is quasi-geostrophic in the lower layer and that the basic velocity profile is sufficiently strong and narrow so that the nondimensional beta parameter is of the order of one or less. Under these conditions the beta-plane approximations are valid and therefore do not constitute an additional assumption.
In the third section we derive the sufficient condition for stability of the perturbations. It is shown that if the gradient of potential vorticity of the basic flow is of one sign throughout the fluid, the basic velocity profile is stable.

In section four we discuss the neutral waves. Our main interest is in the neutral waves that are relevant as far as the stability problem is concerned. Since our basic velocity profile is symmetric, the phase velocity of these neutral waves is equal to the value of the basic velocity at a point where the gradient of potential vorticity of the basic flow is zero. For wave numbers only slightly different from those of these neutral waves there are unstable perturbations. We also consider the neutral waves which correspond to the waves of Rossby (1939) and Haurwitz (1940). For these waves the phase velocities are always less than the minimum basic velocity in the fluid. Finally, we discuss the neutral waves which correspond to the waves of Lord Rayleigh (1913) and Case (1960). These waves have a continuous spectrum of phase velocities such that each wave has a phase velocity equal to the value of the basic velocity somewhere in the fluid. These waves are seldom discussed in meteorological literature.

The unstable waves are considered in the fifth section. We find that the beta effect and the divergence tend to stabilize the flow for a west wind jet. However, if the flow is from east to west away from the jet maximum, the divergence can act as a destabilizing factor.

The final section deals with some applications of the above stability theory. We first consider the case of a tropospheric jet in the atmosphere. So that we may make a comparison with the work of Wiin-Nielsen (1961) we consider a jet with a maximum velocity of 60 m sec\(^{-1}\) and a half width of 1000 km at 45 deg latitude. We find this jet to be unstable. The maximum instability is for a wavelength of about 5500 km and the time for this perturbation to increase by a factor of \(e\) is about 1.8 days for the case of no divergence. For the corresponding case with divergence we have again a wavelength of 5500 km and an \(e\)-folding time of 14 days.

We also apply the above theory to the case of the Gulf Stream. For a current of 1.5 m sec\(^{-1}\) with a half width of 31 km and depth of 550 m we find that the most unstable wavelength is 180 km and the amplifying time 4 days.

2. The perturbation equations of motion and boundary conditions

In this study we consider the stability of the two layer system on a rotating earth shown in Fig. 1. Both layers are homogeneous and incompressible. The upper layer is infinite in depth, has a density \(\rho_1\) and is at rest. The lower layer has a mean depth \(H\), a variable depth \(h\), a density \(\rho_0\) and a basic zonal velocity \(U_0\) which is a function of the latitude. We use a set of Cartesian coordinates with \(x\) directed toward the east, \(y\) directed toward the north and \(z\) taken as the vertical coordinate. The zonal velocity \(U_0\) is of the form:

\[
U_0 = V \text{sech}^2 L^{-1} y + V_0,
\]

where the line \(y = 0\) corresponds to some reference latitude \(\phi_0\) and \(V, L\) and \(V_0\) are unspecified constants.

The effect of the earth's rotation comes in through the Coriolis parameters \(f\) and \(\beta\) where \(f = 2\omega \sin \phi\) and \(\beta = d f / dy = 2 a a^{-1} \cos \phi\). The angular velocity of the earth is \(\omega\), the radius of the earth is \(a\) and the latitude is \(\phi\). The "beta-plane" approximation consists in allowing \(f\) to vary with \(y\), \(f = f_0 + \beta y\), in the expression for the vertical component of the absolute vorticity.

We now examine the stability of the basic flow by superimposing infinitesimal perturbations. In this analysis we assume that the flow is hydrostatic and quasigeostrophic. The perturbations therefore must satisfy the linearized potential vorticity equation. For a discussion of the derivation of this equation and the above assumptions for atmospheric flow see Charney and Stern (1962) and Charney (1960). A corresponding discussion for oceanic flow is given by Stern (1961) and Morgan (1956).

