A Note on Charney's Model of Zonal-Wind Instability[1]

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Burger (1962) has recently demonstrated that Charney's (1947) baroclinic model of the zonal wind, in which the vertical shear is constant, implies exponential instability for disturbances of almost all wavelengths. It is the purpose of this note to elaborate upon this rather unexpected conclusion by showing that there is a transition from a relatively strong instability to a relatively weak instability as the wavelength increases through each of an infinite, discrete set of critical values. (Each point in this set maps into a monotonic curve in a wavelength, wind-shear plane, and Charney had designated the highest of these curves as the "critical stability curve.")

Charney's model, as formulated by Burger, reduces the stability problem to the determination of the roots, say $x=\lambda$, of

$$\Phi(x)=x^2[\Psi'(x)-\alpha\Psi(x)]=0, \tag{1}$$

where

$$\Psi(x)\equiv\Psi(a,2;x) \tag{2}$$

is a confluent hypergeometric function of the second kind in the notation of Erdélyi *et al.* (1953). The parameters a and α are given by

$$a=a(k)=1-\tfrac{1}{2}\alpha_1^{-1}[(g/\gamma RT)+(\beta gs/f^2U_z)], \tag{3}$$

and

$$\alpha=\tfrac{1}{2}[1-(g/2\gamma RT)\alpha_1^{-1}], \tag{4}$$

where

$$\alpha_1=[(gsk^2/f^2)+(g/2\gamma RT)^2]^{\frac{1}{2}}, \tag{5}$$

T and s are mean values of the temperature and stability, U_z is the vertical shear, k is the wave number (wavelength$=2\pi/k$), and the remaining symbols have their usual meanings. We observe that: $a<1$ and $0<\alpha<\frac{1}{2}$; both a and α increase monotonically with k; and a increases monotonically with U_z. The complex wave speed is given by

$$c=-\tfrac{1}{2}(U_z/\alpha_1)\lambda, \tag{6}$$

and complex values of λ (which must occur in conjugate pairs) imply instability.

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Burger proved that $\Phi(x)$ has: either n distinct, positive-real zeros and 2 complex-conjugate zeros if

$$-n<a<1-n \quad (n=0,\ 1,\ \ldots); \tag{7}$$

or n distinct, positive-real zeros and one double zero, $\lambda=0$, if $a=-n$ (moreover, $\lambda=0$ is an admissible zero only if $a=-n$). The aforementioned, critical wavelengths correspond to the points $a=0,\ -1,\ \cdots$.

We first remark that $\Psi(a,2;x)$, and hence also $\Phi(x)$, is an analytic function of x in a plane cut along the negative-real axis and an analytic function of the parameter a for all $x\neq0$ in this cut plane. We infer from this that the zeros of Φ, say $\lambda=\lambda(a)$, are continuous functions of a except at those points at which $\lambda=0$ (since Φ is not analytic in a at $x=0$) or at a multiple root, where $\lambda=\lambda(a)$ has a branch point. Referring to Burger's result that $\lambda=0$ is the only possible multiple zero of $\Phi(x)$, we conclude that $\lambda(a)$ can pass from real to complex values only at the points $a=-n$. We therefore proceed to explore the neighborhood of such a point by expanding Φ, qua function of x and a, about $x=0$ and $a=-n$.

We can obtain the required expansion from the known representation [Erdélyi *et al.*, (1953), §6.7.1(13)]

$$\Gamma(a)\Psi(a,2;x)$$
$$=x^{-1}+\sum_{m=0}^{\infty}[\Gamma(a+m)/\Gamma(a-1)m!(m+1)!]$$
$$\times[\log x+\psi(a+m)-\psi(1+m)-\psi(2+m)]x^m, \tag{8}$$

where ψ denotes the logarithmic derivative of the gamma function, and the principal branch of the logarithm is implied ($\log x$ is real for $x>0$). We observe that $\Gamma(z)$ and $\psi(z)$ have poles at $z=0,\ -1,\ -2,\ \cdots$, in the neighborhood of which their behavior can be inferred from the known identities

$$\Gamma(z)=\pi\csc(\pi z)/\Gamma(1-z) \tag{9}$$

and

$$\psi(z)=\psi(1-z)-\pi\cot(\pi z). \tag{10}$$

We now let

$$\epsilon=-\pi^{-1}\tan(\pi a) \tag{11}$$

and seek the roots of (1) as $\epsilon\to0$ ($a\to-n$). Substituting (9) and (10) into (8) and the result into (1), we find that the first approximation to the roots of (1) in the joint neighborhood of $x=0$ and $a=-n$ is given by

$$\lambda^2=[2\epsilon/(1+n)(2\alpha+n)]+O(\epsilon^{\frac{3}{2}}) \quad (\epsilon\to 0). \qquad (12)$$

