The Spectrum of Nearly Inertial Turbulence in a Stably Stratified Fluid

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(Manuscript received 5 August 1963, in revised form 23 October 1963)

ABSTRACT

The assumptions leading to the classical inertial subrange spectral forms are critically examined. It is shown how, by slightly relaxing the assumptions, a form can be obtained for the spectrum in the presence of other energy sinks. The technique is applied to a stably stratified medium, and the results compared with experiments and other theories.

1. Introduction

In a recent paper Shur (1962) has presented spectra of vertical velocities measured in clear air turbulence. These spectra may be adequately represented as proportional to the inverse third power of wave number at low wave numbers, undergoing a transition to the familiar $-5/3$ power at higher wave numbers. This behavior is not predicted by the expressions of Bolgiano (1959, 1962) for the buoyant subrange. Shur therefore sought (and found) a different explanation, although his derivation "$\cdots$is not rigorous$\cdots$." We present here a rederivation of his result, with some important differences. In particular, the result is shown to be of greater generality than is made clear in Shur's paper, while at the same time the physical model underlying the result is delineated, since the model is felt to be of broader interest.

2. Spectral dependence on energy flux

The inertial subrange of turbulence (Kolmogorov, 1941), when it exists, is characterized completely by the dissipation ϵ. This is the rate at which energy is being dissipated to heat, and also the rate at which energy is being produced, since equilibrium is assumed. It is also the rate at which energy is being transferred through the spectrum, since each narrow wave number range is regarded as receiving energy by inertial transfer from smaller wave numbers at the same rate at which it is inertially transferring energy to larger wave numbers. If ϵ then is regarded as a spectral energy flux, the inertial subrange is characterized by constant spectral energy flux. It is apparent that there are two assumptions here: first, that the spectrum is fully determined by the spectral energy flux, and second that the spectral energy flux is constant. Now, a non-constant spectral energy flux at a particular wave number means that energy is being transferred into or out of the spectrum in that vicinity. Such transfer would ordinarily be expected to introduce other parameters, so that the spectrum could not be regarded as a function of the spectral energy flux only. There is thus a dependence between the two assumptions.

There is, however, a circumstance in which it may be possible to discard the second assumption while keeping the first. If the transfer out of the spectrum is a small fraction of the spectral flux, then the situation may be likened to flow in a slightly porous pipe; the leakage serves mainly to determine the volume flow, and affects the velocity distribution only slightly. The quantity $|\epsilon/(\partial\epsilon/\partial k)|$ serves as a measure of the wave number scale of the variation; it is the wave number "bandwidth" over which appreciable variation in ϵ occurs.[1] We may expect that if this is considerably larger than the wave number, we may apply the approximation that the spectrum is determined locally by the spectral flux, which in its turn is determined by the transfer out. If the transfer out is anisotropic in nature, we expect to find anisotropy in the spectrum; again, if $|(\epsilon/k)/(\partial\epsilon/\partial k)|$ is large compared to unity, we may expect that the spectrum will remain approximately isotropic, since the transfer out is a small part of the "through-put" (ϵ). Such an assumption then is consistent with approximate local isotropy and the use of all the customary forms familiar in the inertial subrange, but with variable ϵ. Turbulence satisfying this assumption could be described as "locally inertial in wave number space."[2]

3. Turbulence in a stratified medium

We wish to apply this reasoning to the case of turbulence with a vertical mean temperature gradient in a gravitational field. We will assume that the temperature gradient is small and nearly uniform. At wave numbers well above the production range and well below the dissipative range, the only source of transfer out of the spectrum is the rate of working against the Archimedian

[1] The argument presented here bears certain resemblances to the argument by Corrsin (1961); in particular the restriction on the magnitude of $|(\epsilon/k)/(\partial\epsilon/\partial k)|$ can be obtained as the ratio of a time characteristic of the leakage and a spectral transfer time.