It is convenient at this point to define the basic nondimensional quantities. We have

\[
\begin{align*}
x^* &= x / L, \quad y^* &= y / L, \quad t^* &= t V / L, \quad h^* &= g^* h / V f_0 L \\
R_o &= V / f_0 L, \quad F^2 = \frac{f_0^2 L^2}{g^* H}, \quad B = \frac{\beta_0 L^2}{V} \quad (2.2) \\
U^* &= \text{sech}^2 y^* + V_0 / V
\end{align*}
\]

where \(g^* = (\rho_0 - \rho_1) g / \rho_1\) is the "reduced" gravity, \(f_0 = 2 a \sin \phi_0\) and \(\beta_0 = 2 a a^{-1} \cos \phi_0\). The three nondimensional numbers defined here are the Rossby number \(R_o\), the inverse of the nondimensional radius of deformation, \(F\) and the nondimensional beta parameter, \(B\). In the analysis which follows both \(F\) and \(B\) will be of order unity. However, the Rossby number must satisfy \(R_o \ll 1\). This condition is necessary for the flow to be quasigeostrophic.

The nondimensional height of the lower layer may
be written in the form:

\[ h^* = \tilde{h}^* + h_1^* \]

where \( \tilde{h}^* \) is the unperturbed interface height and \( h_1^* \) is the variation of the interface height due to the perturbations.

Now we may give the linearized form of the potential vorticity equation for the dependent variable \( h_1^* \). Using the beta plane approximation\(^\text{2}\) this equation becomes:

\[ \frac{\partial}{\partial t^*} (\nabla h_1^* + F^2 h_1^*) + U^* \frac{\partial}{\partial x^*} (\nabla h_1^* - F^2 h_1^*) + \left( B + \frac{d^2 U^*}{d y^*} + F^2 U^* \right) \frac{\partial h_1^*}{\partial x} = 0 \]

where all quantities are nondimensional.

For \( h_1^* \) we assume a solution of the form

\[ h_1^* = h(y^*) e^{ib(x-y^*)} \]

where the phase velocity \( c^* \) may be complex.

\(^2\) For a previous discussion of the beta plane approximation see Long (1960). The basic requirement for this approximation to be valid is that \( B \sim 1 \) while \( L/a \ll 1 \). From the definition of \( B \) we find:

\[ L/a = B \text{ Ro tan } \phi_0. \]

In the above analysis, since the flow is quasigeostrophic and \( B \) is the order of unity, \( L/a \ll 1 \) in mid-latitudes and the beta plane approximation is therefore valid.

If we define:

\[ c = c^* - \frac{V_0}{V}, \quad U = \text{sech}^2 y^*, \quad B_1 = B + \frac{V_0}{V} \]

we obtain from (2.3), after dropping the asterisks, the differential equation

\[ \frac{\partial h}{\partial y} - (k_1^2 + F^2) h + \frac{B_1 - d^2 U/d y^2 + F^2 U}{U - c} h = 0. \]

This is the form of the potential vorticity equation we will use in the following discussion. The basic velocity profile given by (2.4) was used by the writer in a previous study of nondivergent barotropic instability (Lipps, 1962). Its form is shown in Fig. 2.

For boundary conditions we want \( \nu \) (the component of velocity directed along the positive \( y \)-axis) to vanish sufficiently far away from the jet axis. Since the flow is quasigeostrophic, we require \( h \to 0 \) as \( y \to \pm \infty \).

3. A sufficient condition for stability

In this section we give a sufficient condition for the basic velocity profile to be stable. This condition is that the gradient of potential vorticity is of one sign throughout the fluid. The derivation is similar to that first given by Lord Rayleigh (1880) for the case of plane parallel flow.

