Baroclinic Instability of the Zonal Wind: Part II[1]

JOHN W. MILES

Institute of Advanced Studies, Australian National University, Canberra, Australia

(Manuscript received 9 March 1964)

ABSTRACT

An approximate solution of the eigenvalue problem governing the stability of the zonal wind with respect to small disturbances of long wavelength is developed for profiles with strong, positive-definite vertical shear. It is found that certain disturbances, characterized by positive phase velocities, appear to be stable on the basis of a first approximation but are unstable in higher approximations. The results, together with the previously established instability for short wavelengths and/or weak vertical shear, support the conjecture that typical zonal-wind configurations are unstable with respect to small disturbances of almost all wavelengths at almost all windspeeds.

1. Introduction

We consider here baroclinic instability of the zonal wind on the assumptions of: (i) a perfect gas, (ii) adiabatic motion, (iii) gravitational potential proportional to altitude, (iv) small Rossby number, and (v) large Richardson number. As we have shown in an earlier paper [Miles (1964a), hereinafter designated as I], these assumptions permit the stability problem for any zonal-wind configuration to be formulated in terms of the following, dimensionless quantities: $U(z)$ and $\tau(z)$, the vertical profiles of windspeed and temperature; the parameters $1/\beta$ and κ, which are proportional to appropriate scales of wind shear and potential-temperature gradient (or *stability*), respectively; and α and $c=c_r+ic_i$, the wave number and complex wave speed of a spatially periodic, travelling-wave disturbance. Given $U(z)$ and $\tau(z)$, this formulation yields an eigenvalue problem for the determination of $c(\alpha,\beta,\kappa)$, and the configuration is unstable with respect to disturbances of wave number α if c_i, the imaginary part of c, is positive.

The first, and most widely studied, baroclinic model of the zonal wind was proposed and analyzed by Charney (1947). Charney's model (as we shall designate it, although Charney also proposed and analyzed elaborations of this model) is based on (a) the assumption of constant vertical shear and (b) the neglect of the vertical temperature gradient except in the calculation of the potential-temperature gradient [$s(z)$ in (2.4) below], which is approximated as constant. His stability analysis culiminated in a "critical stability curve" in a wavelength (L), wind-shear (Λ) plane ($L\sim 1/\alpha$ and $\Lambda\sim 1/\beta$ in the present notation). This curve was monotonic ($d\Lambda/dL>0$), and Charney concluded that "values of Λ and L at points above the curve correspond to instability and values at points below correspond to stability."

This conclusion (at least for Charney's basic model) has recently been disproved by Burger (1962), who showed that $c_i>0$ for all Λ, L-points other than those on an infinite, discrete set of curves (of which Charney's "critical stability curve" is the dominant member), on each of which $c_i=0$. We shall designate these curves as *singular*. Burger added (implicitly anticipating a subsequent paper) that disturbances corresponding to points on the singular curves also would be unstable but would grow algebraically, rather than exponentially, with time. See also Green (1960).

A further investigation (Miles, 1964b) of Charney's eigenvalue equation revealed that c^2 passes from positive so negative values in first approximation as either wind shear ($1/\beta$) or wave number (α) increases across each of Burger's singular curves. This might appear to suggest a transition from stability to instability in accordance with the conventional principle of exchange of stabilities; in fact, that value of c that is positive-real in first approximation is complex in next approximation, such that $c_r=O(\epsilon)$ and $c_i=O(\epsilon^3)$ as $\epsilon\to 0$, where $\epsilon=0$ on a given singular curve. Moreover, the analysis of I, §8, implies that real values of c cannot lie within the range of U for any configuration for which: (vi) wind speed increases monotonically with altitude and (vii) potential vorticity increases monotonically with latitude; a mode for which the real part of c lies within the range of U then must be unstable.

We can infer from the general formulation of I (in particular, from the character of the singular point at $U=c$) that the eigenvalue equation for a rather general class of configurations has a representation of the form

$$\Phi(c)=A(c)+B(c)\log(-c)=0 \quad (0<\arg c<2\pi), \quad (1.1)$$

[1] This research has been partially supported by the Office of Naval Research under Contract Nonr-4266(00).

where c is the wave speed relative to ground level, A and B are real for real values of c, and the principal branch of the logarithm is implied. It then is obvious that a positive-real zero of Φ must be a simultaneous zero of A and B; otherwise, the logarithm introduces an imaginary term $-i\pi B$. It may happen, however, that

$$A(\hat{c})=0, \quad 0<|B(\hat{c})|\ll 1 \quad (\hat{c}>0). \qquad (1.2\text{a, b})$$