[2] Suggested to the writer by S. Corrsin.

forces. The net rate at which work is done against gravity per unit mass can be obtained from the equations of motion (Richardson, 1920):

$$g\overline{\vartheta u_3}/T_0, \qquad (1)$$

where g is the acceleration of gravity, ϑ the fluctuation in temperature, u_3 the vertical velocity, and T_0 the temperature of an adiabatic atmosphere. The spectral equations show that the rate (per unit mass) at which eddies of size k do work against gravity is given by the spherical average of the cospectrum of $g\overline{\vartheta u_3}/T_0$. The spherical average is taken to remove directional information.

In order to make an estimate for the spectrum of $\overline{\vartheta u_3}$ we must revert for a moment to a more general coordinate system. Suppose that vertical is in an arbitrary direction, so that the gradient of mean potential temperature becomes a vector, $\Theta_{,j}$. We will use cartesian tensor notation in which $\Theta_{,j}$ denotes $\partial\Theta/\partial x_j$. If the Reynolds and Peclet numbers are sufficiently high, and the wave number sufficiently low, the temperature fluctuation equation can be written as

$$\frac{\partial\vartheta}{\partial t}+u_i\vartheta_{,i}+\Theta_{,j}u_j=0. \qquad (2)$$

Let Θ' and Θ'' be the magnitudes of the first and second derivatives of Θ with respect to height. If we assume that

$$\sqrt{2}\Theta''L_E/\Theta'\ll 1, \qquad (3)$$

we may regard the gradient of Θ as approximately uniform. L_E is the Eulerian space integral scale, and we have used the approximation for very large Reynolds number that $L_E\cong u'T_L$, T_L being the Lagrangian time integral scale (Corrsin, 1963). Under these circumstances we can solve (2) to obtain

$$\vartheta(\mathbf{x},t)=-\Theta_{,j}[x_j-a_j(\mathbf{x},t)] \qquad (4)$$

$\mathbf{a}(\mathbf{x},t)$ is the position at $t=0$ of that point which will arrive at $\mathbf{x}$ at time t; thus $\mathbf{x}-\mathbf{a}$ is the Eulerian displacement vector.

The importance of this result is the conclusion that one may draw from it, that, except for multiplication by a constant vector, the temperature fluctuation field is determined solely by the velocity field. Thus, we can write

$$\begin{aligned}\overline{\vartheta(\mathbf{x},t)u_i(\mathbf{x}',t)}&=-\Theta_{,j}\overline{(x_j-a_j)u_i(\mathbf{x}',t)} \qquad (5)\\ &=-\Theta_{,j}A_{ij}\end{aligned}$$

where A_{ij} is a second rank tensor determined completely by the velocity field. This permits statements to be made in considerable detail about the temperature field, based on assumptions regarding the velocity field. For example, if the velocity field is isotropic, then A_{ij} has an isotropic form. If we take the field to be homogeneous, so that (5) is dependent only on $\mathbf{x}-\mathbf{x}'=\mathbf{r}$, and consider a range of values of r ($r^2=\mathbf{r}\cdot\mathbf{r}$) for which the velocity field is locally isotropic, then A_{ij} has the form $A(r)r_ir_j+B(r)\delta_{ij}$ which is even in $\mathbf{r}$; hence the quadrature spectrum of $\overline{\vartheta u_i}$ vanishes. The spectrum can be written

$$\mathcal{S}_{\overline{\vartheta u_i}}=-\Theta_{,j}(\mathcal{A}(k)k_ik_j+\mathcal{B}(k)\delta_{ij}) \qquad (6)$$

where $\mathbf{k}$ is a wave number vector, $k^2=\mathbf{k}\cdot\mathbf{k}$. The requirement for incompressibility can be satisfied by requiring that $k_i\mathcal{S}_{\overline{\vartheta u_i}}=0$ or $\mathcal{A}k^2+\mathcal{B}=0$; hence

$$\mathcal{S}_{\overline{\vartheta u_i}}=-\Theta_{,j}\mathcal{B}(k)\left(\delta_{ij}-\frac{k_ik_j}{k^2}\right). \qquad (7)$$