To derive this condition we must consider the real and imaginary parts of (2.5). We may divide \( h \) and \( c \) into real and imaginary parts:

\[ h = h_r + i h_i \]
\[ c = c_r + i c_i \]

If we multiply the imaginary part of (2.5) by \( h \) and the real part of (2.5) by \( h \), subtract, integrate over all \( y \) and apply the boundary conditions, we find:

\[ c_i \int_{-\infty}^{\infty} \frac{G |h|^2}{1 + |V - c|^2} \, dy = 0. \]

The quantity \( G = B_1 - d^2 V/d y^2 + F^2 V \) is the gradient of the basic potential vorticity. We see that if the gradient of potential vorticity does not change sign anywhere in the fluid, \( c_r = 0 \) and the flow is stable. Stern (1961) derives a similar expression. In his case \( B = 0 \).

4. Neutral waves

In this section we consider the neutral wave solutions to (2.5). We are most interested in the neutral waves which are relevant to the stability problem but there are also some other types of neutral waves which are of interest and will be discussed. The discussion here closely parallels that of the writer’s previous paper on barotropic stability (Lipps, 1962).
Certain neutral waves are of relevance to the stability problem because there are unstable waves with only slightly different wave numbers. In fact, knowing these neutral wave solutions it is possible to calculate $\partial c / \partial k^2$ and thus make an estimate of the instability of the waves with adjacent wave numbers. We treat this question in the next section.

For these neutral waves it can be shown that $U = c$ at $\gamma = \gamma_e$ where the gradient of potential vorticity vanishes. This result follows since the basic velocity profile is an even function of $\gamma$. [See Foote and Lin (1951), and Kuo (1949). Their results concerning this point can easily be extended to cover the present problem provided $B_1$ and $U$ are greater than zero.]

With this information we obtain the two following values for the phase velocity $c$:

$$c = \frac{1}{\mu} \left( \mp \sqrt{\delta^2 - 6B_1} \right),$$  \hspace{1cm} (4.1)

where $\delta = (4 - F^2)/2$.

We now consider the solutions to (2.5) for these values of the phase velocity. If we make a change of variable and set $z = \tanh \gamma$, (2.5) becomes Legendre's equation with the solutions $P_{2\mu}(z)$, where $\mu$ is given by:

$$\mu = \sqrt{4 + k^2 - 6c}.$$  \hspace{1cm} (4.2)

In this expression it is understood that $c$ must take one of the values given by (4.1).

The boundary conditions are that $k = 0$ for $z = \pm 1$. The only solutions meeting these boundary conditions are $P_1(\pm 1)$ and $P_3(\pm 1)$. We consider the solution $P_3(z)$ first:

$$P_3(z) = 1 - z^2 = \text{sech}^2 \gamma.$$  \hspace{1cm} (4.3)

This solution is valid provided $\mu = 2$ or:

$$k^2 = \delta \pm \sqrt{\delta^2 - 6B_1},$$  \hspace{1cm} (4.4)

This expression represents a surface in ($F^2, B_1, k^2$) space along which the solution (4.3) is valid. To simplify matters we consider first $k^2$ as a function of $B_1$ for given values of $F^2$. We obtain the neutral curves shown in Fig. 3 (solid lines), where we have plotted the curves for $F^2 = 0, F^2 = 1$ and $F^2 = 2$. For a given $F^2$ and $B_1$ we see that there are in general two values of $k^2$ for which (4.3) is valid. It will be shown in the next section that for intermediate values of $k^2$ there are amplified waves and for values of $k^2$ outside this wavelength band there are stable waves.

It is equally reasonable to consider $B_1$ as given and plot $k^2$ as a function of $F^2$. We have shown such neutral curves as solid lines in Fig. 4. Here we find again that for a given value of $B_1$ and $F^2$ there are two values of $k^2$ for which (4.4) is valid. It is also true here that there are unstable waves for intermediate values of $k^2$ and stable waves outside this wavelength band. The curve for $B_1 = 0$ in Fig. 4 is relevant when the stability of the Gulf Stream is considered (see Section 6). The lower portion of this curve is a straight line along the $F^2$-axis.