Supposing that $\hat{c}$ is a simple zero of $A(c)$, we then can show (as in §4 below) that (1.1) admits a pair of complex-conjugate zeros in the neighborhood of $c=\hat{c}$ if and only if $B(c)/A'(c)>0$ in that neighborhood, in which case

$$c_i \to \pm\pi(B/A')_{c=\hat{c}} \quad (c_r\to\hat{c}>0). \qquad (1.3)$$

Unfortunately, only the simplest configurations permit an explicit determination of $\Phi(c)$ without the imposition of further approximations [Charney's model yields a representation of Φ in terms of confluent hypergeometric functions, from which power-series representations of A and B may be deduced; see Miles (1964b).] The approximations that most naturally suggest themselves are based on the supposition that α and/or β are/is either small or large. Such approximations were explored in a preliminary way in I, and the results obtained there implied instability as either $\alpha\to\infty$ or $\beta\to\infty$ and suggested instability over much or all of that domain in which $\alpha^2\ll 1$ and $\beta\ll 1$.

The following analysis is devoted to an extension and amplification of the formulation of I, §12, for small α^2 and β. We shall begin, in §§2 and 3 below, with a brief recapitulation of that formulation. We then, in §§4 and 5, shall reduce the determination of A and B in (1.1) to the evaluation of definite integrals with integrands that depend only on $U(z)$. Finally, in §6, we shall apply our results to Charney's model and to the one-parameter family of wind profiles $U(z)=z^\sigma$, $\sigma>0$. We shall find that even such apparently simple profiles present a surprisingly wide range of possibilities; in particular, stable regions may be embedded between unstable regions in an α,β-plane. Stability appears to be exceptional, however, and typical flows are unstable for either sufficiently small or sufficiently large values of β/α^2 if both α^2 and β are small. Coupled with the asymptotic results of I, these results lend support to our previous conjecture that, subject to the assumptions (i)–(v) above, typical zonal-wind configurations are unstable with respect to small disturbances of almost all wavelengths at almost all wind speeds.

2. Basic formulation

We let p, ρ and T denote pressure, density and temperature; p_0, ρ_0 and T_0 reference values at ground level; R the gas constant for air; γ the specific-heat ratio; g the acceleration of gravity; H the scale height of the atmosphere, such that

$$p_0=R\rho_0T_0=\rho_0 gH; \qquad (2.1)$$

z a dimensionless pressure altitude and τ a dimensionless temperature, such that

$$p/p_0=1-z,\ T/T_0=\tau(z),\ \rho/\rho_0=(1-z)/\tau(z); \qquad (2.2\text{a, b, c})$$

and $\hat{U}U(z)$ the zonal-wind velocity, with $\hat{U}$ as an appropriate scale. Invoking the hydrostatic approximation, we find that the true altitude is given by

$$h(z)=H\int_0^z(1-z)^{-1}\tau(z)dz \qquad (2.3)$$

and the dimensionless stability—i.e., the vertical gradient of the logarithmic potential temperature multiplied by H—by

$$s(z)=\frac{d\log(p^{1/\gamma}\rho^{-1})}{d(h/H)}=\left[\left(\frac{\gamma-1}{\gamma}\right)+\frac{(1-z)\tau'}{\tau}\right]\tau^{-1}. \qquad (2.4)$$

We shall assume that

$$\frac{(1-z)\tau'(z)}{\tau(z)}=\frac{d\log T}{d(h/H)}>-\left(\frac{\gamma-1}{\gamma}\right), \qquad (2.5)$$

so that $s(z)>0$ in $z=[0,1]$. We also introduce the reference value

$$\kappa=s(0) \qquad (2.6)$$

and the weighting function

$$P(z)=\kappa(1-z)^2/s(z)\tau^2(z) \qquad (2.7\text{a})$$

$$=\kappa(1-z)^2/[(\gamma-1)\gamma^{-1}\tau(z)+(1-z)\tau'(z)]. \qquad (2.7\text{b})$$