Thus, λ is respectively real or imaginary *in first approximation* as $\epsilon\to 0$ through positive or negative values. This appears to suggest that the principle of exchange of stabilities holds, such that there is a transition from a pair of conjugate-imaginary wave speeds to a pair of real, oppositely signed wave speeds as $-a$ increases through n (ϵ increases through zero). In fact, the presence of the logarithmic term in $\Phi(x)$ implies that Φ cannot have negative-real zeros, and we find that the second approximation to the root that is given in first approximation by the negative-real square root of (12) for $\epsilon>0$ has the imaginary part

$$\lambda_i=\mp\pi[2(1+n)(2\alpha+n)]^{-\frac{1}{2}}\epsilon^{\frac{3}{2}}+O(\epsilon^2) \quad (\lambda_r\to 0-) \qquad (13)$$

as the negative-real axis is approached from above or below ($\arg\lambda\to\pm\pi$). We conclude that $\Phi(x)$ has the following zeros that tend to zero as $a\to -n$ ($n=0, 1, \cdots$): either a pair of conjugate-imaginary zeros for $a+n\to 0+$; or a pair of complex-conjugate zeros with negative-real part *and* one positive-real zero as $a+n\to 0-$. Referring to Burger's results, we recall that $\Phi(x)$ also has n positive-definite zeros for $|a+n|<1$.

Now let us consider the metamorphosis of the eigenvalues—i.e., the roots of (1)—as the wavelength increases from zero or, equivalently, as k decreases from ∞. We infer from (3) and (5) that

$$a(k)\sim 1- \quad (k\to\infty), \qquad (14)$$

and we define the sequence $k_0, k_1, \cdots$ such that

$$a(k_n)=-n, \quad 0\leqq n<\beta\gamma RTs/f^2U_z. \qquad (15a, b)$$

The only eigenvalues for $k>k_0$ then are a complex-conjugate pair that are approximated by (12) as $k\to k_0+$. We also remark that an asymptotic solution of (1) yields (cf. Miles, 1964, §13)

$$\lambda_r=O(1), \quad \lambda_i=O(1/k) \quad (k\to\infty). \qquad (16)$$

As k decreases through k_0, a new eigenvalue appears and remains positive-definite for $k<k_0$; similarly, and in accordance with the conclusions of the preceding paragraph, a new, positive-real eigenvalue appears as k decreases through each of $k_1, k_2, \cdots, k_N$, where N is the integral part of $\beta\gamma RTs/f^2U_z$. (We note that $N=0$ if $U_z>\beta\gamma RTs/f^2$.) The complex-conjugate eigenvalues degenerate to zero at each of $k_0, k_1, ..., k_N$; are approximated by (12), such that $\lambda_i=O(|\epsilon|^{\frac{1}{2}})$, as $k\to k_n+$; and are approximated by (12) and (13), such that $\lambda_r=O(\epsilon^{\frac{1}{2}})$ and $\lambda_i=O(\epsilon^{\frac{3}{2}})$, as $k\to k_n-$. Referring to (12) and (13), we remark that λ_i is a monotonically decreasing function of n for $k\to k_n+$. This suggests that maximum instability should be observed for $k>k_0$ [in accord with Charney's original prediction; cf. Green (1961)]

We propose the designations *strong* and *weak* for instabilities in the neighborhoods of $k=k_n+$ and $k=k_n-$, respectively. We have demonstrated (above) that there is a sharp transition from strong to weak instability as k decreases through k_n; moreover, we can infer from the continuity of $\lambda(a)$, $a\neq -n$, that there is a gradual transition from weak to strong instability as k decreases from k_n- toward $k_{n+1}+$. [We also can establish that $k\to\infty$ implies $\epsilon\to 0+$, corresponding to weak instability; cf. Miles (1964), §13].

Weak instability, as we have defined it, depends on the existence of a critical layer (where the local wind speed is equal to the phase velocity of the disturbance) and therefore appears to derive its energy from the mean flow through a concentrated jump in a cross correlation between two components of the disturbance (compare the role of Reynolds stress in homogeneous shear flows). Strong instability, on the other hand, does not appear to depend directly on the existence of a critical layer and derives its energy primarily from mean flow through the process described by Eady (1949) as "quasi-horizontal overturning." We emphasize that these last statements are both qualitative and conjectural; nevertheless, the distinction between the two types of instability appears to be worth drawing and to invite further exploration.

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