**Fig. 3.** Curves of neutral stability. The solid lines are neutral curves for the $P_3^1(z)$ neutral disturbance. The dashed line is a neutral curve for the $P_3^2(z)$ disturbance.

**Fig. 4.** Curves of neutral stability. Here we plot the curves of neutral stability in the ($F^2, B_1$)-plane instead of the ($B_1, F^2$)-plane as in Fig. 3. The solid lines are the neutral curves for the $P_3^1(z)$ disturbance and the dashed lines are neutral curves for the $P_3^2(z)$ disturbance.
We now consider the disturbances corresponding to the $P_2^{-1}(z)$ solution:

$$P_2^{-1}(z) = (1-z^2)^{1/2} = \text{sech} \, y \, \tanh \, y.$$  \hfill (4.5)

This solution is valid provided $\mu = 1$ or

$$k^2 = -3 + \sqrt{3^2 - 6B_1}.$$ \hfill (4.6)

Here we have only the plus sign in front of the radical since the minus sign would correspond to negative values of $k^2$. In Fig. 3 we have plotted as the dashed line the neutral curve corresponding to (4.6) for $F^2 = 0$. The $F^2 = 1$ curve corresponds to a point at $k^2 = 0$, $B_1 = 0$. The $F^2 = 2$ curve does not exist for any positive $k^2$. In the next section we will show that for a given $F^2$ and $B_1$ there are unstable disturbances for values of $k^2$ less than that given by (4.6) and for greater values of $k^2$ there are stable disturbances.

Again we may plot $k^2$ versus $F^2$ for given values of $B_1$. Such neutral curves are shown in Fig. 4 as dashed curves for $B_1 = 0$, $B_1 = \frac{1}{6}$ and $B_1 = \frac{1}{3}$.

The neutral waves discussed above are relevant to the stability problem, but there are also other two types of neutral waves. The first type we will call the Rossby (1939)–Haurwitz (1940) neutral waves. These waves are periodic in $y$ as $y \to \pm \infty$ and have phase velocities less than the minimum zonal velocity. In fact as $y \to \pm \infty$, $h \to A \cos \, my + B \sin \, my$ where

$$k^2 + m^2 = \frac{B_1}{c}.$$ \hfill (4.7)

Kuo (1949) and Lipps (1962) find similar waves.

The final type of neutral wave is one that is often neglected although it was first discussed by Lord Rayleigh (1913). These waves have a continuous spectrum of phase velocity such that for each wave $U = c$ for some point or points within the fluid. Case (1960) finds that the disturbance formed by the sum of these disturbances usually dies off at 1/t or faster.

5. Amplified waves

In this section we consider the amplified waves for wave numbers very close to those of the neutral waves given by (4.4) and (4.6). We expand $c$ in a Taylor series of the form

$$c = c_0 + \frac{\partial c}{\partial S} \Delta S + \frac{\partial c}{\partial B_1} \Delta B_1 + \frac{\partial c}{\partial F^2} \Delta F^2 + \cdots,$$ \hfill (5.1)

where $c_0$ is the value of the phase velocity along the neutral curve, and $S = -k^2$. In the work below we will consider that $F^2$ is held fixed. Thus the term $(\partial c/\partial F^2)\Delta F^2$ vanishes.

To calculate $\partial c/\partial S$ and $\partial c/\partial B_1$ along the neutral curves we use the approach previously used by the writer (Lipps, 1962). We find:

$$\frac{\partial c}{\partial S} = \frac{Gh^2}{\int_{-\infty}^{\infty} \frac{1}{(U - c_0)^2} dy},$$ \hfill (5.2)

$$\frac{\partial c}{\partial B_1} = \frac{Gh^2}{\int_{-\infty}^{\infty} \frac{1}{(U - c)^2} dy},$$ \hfill (5.3)

where $h$ is a neutral solution given either by (4.3) or (4.5), $c_0$ is the relevant phase velocity, and $G = B_1 - dU/dy^2 + F^2 U$.