We note that

$$P(z)=1 \qquad (z=0) \qquad (2.8\text{a})$$

$$>0 \qquad (0\leqq z<1) \qquad (2.8\text{b})$$

$$\sim P_1(1-z)^2 \quad (z\to 1), \qquad (2.8\text{c})$$

where

$$P_1=\kappa\gamma/(\gamma-1)\tau(1). \qquad (2.9)$$

We then can show (I, §§5 and 6) that small disturbances in the potential energy of a particle are governed by the eigenvalue problem

$$(U-c)[(P\psi')'-\alpha^2\psi]+[\beta-(PU')']\psi=0, \qquad (2.10)$$

$$(U-c)\psi'-(U'-\kappa c)\psi=0 \quad (z=0), \qquad (2.11)$$

and

$$P\psi\psi'=0 \quad (z=1), \qquad (2.12)$$

where $\psi(z)$ is the complex amplitude of a travelling wave of wavelength L and complex wave speed $\hat{U}c$,

$$\alpha=(\kappa gH)^{\frac{1}{2}}f^{-1}(2\pi/L) \qquad (2.13)$$

is a dimensionless wave number,

$$\beta=(\kappa gH/f\hat{U}a)\cot\varphi \qquad (2.14)$$

is an appropriate, inverse measure of vertical shear, $f=2\Omega\sin\varphi$ is the Coriolis parameter, a is the mean radius of the earth, and φ is the latitude. The conventional Rossby and Richardson numbers are given by

$$\mathrm{Ro}=(\hat{U}/f)(2\pi/L)=(\kappa gH)^{\frac{1}{2}}(\beta fa)^{-1}\alpha\cot\varphi \quad (2.15)$$

and

$$\mathrm{Ri}=\kappa gH/\hat{U}^2=(\alpha/\mathrm{Ro})^2. \quad (2.16)$$

We also note that $\beta-(PU')'$ in (2.10) is proportional to the northerly gradient of the potential vorticity.

We emphasize that the only assumptions implicit in the formulation of this section are (i)–(v) in §1 above. In particular, we have not assumed that the disturbance represented by ψ travels along parallels of latitude; however, that disturbance that has the maximum rate of growth for a given wavelength L is so directed.

3. Integral-equation formulation

We now introduce the new dependent variable $F(z)$ according to

$$\psi(z)=Y(z)V(z)F(z), \quad (3.1)$$

where

$$V(z)=U(z)-c, \quad (3.2)$$

and $Y(z)$ is to be determined such that (after an appropriate normalization)

$$\psi(z)\sim Y(z)V(z) \quad (z\to 1) \quad (3.3a)$$

and

$$Y(0)=1. \quad (3.3b)$$

We shall assume that

$$U'(z)>0 \quad (0<z<1), \quad U_0=U(0)=0; \quad (3.4a, b)$$

however, our results hold for $U_0\neq 0$ if $\kappa|U_0|\ll 1$ and $U(z)$ and c are replaced by $U(z)-U_0$ and $c-U_0$.

Substituting (3.1)–(3.3) into (2.10)–(2.12), we obtain

$$(PY^2V^2F')'+GF=0, \quad (3.5)$$

$$c^2[F'+(Y'-\kappa)F]=0 \quad (z=0), \quad (3.6)$$

and

$$F(1)=1, \quad (3.7)$$

where

$$G(z)=Y^2(\beta V-\alpha^2V^2)+Y(PY'V^2)'. \quad (3.8)$$

We then can transform (cf. I, §11) the differential equation (3.5) and the boundary condition (3.7) to the integral equation

$$F(z)=1-\int_z^1 [P(z_1)Y^2(z_1)V^2(z_1)]^{-1}dz_1 \times\int_{z_1}^1 G(z_2)F(z_2)dz_2 \quad (3.9)$$

and the boundary condition (3.6) to the eigenvalue equation

$$\Phi(c)=0, \quad (3.10)$$

where

$$\Phi(c)=c^2[F'+(Y'-\kappa)F]_0 \quad (3.11a)$$

$$=\int_0^1 GFdz+(Y_0'-\kappa)F_0c^2 \quad (3.11b)$$

$$=\int_0^1 [(\beta V-\alpha^2V^2)Y^2F-PY'V^2(YF)']dz -\kappa F_0c^2, \quad (3.11c)$$

and the subscript zero implies evaluation at $z=0$ [see second paragraph of §4 regarding advantages of either (3.11b) or (3.11c) relative to (3.11a)].

