Baroclinic Instability of the Zonal Wind: Part III[1]

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ABSTRACT

A zonal-wind configuration in which wind speed is proportional to pressure-altitude and stability is proportional to the square of the density is posed. A solution to the eigenvalue problem governing the stability of this configuration with respect to small disturbances is obtained in terms of hypergeometric functions. It is proved that one and only one (exponentially) unstable mode exists for each point in a wavelength, wind-shear plane.

1. Introduction

The general formulation and structure of the eigenvalue problem governing the baroclinic instability of the zonal wind with respect to small disturbances has been considered in Part I (Miles, 1964a) of this series of papers. The explicit representation of the dominant eigenvalues for long wavelengths and strong, positive-definite vertical shear has been considered in Part II (Miles, 1964b). These papers, hereinafter designated by *I* and *II*, were motivated by Green's (1960) discovery that Charney's (1947) constant-shear model implies instability at almost all wavelengths and wind speeds. The assumptions of (i) a perfect gas, (ii) adiabatic motion, (iii) gravitational potential proportional to altitude, (iv) small Rossby number, and (v) large Richardson number are basic in each of these four papers and in the following development; in addition, the assumptions that (vi) wind speed increases monotonically with altitude and (vii) potential vorticity increases monotonically with latitude hold for typical configurations. Charney's constant-shear configuration implies that the wind speed increases linearly to infinity at the top of the atmosphere.

Our previous investigations imply that any configuration for which assumptions (i)–(vii) hold is unstable for sufficiently small values of either wavelength (*I*, §13) or vertical shear (*I*, §14) and also for sufficiently large values of both wavelength and shear (*II*, §§5 and 6), although stability bands are possible in this last domain for rather atypical configurations; these limits correspond to $\alpha\rightarrow\infty$, $\beta\rightarrow\infty$ and α^2, $\beta\rightarrow 0$, respectively, in the general formulation of *I* and in §2 below. It also appears from the limiting results of *II* that a wavelength, wind-shear plane for a typical configuration is divided into domains of relatively strong and relatively weak instability, with a fairly sharp transition across a certain critical curve. Charney's model yields an infinite sequence of such critical curves, on each of which small disturbances are stationary (with respect to an observer at ground level) and only algebraically, rather than exponentially, unstable (Miles, 1964c).

There remains the possibility that at least some of the results for Charney's model are anomalous consequences of his assumptions that wind speed increases without limit and that the vertical temperature gradient can be neglected except where it enters the calculation of the stability (which also is assumed to be constant). The former assumption definitely appears to be questionable on physical grounds (even though it simplifies the mathematical problem). The latter assumption, although intuitively acceptable, is in some degree inconsistent.

With those remarks in mind, we shall pose a model in which wind speed is linear in pressure, rather than true, altitude; and in which the temperature varies continuously, such that the stability is proportional to the square of the density.

These assumptions appear to lead to the simplest possible eigenvalue problem in which each singularity of the governing differential equation is regular. There must be at least three singular points for this differential equation, corresponding to zero density, infinite wind speed, and the critical layer at which wind speed equals wave speed ($U=c$), although the latter singularities generally lie outside of the physical domain and are accessible only through analytic continuation. The simplest differential equation having only regular singularities therefore is the hypergeometric equation, as in §3 below. (Charney's model, in which the singular points of zero density and infinite wind speed coincide, leads to the confluent hypergeometric equation, with

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one regular singularity at $U=c$ and one irregular singularity at $U=\infty$.) We shall show that this model yields an infinite sequence of singular curves, each of which is a locus of stationary modes; however, these curves are not stability boundaries, and the stationary modes merely terminate the trajectories of a class of neutral modes, which cannot be continued across the singular curves. We also shall show that there is exactly one unstable mode for each point (including points on the aforementioned singular curves) in a wavelength, wind-shear plane. It therefore appears that the only anomalous consequence of Charney's model is the vanishing of the exponential rate of growth of the unstable disturbance on the sequence of singular curves discovered by Green (see also Burger, 1962).

2. Formulation

We let p, ρ and T denote pressure, density and temperature; p_0, ρ_0 and T_0 reference values at ground level; R the gas constant for air; γ the specific-heat ratio; g the acceleration of gravity; H the scale height of the atmosphere, such that

$$p_0=R\rho_0T_0=\rho_0gH; \tag{2.1}$$

and

$$z=1-(p/p_0). \tag{2.2}$$

We assume the temperature distribution

$$T/T_0=\tau(z)=\tau_1+(1-\tau_1)(1-z)^{(\gamma-1)/\gamma} \tag{2.3}$$

and the (westerly) wind-speed distribution

$$U=\hat{U}z, \tag{2.4}$$

where $\hat{U}$ is the limiting wind speed at the top of the atmosphere. Starting from these definitions and invoking the hydrostatic approximation and the perfect-gas law ($p=R\rho T$), we find that the true altitude is given by

$$\frac{h}{H}=\int_p^{p_0}\frac{dp}{\rho gH}=\int_0^z\frac{\tau(z)dz}{1-z}=-\tau_1\log(1-z)$$
$$+\gamma(\gamma-1)^{-1}(1-\tau_1)[1-(1-z)^{(\gamma-1)/\gamma}]. \tag{2.5}$$