Each of these expressions has two singularities in the denominator where $U = c_0$, and we must integrate around these points in the complex $y$-plane. To decide whether to integrate above or below these points we use the criterion of Foote and Lin (1951). By taking the limit of the viscous solution in the limit of vanishing viscosity they find that the path of integration should be taken above the point where $dU/dy < 0$ and below the point where $dU/dy > 0$. A similar situation holds for the singularities in the numerator of (5.3).

We evaluate the expressions (5.2) and (5.3) along the neutral curves $F^2 = 0$ and $F^2 = 1$ in Fig. 3. The case of $F^2 = 0$ has already been considered by the writer (Lipps, 1962). The present $B$ corresponds to $\frac{1}{4} \chi$ in that paper; otherwise the results are the same. After calculating the values of $\partial c/\partial S$ and $\partial c/\partial B_1$ along the neutral curves in Fig. 3 we can then find an estimate of the $c_1 = 0.025$ curves from the Taylor series (5.1).

We first give the results previously obtained for the $F^2 = 0$ case so that they may be compared with the new results for $F^2 = 1$. For $F^2 = 0$, $B_1 = B$ so that we may drop the subscript on $B$. In Fig. 5 we show the $P_2^{-1}(z)$ neutral curve for $F^2 = 0$ and the estimated $c_1 = 0.025$ curve found from the calculated values of $\partial c/\partial S$ and $\partial c/\partial B_1$ along the neutral curve. The disturbances relevant to this figure for which $h$ is an even function of $y$ we call the symmetric disturbances. We note that, for a given $B$, the disturbances with wavelengths between those of the neutral waves are unstable. Disturbances outside this wavelength band are stable.

In Fig. 6 we show the $P_2^{-1}(z)$ neutral curve for $F^2 = 0$ and the estimated $c_1 = 0.025$ curve. For disturbances which are relevant to this figure $h$ is an odd function of $y$. We call these the asymmetric disturbances. We note that, for a given $B$, disturbances with wavelengths longer than that of the neutral wave are unstable and disturbances with shorter wavelengths are stable.
Now we consider the case $F^2=1$. In this case the $P_3^1(z)$ neutral curve is reduced to the point $k^2=0, B_1=0$. It can be shown from the value of $\partial c/\partial S$ that there are no amplifying waves for $k^2>0$ which are related to the $P_3^2(z)$ neutral solution. Thus we need not consider this neutral solution any further. We will, however, find unstable solutions near the $P_3^2(z)$ neutral curve. Hence it is apparent that the symmetric disturbances are more unstable than the asymmetric disturbances. The same result is also true for $F^2=0$ in the sense that the symmetric disturbances have larger growth rates than the asymmetric disturbances. This result is often assumed in similar stability investigations [c.f., Kuo (1949) and Wiin-Nielsen (1961)].

The $P_3^2(z)$ neutral curve is shown in Fig. 7. In order to find the $c_1=0.025$ curve we calculate $\partial c/\partial S$ and $\partial c/\partial B_1$ along the neutral curve. From (5.2) and (5.3) we find

$$\frac{\partial c}{\partial S} = \frac{1}{3\left[-6c_0 - \frac{1}{2} + \frac{c_0(3c_0 - 3/4)}{\sqrt{1-c_0}}(\log \eta - \pi i)\right]}$$

(5.4)

$$\frac{\partial c}{\partial B_1} = \frac{1}{2\sqrt{1-c_0}}$$

where

$$\eta = \frac{1-\sqrt{1-c_0}}{1+\sqrt{1-c_0}}.$$

Along the upper branch of the neutral curve in Fig. 7 the phase velocity is

$$c_0 = \frac{1}{4}(1 + \sqrt{1-(8/3)B_1}),$$

and along the lower branch of this curve it is

$$c_0 = \frac{1}{4}(1 - \sqrt{1-(8/3)B_1}).$$
Thus we again infer that for wave numbers between those of the two branches of the neutral curve we have unstable disturbances. From the values of $\partial c/\partial S$ and $\partial c/\partial B_1$ given by (5.4) we estimate the curve $c_i=0.025$ shown in Fig. 7. These values of $\partial c/\partial S$ and $\partial c/\partial B_1$ are shown in Table 1.