The explicit determination of $Y(z)$ requires the specification of the behavior of $P(z)$ and $U(z)$ in the neighborhood of $z=1$. We shall assume that $U(z)$ either (a) is bounded or (b) has a linear asymptote with respect to true altitude; appropriate choices of the velocity scale $\hat{U}$ then imply: either

$$U(z)\sim 1 \quad (3.12a)$$

or

$$U(z)\sim -\log(1-z) \quad (z\to 1). \quad (3.12b)$$

Invoking (2.8c) we find that (3.3) can be satisfied by (see I, §10 for details) either

$$Y(z)=(1-z)^{\mu-\frac{1}{2}} \quad (U\sim 1) \quad (3.13a)$$

or

$$Y(z)=(1-z)^{\alpha_1-\frac{1}{2}}(V/V_0)^{(1+\beta_1-2\alpha_1)/2\alpha_1}, \quad (3.13b)$$

where

$$\mu=[\alpha_1^2-\beta_1(1-c)^{-1}]^{\frac{1}{2}}=\mu_r+i\mu_i \quad (\mu_r>0), \quad (3.14)$$

$$\alpha_1=[(\alpha^2/P_1)+\tfrac{1}{4}]^{\frac{1}{2}}, \quad (3.15)$$

and

$$\beta_1=\beta/P_1. \quad (3.16)$$

The boundary condition (3.7), together with an appropriate choice of $Y(z)$, ensures the convergence of the integrals in (3.9) and (3.11) provided that the singular point $z=z_c$, at which

$$U(z_c)=c, \quad (3.17)$$

does not lie on the path(s) of integration. We find [I, §10] that the solution to the differential equation (3.5) in the neighborhood of this singular point must exhibit the behavior

$$F(z)\to C\{(z-z_c)^{-1}-[\beta-(PU')'](PU')^{-1} \times\log(z-z_c)+O(z-z_c)^0\} \quad (z\to z_c), \quad (3.18)$$

where C is a constant and the coefficient of the logarithm is evaluated at $z=z_c$. The principal branch of $\log(z-z_c)$ is implied, and we infer from the restriction (3.4a) that

$$[\log(z-z_c)]_i \to \mp\pi \quad (z<z_c,\ c_i\to 0\pm). \quad (3.19)$$

This logarithmic branch point at $z=z_c$ implies a logarithmic branch point for $\Phi(c)$ at $c=0$ and rules out positive-real values of c except as permitted by (4.12) below. We shall adopt the convention

$$0 \leqq \arg c < 2\pi, \quad (3.20)$$

in consequence of which (3.19) implies

$$\log(-c)=\log c - i\pi \quad (3.21)$$

in the subsequent expressions for $\Phi(c)$.

4. Iterative solution

We can construct a solution to the integral equation (3.9) by iteration, starting from the zero-th approximation

$$F^{(0)}(z)=1 \quad (4.1)$$

in order to satisfy (3.7). Regarding the result as an expansion in the parameter

$$\delta=\max\{\alpha^2,\beta\}, \quad (4.2)$$

we find that

$$F(z)=F^{(n)}(z)+O(\delta^{n+1}) \quad (\delta\to 0), \quad (4.3)$$

where $F^{(n)}$ is the result of n iterations. [It appears that convergence could be proved either through a consideration of F as a function of the parameters α^2 and β or along the lines of Drazin and Howard (1962).]

Similarly, we define the nth approximation $\Phi^{(n)}$, such that

$$\Phi(c)=\Phi^{(n)}(c)+O(\delta^{n+1}). \quad (4.4)$$

Remarking that

$$Y(z)=1+O(\delta) \quad (z<1,\ \delta\to 0) \quad (4.5)$$

and hence, from (3.8), that

$$G(z)=O(\delta) \quad (\delta\to 0), \quad (4.6)$$

we infer that the substitution of the approximation $F^{(n)}$ into either (3.11b) or (3.11c), but not (3.11a), yields the higher approximation $\Phi^{(n+1)}$ provided that

$$\kappa=O(\delta). \quad (4.7)$$

Observing that $\kappa\ll 1$ for typical configurations, we shall impose the restriction (4.7) in order to simplify the subsequent error terms; otherwise it would be necessary only to replace $O(\delta^{n+1})$ by $O(\delta^{n+1},\kappa\delta^n)$ in certain of these terms.

We anticipate that $\Phi(c)$ must be of the form (1.1)—viz.,

$$\Phi(c)=A(c)+B(c)\log(-c), \quad (4.8)$$

where: A and B are real for real values of both c and μ [we note that $c_i=0$ implies $\mu_i=0$ for sufficiently small β, but in a more general treatment we would have to account for the branch point at $c=1-(\beta_1/\alpha_1^2)$];

$$A=O(\delta), \quad B=O(\delta^2) \quad (\delta\to 0); \quad (4.9\text{a, b})$$

and ($A(c)$ and $B(c)$ are analytic functions of c in the neighborhood of $c=\hat{c}$, where