We then may calculate the dimensionless stability—i.e., the logarithmic gradient of the potential temperature with respect to h/H—according to

$$s(z)=\frac{d\log(p^{1/\gamma}\rho^{-1})}{d(h/H)}=\frac{\kappa}{\tau^2(z)}, \tag{2.6}$$

where

$$\kappa=s(0)=\tau_1(\gamma-1)/\gamma. \tag{2.7}$$

Substituting the distributions (2.3), (2.4) and (2.6) into the general formulation of I, §6 [$P(z)=(\rho/\rho_0)^2\times(\kappa/s)=(1-z)^2$, $Q(z)=1$, $U(z)=z$ in I (6.1)–(6.8)], we find that small disturbances in the potential energy of a particle are governed by the differential equation

$$(z-c)[(1-z)^2\psi'(z)]'+[\beta+2(1-z)$$
$$-\alpha^2(z-c)]\psi(z)=0, \tag{2.8}$$

subject to the boundary conditions

$$c\psi'+(1-\kappa c)\psi=0\quad(z=0), \tag{2.9}$$

and

$$(1-z)^2\psi'\psi\rightarrow 0\quad(z\rightarrow 1). \tag{2.10}$$

Here, $\psi(z)$ is the complex amplitude of a travelling wave of wavelength L and complex wave speed $\hat{U}c$,

$$\alpha=(\kappa gH)^{\frac{1}{2}}f^{-1}(2\pi/L) \tag{2.11}$$

is a dimensionless wave number,

$$\beta=(\kappa gH/f\hat{U}E)\cot\varphi, \tag{2.12}$$

is an inverse measure of vertical shear, $f=2\Omega\sin\varphi$ is the Coriolis parameter, E is the mean radius of the Earth, and φ is the latitude.

3. Hypergeometric solution

The differential equation (2.8) has three regular singularities at $z=c$, 1, ∞. We therefore can represent its solution by the Riemann symbol

$$\psi=P\begin{Bmatrix} c & 1 & \infty & \\ 0 & -\frac{1}{2}+\mu & \frac{1}{2}+\nu & z \\ 1 & -\frac{1}{2}-\mu & \frac{1}{2}-\nu & \end{Bmatrix}, \tag{3.1}$$

where

$$\mu=\mu(c)=\alpha_1(1-c)^{-\frac{1}{2}}(c_1-c)^{\frac{1}{2}}.\quad \mathrm{Rl}\{\mu\}>0, \tag{3.2a,b}$$

$$\nu=(\alpha_1{}^2+2)^{\frac{1}{2}}>0, \tag{3.3}$$

$$\alpha_1=(\alpha^2+\tfrac{1}{4})^{\frac{1}{2}}>0, \tag{3.4}$$

and

$$c_1=1-(\beta/\alpha_1{}^2). \tag{3.5}$$

We shall choose the cut for $\mu(c)$ along $c=[c_1,1]$; then (3.2b) holds everywhere in the cut plane, and μ is positive-real for c in $(-\infty, c_1)$ or $(1,\infty)$.

Invoking the boundary condition (2.10) and the restriction (3.2b), we must restrict the general solution of (3.1) to have the exponent $-\frac{1}{2}+\mu$ in the neighborhood of $z=1$. We therefore can express ψ as the product of $(1-z)^{\mu-\frac{1}{2}}$ and an analytic function of $1-z$ in $|1-z|<|1-c|$. We also shall include an explicit factor $z-c$, anticipating that this will simplify the eigenvalue equation of the following section [see (3.11) below for the form of the solution if the factor $z-c$ is not explicitly included].

Motivated by these considerations, we introduce the transformation

$$\psi(z)=w^{\mu-\frac{1}{2}}(1-w)F(w),\quad w=(1-z)/(1-c), \tag{3.6a,b}$$

under which (2.8) goes over to the hypergeometric equation

$$w(1-w)F''(w)+[(2\mu+1)(1-w)-2w]F'(w) -[(1+\mu)^2-\nu^2]F(w)=0. \quad (3.7)$$

The singularities at $z=1$ and $z=c$ now appear at $w=0$ and $w=1$, respectively; the physical domain $z=[0,1]$ is mapped on the straight line $w=[W,0]$, where

$$W=1/(1-c); \quad (3.8)$$

and

$$F(w)=F(a,b;a+b-1;w), \quad (3.9)$$

is a hypergeometric series with

$$a=1+\mu-\nu,\quad b=1+\mu+\nu,\quad a+b-1=1+2\mu. \quad (3.10a,b,c)$$

We emphasize that, although ψ generally has a branch point at $w=0$ in consequence of the evanescence of the density at $z=1$, $F(w)$ is analytic in a plane cut along $w=[1,\infty]$. On the other hand, both ψ and F have branch points at $w=1$ (although ψ is bounded there) unless F degenerates to a polynomial (as in §5 below).