Table 1. The values of $\partial c/\partial S$ and $\partial c/\partial B_1$ calculated along the $P^2(z)$ neutral curve in Fig. 7 for $F^2=1$.

<table>
<thead>
<tr>
<th>$k^2$</th>
<th>$B_1$</th>
<th>$\partial c/\partial S$</th>
<th>$\partial c/\partial B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.00</td>
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<td>-0.0659</td>
<td>-0.201</td>
</tr>
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</tr>
<tr>
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</tr>
<tr>
<td>0.00</td>
<td>0</td>
<td>-0.667</td>
<td>-1.000</td>
</tr>
</tbody>
</table>

On comparison of Fig. 5 with Fig. 7 it is evident that the effect of divergence tends to stabilize the jet. If $F^2=0$ the jet becomes unstable for $B_1 < \frac{3}{4}$. However, if $F^2=1$ the jet becomes unstable for $B_1 < \frac{3}{5}$. In an example given later it will be seen that the growth rates are larger for $F^2=0$ than for $F^2=1$ for a west wind jet. Thus the stabilizing effect of divergence for a west wind jet manifests itself in two ways. When divergence is present, the jet is stable for smaller values of $B_1$ and the amplification rates for an unstable jet are smaller. If, however, we have east winds away from the center of the jet, $B_1$ is less than $B$ and it is not immediately clear that the effect of divergence is stabilizing. This effect will be discussed more fully in the next section.

It should be noted that $B_1 \geq \frac{3}{5}$ for $F^2=0$ and $B_1 \geq \frac{3}{5}$ for $F^2=1$ are just the necessary conditions for the gradient of potential vorticity to be non-negative throughout the fluid. Thus these results are in agreement with equation (3.2) which shows that if the gradient of potential vorticity does not change sign, the basic flow is stable. In this case equation (3.2) gives both a necessary and sufficient condition for stability.

6. Geophysical applications

In this section we apply the above stability theory to jets in the troposphere and to the Gulf Stream. In the former case the lower layer corresponds to the troposphere and the upper layer represents an inert stratosphere. For the Gulf Stream it is the upper layer that is of finite depth, while the lower layer is considered to be infinite in depth and inert. It can be shown that the perturbation potential vorticity equation takes the same form for the Gulf Stream stability problem as for the atmospheric problem. Thus the analysis above applies if the basic velocity in the upper level is of the form $sech^2$.

The above theory will be applied to a jet at latitude 45 deg so that we may compare our results with those of Wiin-Nielsen. We first consider the effect of divergence on the sufficient condition for stability. To this end we show Fig. 8. Here we allow $V$, $V_0$, and $L$ of the dimensional basic velocity (2.1) to vary, but hold the other parameters fixed. We define $q^2 = \frac{f_0^2}{g} H$ and set it equal to $10^{-12}$ m$^{-2}$. At 45 deg $\beta$ has the value $1.619 \times 10^{-11}$ m$^{-1}$ sec$^{-1}$. These values are used in Fig. 8. For the curves shown, the gradient of potential vorticity is positive everywhere except at one point where it is zero, so along these curves the jet is stable. If, however, the value of $L^2$ or $V_0$ is decreased or $V$ is increased from the value at any point along one of these curves, the jet becomes unstable. Thus the lines shown can be said to represent conditions under which the jet is marginally stable. We have plotted curves for $V_0 = 10$ m per sec, $V_0 = -10$ m per sec. The dashed curve is the marginally stable curve for the case of no divergence. It is a straight line in this figure. The figure shows that the effect of divergence tends to make the jet more stable for positive values of $V_0$. For negative values of $V_0$, which correspond to an east wind on either side of the jet, the situation is different. In this case the jet is more unstable in the presence of divergence than with no divergence for sufficiently narrow jets. For sufficiently broad jets the effect of divergence is again stabilizing.