$$A(\hat{c})=0 \quad (\delta\to 0). \quad (4.10)$$

We distinguish among three possibilities: (a) $\hat{c}$ is complex with $\hat{c}_i>0$ at $\delta=0+$; (b) $\hat{c}<0$; (c) $\hat{c}>0$. We then can show that $\Phi(c)$ has, respectively: (a) a pair of complex-conjugate zeros that tend to $\hat{c}$ and $\hat{c}^*$ as $\delta\to 0$; (b) a negative-real zero that tends to $\hat{c}<0$ as $\delta\to 0$; (c) a pair of complex-conjugate zeros, such that

$$c_r\to\hat{c}>0, \quad c_i\to\pm\pi D(\hat{c})=O(\delta),$$
$$\arg c\to \begin{matrix}0+\\2\pi-\end{matrix}, \quad (\delta\to 0), \quad (4.11\text{a})$$

if and only if

$$D(\hat{c})=[B(c)/A'(c)]_{c=\hat{c}}>0. \quad (4.11\text{b})$$

But if $D(\hat{c})<0$, we obtain the contradictory result $c_i<(>)0$ on the provisional hypothesis that $\arg c\to 0+(2\pi-)$, and (the principal branch of) $\Phi(c)$ does not have any zero that tends to $\hat{c}>0$. Finally, we have the exceptional case

$$A(\hat{c})=B(\hat{c})=0 \quad (\hat{c}>0), \quad (4.12)$$

for which $\Phi(c)$ admits a positive-real zero.

We shall designate those instabilities for which $c_i=O(1)$ as *strong* and those for which $c_i=O(\delta)$ as *weak*. As we have remarked elsewhere (Miles, 1964b), weak instability resembles that in conventional, inviscid shear flows in the sense that it depends on the existence of a critical layer at which $U=c_r$.

5. Dominant approximations

We shall carry out the program of the preceding section only to the point of determining the dominant approximations to both c_r and c_i as $\delta\to 0$.

We first determine $\Phi^{(1)}(c)$. Substituting (4.1) and (4.5) into (3.11c) and neglecting terms of $O(\delta^2)$, we obtain

$$\Phi^{(1)}(c)\equiv A^{(1)}(c)=\int_0^1(\beta V-\alpha^2V^2)dz-\kappa c^2. \quad (5.1)$$

Substituting $V=U-c$ into (5.1) and introducing the mean values

$$U_1=\int_0^1 Udz \quad \text{and} \quad U_2=\int_0^1 U^2dz, \qquad (5.2\text{a,b})$$

we rewrite the quadratic $A^{(1)}(c)$ in the alternative forms

$$A^{(1)}(c)=(\beta U_1-\alpha^2U_2)+(2\alpha^2U_1-\beta)c-(\alpha^2+\kappa)c^2 \qquad (5.3\text{a})$$

$$=-(\alpha^2+\kappa)(c-c_+)(c-c_-), \qquad (5.3\text{b})$$

where

$$c_\pm=(\alpha+\kappa)^{-1}\{\alpha^2U_1-\tfrac{1}{2}\beta\pm[\tfrac{1}{4}\beta^2-\alpha^4(U_2-U_1^2)+\kappa(\beta U_1-\alpha^2U_2)]^{\frac{1}{2}}\}. \qquad (5.4)$$

Let

$$\beta'=2\{(\alpha^2+\kappa)^{\frac{1}{2}}[\alpha^2(U_2-U_1^2)+\kappa U_1^2]^{\frac{1}{2}}-\kappa U_1\} \qquad (5.5)$$

and

$$\beta''=(U_2/U_1)\alpha^2 \qquad (5.6)$$

(note that $\beta'=\beta''=0$ for $\alpha=0$); then: (i) $c_\pm$ are complex-conjugates for $\beta<\beta'$; (ii) $c_+>c_->0$ for $\beta'<\beta<\beta''$ and $U_2<2U_1^2$; (ii)′ $c_-<c_+<0$ for $\beta'<\beta<\beta''$ and $U_2>2U_1^2$; (iii) $c_+>0>c_-$ for $\beta>\beta''$. Referring to the discussion of the preceding section, we conclude that

$$c_i=\pm\tfrac{1}{2}(\alpha^2+\kappa)^{-1}(\beta+\beta'+4\kappa U_1)^{\frac{1}{2}}(\beta'-\beta)^{\frac{1}{2}}+O(\delta) \qquad (\beta<\beta',\ \delta\rightarrow 0). \qquad (5.7)$$

We now suppose that $\beta>\beta'$ and that $c\rightarrow c_r>0$ as $\delta\rightarrow 0$. It then suffices for the determination of $c_i=O(\delta)$ to determine only the imaginary part of $F_i^{(1)}$ as z_c tends to the positive-real axis. Substituting (4.1) and (4.5) into the right-hand side of (3.9), we obtain

$$F_i(z)=-Im\int_z^1 (PV^2)^{-1}dz_1\int_{z_1}^1 Gdz_2 + O(\delta^2) \quad (c\rightarrow c_r>0). \qquad (5.8)$$