$\mathcal{B}$ is real (due to the symmetry); hence $\mathcal{S}_{\overline{\vartheta u_i}}=Co_{\overline{\vartheta u_i}}$. Averaging over a spherical shell, we can write

$$\oint Co_{\overline{\vartheta u_i}}d\sigma=-\Theta_{,i}(4/3)\pi k^2\mathcal{B}(k). \qquad (8)$$

Now, $k^2\mathcal{B}(k)$ is determined solely by the velocity spectrum. If this is locally inertial, then $k^2\mathcal{B}(k)$ is dependent only on ϵ and k. On dimensional grounds, then, it must have the form (reverting to $\Theta_{,i}=\Theta'\delta_{3i}$),

$$\oint Co_{\overline{\vartheta u_3}}d\sigma=-\Theta'c\epsilon^{1/3}k^{-7/3} \qquad (9)$$

where c is a constant which we may hope is of order unity.

We will use this form in what follows, but it should be noted that (9) is dependent only on (5) and the assumption of locally inertial behavior in wave number space, and not on the assumption of isotropy inherent in (6), (7) and (8); for (9) could have been obtained directly from (5) by taking

$$\oint Co_{\overline{\vartheta u_i}}d\sigma=-\Theta'\oint Co_{A_{33}}d\sigma \qquad (10)$$

and, since A_{33} depends only on the velocity field, and $\oint Co_{A_{33}}d\sigma$ depends only on k, local inertiality and dimensional reasoning are all that are required for (9).

The rate at which energy is leaving the spectrum is given by

$$\frac{\partial\epsilon}{\partial k}=-\frac{g}{T_0}\Theta'c\epsilon^{1/3}k^{-7/3}. \qquad (11)$$

This is a non-linear first order equation for the spectral flux. It can be solved to give

$$\begin{aligned}\epsilon^{2/3}&=\epsilon_0^{2/3}+(g/T_0)\Theta'(c/2)k^{-4/3}\\ &=\epsilon_0^{2/3}(1+(g/T_0)\Theta'\epsilon_0^{-2/3}(c/2)k^{-4/3})\\ &=\epsilon_0^{2/3}\{1+(k/k_b)^{-4/3}\} \qquad (12)\end{aligned}$$

where the condition has been applied that, for large wave number, ϵ must go to ϵ_0, the viscous dissipation.[3] The spectrum is given by

$$E=\alpha\epsilon_0^{2/3}\{1+(k/k_b)^{-4/3}\}k^{-5/3} \tag{13}$$

which has the expected behavior.

The criterion on the applicability of local inertiality becomes

$$\left|\frac{k}{\epsilon}\frac{\partial\epsilon}{\partial k}\right|=2\frac{1}{1+(k/k_b)^{4/3}}\ll 1. \tag{14}$$

The criterion is certainly satisfied for $(k/k_b)^{4/3}\gg 1$; while for $k\sim k_b$, times whose ratio is represented by (14) are approximately equal. For $(k/k_b)^{4/3}\ll 1$, which is the region in which we expect (13) to approach proportionality to k^{-3}, (14) approaches the value 2. The success of (13) in correlating Shur's data (1962) seems to imply that $|(k/\epsilon)/(\partial\epsilon/\partial k)|\ll 1$ may be too strong a criterion, and in fact $|(k/\epsilon)/(\partial\epsilon/\partial k)|\sim 1$ or smaller may be adequate.[4]

We can now make an estimate of c. Let us assume that the Reynolds number is so large that the spectrum can be approximated as rising sharply from zero at k_0, having the form (13) until a dissipative wave number k_d, and falling abruptly to zero there. Let us assume further that $k_b\gg k_0$. Then we may integrate (9) to obtain $-\overline{\vartheta u_3}/\Theta'=K_h$, the eddy diffusivity for heat. At k_0, ϵ rises to $\epsilon_0(k_b/k_0)^2\cong\epsilon_T$ say, the total energy available for dissipation. If we take $\epsilon_T=\overset{*}{u}{}^3k_0$, and $K=\overset{*}{u}/k_0$, the eddy diffusivity for momentum, then c is equal to $2K_h/K$, i.e., twice the inverse of the turbulent Prandtl number.