We now consider the effect of divergence for a particular velocity profile centered at 45 deg. For this example we take $q^2 = 10^{-12}$ m$^{-2}$, $L = 10^5$ m, $V_0 = 0$, and $V = 60$ m per sec. Thus $F^2 = 1$ and $B_1 = 0.27$. This basic velocity profile is somewhat similar to one of the cases considered by Wiin-Nielsen. We will estimate the amplification rate

![Fig. 8. Stability of sech^2 jet at 45 deg. The quantities $V$, $V_0$, and $L^2$ are defined by (2.1). The units of the velocities $V$ and $V_0$ are m sec$^{-1}$, and the units of $L^2$ are m$^2$. For explanation see text.](image-url)
for the most unstable disturbance for the same basic velocity profile with $F^2=0$. It will be found for this particular example that the nondivergent model gives much larger values for the amplification rates.

In Fig. 9 we have plotted $c_t$ versus $k^2$ for the jet with $F^2=1$ and $B_1=0.27$. Here $c_t$ and $k^2$ are nondimensional. The dashed lines are the slopes $\partial c_t/\partial k^2$ calculated from $\partial c/\partial S$. The triangles represent values of $c_t$ obtained from calculating $\partial c/\partial B_1$ along the neutral curve in Fig. 7. From this curve we find a maximum amplification rate such that the most unstable disturbance will increase by a factor of $e$ in about 14 days. The wavelength of this disturbance is near 5500 km.

We now consider the case with no divergence. Fig. 10 shows a plot of $c_t$ versus $k^2$. It is immediately evident that this jet is much more unstable. The squares represent points from the $c_t=0.025$ curve in Fig. 5. Otherwise the notation is the same as in Fig. 9. We plot the curve below the triangle at $k^2=2$, since the data of Lessen and Fox (1955) for the case $B=0$ indicate that this value of $c_t$ is probably too large. From this curve we find that the most unstable disturbance will amplify by a factor of $e$ in 1.8 days. The wavelength of this disturbance is again near 5500 km.

In Fig. 11 we show the basic potential vorticity profiles for the cases $F^2=0$ and $F^2=1$ with $B_1=0.27$. It can be seen that the $F^2=0$ potential vorticity profile is the most unstable.

These results can be compared with those of Wiin-Nielsen (1961). He considers the stability of a sinusoidal type zonal velocity profile for divergent barotropic flow confined between walls at given latitude circles. The beta-plane approximation is used so that the problem is essentially reduced to that of the stability of a basic flow in a channel where the effect of the variable Coriolis parameter has been included. A solution for the complex phase velocity is obtained by assuming that the perturbations have only two Fourier components in the meridional direction.

Therefore when a comparison is made between the theory presented above and that of Wiin-Nielsen it should be remembered that both theories give only approximate results for the amplification rates. The amplification rates calculated in this paper are approximate because we consider only the first derivatives $\partial c/\partial S$ and $\partial c/\partial B_1$ in estimating $c_t$, while in Wiin-Nielsen's work the calculated phase velocities are approximate because no more than the first two Fourier components are considered. It also should be remembered that the two physical models are different. The zonal velocity has a different shape for each model and the walls in Wiin-Nielsen's model will have an effect on the stability of the flow. Hence we can expect at most qualitative agreement between the two models.

Wiin-Nielsen considers the effect of divergence for different channel widths. In one set of calculations he considers a channel width of 3000 km. We will use this channel width in comparing results from his theory and the above analysis. The jet is centered at 45 deg in both analyses. For the case of no divergence Wiin-Nielsen’s re-
We now consider the stability of the Gulf Stream. The actual distribution of temperature and velocity is shown by Stommel (1958) in his Figs. 32–34. We idealize this picture by considering a two-layer incompressible fluid. The upper layer corresponds to the layer in the ocean above the 10°C isotherm. In this layer we take the basic velocity to be of the form sech²⁷. The lower layer has an infinite depth and is inert. This situation is shown in Fig. 12. The velocity in the upper layer is taken to be into the paper. The coordinates are taken so that x is in the direction of the basic flow, the y-axis points to the left across the stream and the z-axis is in the vertical. In Fig. 12 the height of the free surface is h₁, the height of the interface is h₀, and the thickness of the upper layer is h₂.