Expanding the integrand in a Laurent series about $z_1=z_c$ or, equivalently, $V(z_1)=0$, and neglecting terms of $O(\delta^2)$, we obtain

$$F_i^{(1)}=-\left\{\frac{d}{dU}\left[(PU')^{-1}\int_z^1 Gdz\right]\right\}_{V=0}\times(\arg V)\,|_z{}^1 \qquad (5.9\text{a})$$

$$=KI[1+O(\delta)](\arg c-\pi) \quad (z<z_c) \qquad (5.9\text{b})$$

$$=0 \quad (z>z_c), \qquad (5.9\text{c})$$

where

$$K=K(c)=[(d/dU)(PU')^{-1}]_{U=c_r} \qquad (5.10\text{a})$$

$$=-[(PU')'/P^2U'^3]_{z=z_c}, \qquad (5.10\text{b})$$

$$I=I(c)=\int_{z_c}^1(\beta V-\alpha^2V^2)dz, \qquad (5.11)$$

and z_c is approximated by its real part. Substituting $F=1+iF_i^{(1)}$ into (3.11b) and placing the result in the form of (4.8), we obtain

$$\Phi(c)=A^{(1)}(c)+iB^{(2)}(c)(\arg c-\pi)+\mathcal{E}, \qquad (5.12)$$

where $A^{(1)}(c)$ is given by (5.3),

$$B^{(2)}(c)=K(c)I(c)\left[\int_0^{z_c}(\beta V-\alpha^2V^2)dz-\kappa c^2\right] \qquad (5.13\text{a})$$

$$=K(c)I(c)[A^{(1)}(c)-I(c)], \qquad (5.13\text{b})$$

and

$$\mathcal{E}_r=O(\delta^2), \quad \mathcal{E}_i=O(\delta^3). \qquad (5.14)$$

Substituting (5.3b) and (5.13b) into (4.11) and letting $c_r\rightarrow c_+>0$ on the supposition that $K(c_+)>0$, we obtain the consistent result

$$c_i=\pm\pi[K(c_+)I^2(c_+)/(\alpha^2+\kappa)(c_+-c_-)]+O(\delta^2)$$
$$\left(c_r\rightarrow c_+>0,\quad \arg c\rightarrow \begin{matrix}0+\\ 2\pi-\end{matrix},\quad K>0\right); \qquad (5.15)$$

on the other hand, $c_r\rightarrow c_->0$ does not yield an admissible zero of $\Phi(c)$ if $K(c_-)>0$. Conversely, we can interchange $c_+>0$ and $c_->0$ in (5.15) if $K(c_-)<0$, whereas $c_r\rightarrow c_+>0$ does not yield an admissible zero of $\Phi(c)$ if $K(c_+)<0$.

The approximations of (5.7) and (5.15) are not uniformly valid in the neighborhood of $\beta=\beta'$ ($c_+=c_-$), but it is obvious that (5.12) admits a pair of complex-conjugate zeros, such that

$$c_i=O(\delta^{\frac{1}{2}})(c_r\rightarrow c_+=c_->0). \qquad (5.16)$$

Recapitulating, we have tabulated the admissible zeros of $\Phi(c)$ as $\delta\rightarrow 0$ in Table 1. Instability in category (i) is strong, with $c_i=O(1)$ given by (5.7); instability in categories (iia, b, c) and (iiia) is weak, with $c_i=O(\delta)$ given by (5.15). The transition from strong to weak instability is relatively sharp, with an intermediate point at which $c_i=O(\delta^{\frac{1}{2}})$, according to (5.16); however, (5.16) does not hold if, as in §6a below, $I=0$ at $\beta=\beta'$, and the transition then is even sharper.

TABLE 1. The admissible zeros of $\Phi(c)$ as $\delta\rightarrow 0$; $c_\pm$ are given by (5.4); c_i is given by either (5.7) or (5.15).