We may relate k_0 and k_b under the same assumptions by defining a flux Richardson number as $R_f=(\epsilon_T-\epsilon_0)/\epsilon_T$; that is, the fraction of the energy being dissipated which is dissipated against Archimedian forces. Then $k_0/k_b=\sqrt{1-R_f}$. Note that in order to have k_0/k_b moderately small, it is necessary to have R_f quite close to unity; this seems to be consistent with strong stratification, in which the large amplitude slow excursions associated with the wave-like motion might be expected to dissipate the majority of the energy through thermal conduction, leaving little to be dissipated by viscous action at larger wave numbers.

Exactly the same reasoning can be applied to the Lagrangian time spectrum; we find

$$\epsilon=\epsilon_0 e^{(\omega/\omega_b)^{-2}},\qquad \omega_b^2=\frac{g}{T_0}\frac{\Theta'\beta}{2} \tag{15}$$

(where β is a constant which may be of order unity). However, we also find

$$\frac{\omega}{\epsilon}\frac{\partial\epsilon}{\partial\omega}=-2\left(\frac{\omega_b}{\omega}\right)^2. \tag{16}$$

Hence (15) is presumably not an acceptable representation of the spectrum below ω_b.

4. Discussion

We may conclude that, if the phenomenon measured by Shur (1962) is that described here, then in light of the failure of (14), the dominant influence in determining the form of the spectrum is the spectral flux, even where this is far from constant.

The question of the connection of this result with the measurements of Shur, and their interpretation, must be raised. It has been suggested that the low wave number results of Shur are random internal waves, rather than turbulence. This is a semantic question, and relates to the proper definition of the phenomenon "turbulence." If this is taken to be any random, three-dimensional velocity field with a continuous spectrum, displaying spectral transfer and dissipation at high wave numbers, then the case of internal waves is also included. Spectrally, in a stably stratified medium, we may expect behavior to be most wave-like at low wave numbers; with an increase in wave number, inertia forces become important and breaking begins; when inertia forces dominate, the field is no longer wave-like; at the highest wave numbers, viscous forces become important. No matter what the definition, the motion at the lowest wave numbers is nearly irrotational (since the vorticity spectrum is proportional to the second moment of the velocity spectrum). It is to be anticipated that energy imported to, or produced in, a temperature inversion will excite a quite intense band of internal waves at low wave numbers in addition to an inertial and viscous field at higher wave numbers, and the same techniques can be used (other things being equal) to analyze the field at low wave numbers as at high.

It is instructive to contrast the present work with that of Bolgiano (1959, 1962). In his work it is assumed that there is a wave number range in which the form of the spectrum is determined solely by the net rate of dissipation of mean-square specific buoyancy forces, since this is a convenient measure of the mean-square intensity of specific buoyancy forces. This assumption leads to a dependence of E on $k^{-11/5}$. Implicit in this reasoning, it would seem, is the assumption that the rate of production of mean-square specific buoyancy forces is uniform in this wave number range. The present model, on the other hand, permits this production to vary with wave number. In addition, it is assumed in this paper that

[3] $k_b=(c/2)^{3/4}k_0$ where k_0 is the wave number defined by Dougherty (1961).

[4] The ideas presented here can also be used to derive the results of Corrsin (1961); the criterion (14) is satisfied by the spectra in that paper.

the dominant influence is the rate at which work is done by the specific buoyancy force (since it will not affect the energy distribution if it does no work). Finally, it is assumed herein that the form of the spectrum is dependent only on the spectral energy flux, even when this is not constant.

The emphasis on the rate of working of the specific buoyancy force as against emphasis on the intensity of that force, places this work in agreement with that of Dougherty (1961) (who did not determine a form for the spectrum, however). Dougherty also contrasts his argument with that of Bolgiano in much the same terms as above.

Finally, the work of Monin (1962) should be mentioned. By a Heisenberg type approximation, Monin obtains an interpolation form between the $k^{-5/3}$ law and the law of Bolgiano. The latter is again a result of assuming dependence on the rate of dissipation of specific buoyancy forces. Another choice of parameters would give another asymptote.

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