After making the same assumptions as in Section 2 we arrive at the perturbation potential vorticity equation given by (2.5) with B₁ = 0. The dependent variable is the thickness of the upper layer h. The B₁ term drops out because we take the basic velocity profile to vanish as y → ±∞ and B₁ is negligible as can be seen from the dimensions of the current given below. [For a derivation of (2.5) for the Gulf Stream see Stern (1961), who also treats the stability problem.]

For this problem we again have F² = f₀L²/g⁺H where H is the mean depth of the upper layer and the other symbols have the same meaning as before. Likewise all the other nondimensional variables have the same form as previously.

Now we determine the relevant values of R₀ and F². If we give to V the value of the mean velocity in the upper layer at the point of maximum current we find V ≥ 1.5 m per sec. From Stommel, Fig. 32, we find for L the value 31 km. For f₀ we take the value of the Coriolis parameter at 38 deg which is 0.897 × 10⁻⁴ sec⁻¹.

Hence we find for the Rossby number R₀ = 0.540.

The value of F² is somewhat more difficult to find since it is evident from Fig. 33 of Stommel that the mean density varies continuously with depth and is not discontinuous as assumed in the above model. We therefore find F² by the following somewhat indirect procedure. Stommel states that the main thermocline (10°C isotherm) slopes from 200 m below the surface to the left of the current to 900 m below the surface to the right of the current. To find F² we likewise require the thickness of the upper layer to vary from 200 to 900 m across the current.

In order to discuss the variation in thickness of the upper layer we need an expression for the basic height of this layer. Such an expression can be obtained since the flow is quasigeostrophic and the mean height of the layer is H. The nondimensional form of this height is:

\[ \hat{h}(y) = (F²R₀)^{-1} \tanh y \]  

(6.1)

where the first term on the right is the nondimensional form of the mean height H and the second term represents the variation in height needed to maintain the
basic flow in geostrophic balance. The nondimensionalization is carried out as in (2.2).

For the variation in thickness that Stommel describes we must have:

$$F^2 \text{Ro} = 7/11.$$  

Hence we find $F^2 = 1.2$. [The corresponding value of $(\rho_0 - \rho_1) \rho_0^{-1}$ is 1/840. The value usually stated is 1/150 (e.g., Stommel, p. 111).] Therefore we will take $F^2 = 1$ for this analysis. From the above stability theory we plot in Fig. 13 $c_1$ versus $k^2$ for the case $F^2 = 1$, $B_1 = 0$. At $k^2 = 0$, $c_1 = 0$ we have $\partial c_1 / \partial k^2 = 0$ as is shown by the horizontal dashed lines. The fastest growing disturbance will amplify by a factor of $e$ in 4 days. The wavelength of this disturbance is about 180 km.

This amplification rate seems rather large. The reason may be that in the actual ocean there are dynamical processes which are neglected here, such as the action of friction. It also should be noted that the quasigeostrophic approximation is poor for both the atmospheric jet and the Gulf Stream current. In both cases $\text{Ro} \approx \frac{1}{2}$. Perhaps for a more exact theory the instability of the Gulf Stream would not be so great. Again, it is obvious that the two-level model itself is a crude approximation to the vertical structure of these fluid motions and this approximation may bring a significant inaccuracy into the calculated amplification rates. In this regard it is interesting to note that the current is much more stable for $F^2 = 2$. If we keep $V$ as 1.5 m sec$^{-1}$ and $L$ as 31 km but take $F^2 = 2$, we find that the most unstable wavelength is near 160 km. The amplitude of this wave will increase by a factor of $e$ in 18 days. Hence it is apparent that a relatively small change in $F^2$ can make a large change in the stability of the Gulf Stream.

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