Category	β-range	$K(c_+)$	$K(c_-)$	Neg.-real c	Compl.-conj. c	Total c
(i)	$0, \beta'$	—		—	$c_\pm=O(1)$	2
(iia)	β', β''	>0	>0	—	$c_+\pm iO(\delta)$	2
(iib)		<0	<0	—	$c_-\pm iO(\delta)$	2
(iic)		>0	<0	—	$c_\pm\pm iO(\delta)$	4
(iid)		<0	>0	—	—	0
(ii)′		—		$c_\pm$	—	2
(iiia)	$>\beta''$	>0	—	c_-	$c_+\pm iO(\delta)$	3
(iiib)		<0	—	c_-	—	1

The most typical configurations appear to be those for which $K(c)>0$ for $c>0$ and $U_2<2U_1^2$. The categorical sequence as β increases from zero then is (i), (iia), (iiia), and we have instability for all (sufficiently small) β, although c_i decreases sharply from $O(1)$ to $O(\delta)$ as β increases through β'. There are, however, many other possibilities, as we shall see from the examples of the following section.

6. Examples

We now consider some (apparently) simple wind profiles on the basis of the additional approximations

$$\kappa \ll \delta \tag{6.1}$$

and

$$P(z)=(1-z)^2. \tag{6.2}$$

The approximation (6.1), which now replaces the weaker restriction (4.7), holds for typical configurations and has been invoked in most of the previous investigations of zonal-wind instability. The approximation (6.2) holds for slowly varying $\tau(z)$ and also has been invoked in these previous investigations. We add that (6.2) is exact for

$$\tau(z)=\tau_1+(1-\tau_1)(1-z)^{(\gamma-1)/\gamma}. \tag{6.3}$$

Invoking (6.1) in (5.4)–(5.7) and introducing

$$\eta=\beta/2\alpha^2=\tfrac{1}{2}(f/\hat{U}a)(L/2\pi)^2\cot\varphi, \tag{6.4}$$

we obtain

$$c_\pm=U_1-\eta\pm(\eta^2-\eta'^2)^{\frac{1}{2}}, \tag{6.5}$$

$$\eta'=(U_2-U_1^2)^{\frac{1}{2}}, \tag{6.6}$$

$$\eta''=U_2/2U_1, \tag{6.7}$$

and

$$c_i=\pm(\eta'^2-\eta^2)^{\frac{1}{2}}+O(\delta)\quad(\eta<\eta'). \tag{6.8}$$

We remark that $2\hat{U}\eta$ is the Rossby-wave speed.

Substituting (6.2) into (5.10), which is the only point at which $P(z)$ enters our calculation, we obtain

$$K=[2U_c'-(1-z_c)U_c'']/[(1-z_c)U_c']^3, \tag{6.9}$$

where the subscript c implies $z=z_c$.

6a. Charney's model. Charney's model implies approximations equivalent to (6.1) and (6.2) and (with $\hat{U}$ as the windspeed at $p/p_0=1/e$)

$$U(z)=-\log(1-z). \tag{6.10}$$

Substituting (6.10) into (5.2), (6.5)–(6.7), (6.9) and (5.11), we obtain $U_1=1$, $U_2=2$, $\eta'=\eta''=1$,

$$c_\pm=1-\eta\pm(\eta^2-1)^{\frac{1}{2}}, \tag{6.11}$$

$$K(c)=e^c, \tag{6.12}$$

and

$$I(c)=e^{-c}(\beta-2\alpha^2)=\alpha^2(\eta-1)e^{-c}. \tag{6.13}$$

It follows that there is a direct transition between categories (i) and (iiia) of Table 1 at $\eta=\eta'=\eta''=1$ ($\beta=2\alpha^2$). Taking the imaginary part of (6.11) for $\eta<1$ and substituting (6.11)–(6.13) into (5.15) for $\eta>1$, we obtain

$$c_i=\pm(1-\eta^2)^{\frac{1}{2}}\quad(\eta<1) \tag{6.14a}$$

$$=\pm 2\pi\alpha^2(\eta-1)^{\frac{3}{2}}(\eta+1)^{-\frac{1}{2}}\times\exp[(\eta-1)-(\eta^2-1)^{\frac{1}{2}}]\quad(\eta>1). \tag{6.14b}$$

We emphasize that (5.16) yields $c_i=0$ at $\eta=1$ in consequence of the factor $\eta-1$ in I (cf. remarks in penultimate paragraph of §5.)

Letting $\eta\rightarrow 1\mp$ in (6.14) and restoring α and β through (6.4), we obtain

$$\alpha c_i=(2\alpha^2-\beta)^{\frac{1}{2}}\qquad(\beta\rightarrow 2\alpha^2-) \tag{6.15a}$$

$$=\tfrac{1}{2}\pi(\beta-2\alpha^2)^{\frac{3}{2}}\quad(\beta\rightarrow 2\alpha^2+) \tag{6.15b}$$

for the dimensionless rate of growth of an unstable disturbance in the neighborhood of $\beta=2\alpha^2$. This result agrees (for α^2, $\beta\rightarrow 0$) with that obtained from the exact, confluent-hypergeometric solution for Charney's model in the neighborhood of $c=0$ (Miles, 1964b).

6b. Monomial wind profiles. A simple, one-parameter family of wind profiles, each of which satisfies the restrictions (3.4a, b), is given by

$$U(z)=z^\sigma\quad(\sigma>0). \tag{6.16}$$

Substituting (6.16) into (5.2), (6.5)–(6.7), and (6.9), we obtain

$$\eta'=\sigma(1+\sigma)^{-1}(1+2\sigma)^{-\frac{1}{2}}, \tag{6.17}$$

$$\eta''=\tfrac{1}{2}(1+\sigma)(1+2\sigma)^{-1}, \tag{6.18}$$

$$c_\pm=(1+\sigma)^{-1}-\eta\pm[\eta^2-\sigma^2(1+\sigma)^{-2}(1+2\sigma)^{-1}]^{\frac{1}{2}}, \tag{6.19}$$

and

$$K=\sigma^{-2}c^{(1-2\sigma)/\sigma}(1-c^{1/\sigma})^{-3}[(1+\sigma)c^{1/\sigma}-(\sigma-1)] \tag{6.20a}$$

$$\lesseqgtr 0,\quad c\lesseqgtr[(\sigma-1)/(\sigma+1)]^\sigma\quad(\sigma>1). \tag{6.20b}$$

Introducing $\eta_\pm$, such that

$$K(c_\pm)=0\quad(\eta=\eta_\pm), \tag{6.21}$$

we find that $\eta_-=\eta''$ at $\sigma=1$, $\eta_\pm=\eta'$ at $\sigma=1.54$, $\eta_+=\eta''$ at $\sigma=1.81$, and $c_-<c_+<0$ in (η',η'') for $\sigma>2.41$ ($U_2>2U_1^2$). Referring to Table 1, we then obtain the following categorical sequences as η increases from zero:

$0<\sigma\leqq 1$:	(i)/η'/(iia)/η''/(iiia)
$1<\sigma<1.54$:	(i)/η'/(iia)/η_-/(iic)/η''/(iiia)
$1.54<\sigma<1.81$:	(i)/η'/(iib)/η_+/(iic)/η''/(iiia)
$1.81<\sigma<2.41$:	(i)/η'/(iib)/η''/(iiib)/η_+/(iiia)
$2.41<\sigma$:	(i)/η'/(ii)′/η''/(iiib)/η_+/(iiia).

Denoting strong instability [$c_i=O(1)$] by S, weak instability [$c_i=O(\delta)$] with $c\rightarrow c_\pm$ by $W_\pm$, and neutral stability $[c_i=0]$ by N, we also can write the foregoing sequences in the form:

$$
\begin{aligned}
0<\sigma\leqq 1: &\quad S/\eta'/W_+\\
1<\sigma<1.54: &\quad S/\eta'/W_+/\eta_-/W_\pm/\eta''/W_+\\
1.54<\sigma<1.81: &\quad S/\eta'/W_-/\eta_+/W_\pm/\eta''/W_+\\
1.81<\sigma<2.41: &\quad S/\eta'/W_-/\eta''/N/\eta_+/W_+\\
2.41<\sigma: &\quad S/\eta'/N/\eta_+/W_+.
\end{aligned}
$$

We therefore have instability for all α^2 and β as $\delta\rightarrow 0$ if $\sigma<1.81$ and stability for, and only for, $\eta''<\eta<\eta_+$ if $1.81<\sigma<2.41$ or $\eta'<\eta<\eta_+$ if $\sigma>2.41$.

REFERENCES

Burger, A. P., 1962: On the non-existence of critical wavelengths in a continuous baroclinic stability problem. *J. atmos. Sci.* **19**, 31–38.

Charney, J. G., 1947: The dynamics of long waves in a baroclinic westerly current. *J. Meteor.*, **4**, 135–162.

Drazin, P. G., and L. N. Howard, 1962: The instability to long waves of unbounded parallel inviscid flow. *J. fluid Mech.*, **14**, 257–283.

Green, J. S. A., 1960: A problem in baroclinic instability. *Quart. J. R. meteor. Soc.*, **86**, 237–251.

Miles, J. W., 1964a: Baroclinic instability of the zonal wind. *Rev. Geophys.*, **2**, 155–176.

——, 1964b: A note on Charney's model of zonal-wind instability. *J. atmos. Sci.*, **21**, 451–452.