We note that the solution of (3.6a) also can be expressed in the form

$$\psi(z)=w^{\mu-\frac{1}{2}}F(a-1,b-1;a+b-1;w). \quad (3.11)$$

4. Eigenvalue equation

It remains to satisfy the boundary condition (2.9). Substituting (3.6) and (3.8) therein and introducing

$$\Phi(w;\mu,\nu)\equiv\Phi(w)=(1-w)^2[wF'(w)+\lambda F(w)] \quad (4.1)$$

and

$$\lambda=\mu+\kappa-\tfrac{1}{2}, \quad (4.2)$$

we can place the result in the alternative forms

$$W^{\mu-\frac{3}{2}}\Phi(W;\mu,\nu)=0,\quad (1-c)^{\frac{3}{2}-\mu}\Psi(c;\alpha_1,c_1)=0, \quad (4.3a,b)$$

where

$$\Psi(c;\alpha_1,c_1)\equiv\Psi(c)=\Phi[(1-c)^{-1}; \alpha_1(1-c)^{-\frac{1}{2}}(c_1-c)^{\frac{1}{2}},(\alpha_1{}^2+2)^{\frac{1}{2}}]. \quad (4.4)$$

We regard (4.3a) and (4.3b) as eigenvalue equations for the determination of $W(\mu,\nu)$ and $c(\alpha,\beta)$, respectively, such that

$$(1-c)^{-1}=W[\alpha_1(1-c)^{-\frac{1}{2}}(c_1-c)^{\frac{1}{2}},(\alpha_1{}^2+2)^{\frac{1}{2}}] =W\{[\alpha^2+\tfrac{1}{4}-\beta(1-c)^{-1}]^{\frac{1}{2}},(\alpha^2+\tfrac{9}{4})^{\frac{1}{2}}\} \quad [c=c(\alpha,\beta)]. \quad (4.5)$$

We infer from the known properties of the hypergeometric equation that $\Phi(w;\mu,\nu)/\Gamma(2\mu+1)$ is an analytic function of w in a plane cut along $w=[1,\infty]$ and an entire function of each of the parameters μ and ν for $w\neq1$ or ∞. Remarking that $\Gamma(2\mu+1)$ has no poles in the restricted domain permitted by (3.2), we then can infer from (4.3b) and (4.4) that $c_r(\alpha,\beta)$ and $c_i(\alpha,\beta)$, the real and imaginary parts of $c(\alpha,\beta)$, are continuous (but not generally single-valued) functions of each of the positive-real variables α and β except at the following singular points: (a) $c=0$ ($W=1$) and $c=1$ ($W=\infty$), where $\Phi(w)$ has branch points; (b) the multiple zeros, if any, of $\Psi(c)$ [a multiple zero of $\Phi(w)$ does not, in general, imply a multiple zero of $\Psi(c)$]; (c) $c=c_1$, where $\mu(c)$ has a branch point (in addition to the branch point at $c=1$).

We designate a locus of singular points of given type [i.e., (a), (b), or (c) of the preceding paragraph] in an α,β-plane as a *singular curve*, say $\beta=\hat{\beta}(\nu)$. It is only at such a singular curve that the trajectory of a particular eigenvalue, $c=c(\alpha,\beta)$, can either originate or terminate for $\alpha,\beta>0$. Stability boundaries, if any, must be singular curves of either type (a) or type (b), for complex eigenvalues can occur only in complex-conjugate pairs, each of which must coalesce in a real eigenvalue on such a boundary and either cannot be continued across the boundary [type (a)] or must have as its continuation a pair of real eigenvalues [type (b)]. On the other hand, singular curves of types (a) and (b) are not necessarily stability boundaries. A singular curve of type (c) cannot be a stability boundary [unless it also is a singular curve of type (a) or type (b)] but is simply a boundary across which a real eigenvalue cannot be continued in consequence of the restriction (3.2b).

We can deduce from theorems (ii), (iii), and (v) of *I*, §8, that complex c must lie within the unit circle $|c|<1$ or, more precisely, within a circle of radius

$$C=1-\alpha^2(2\alpha^2+\beta)^{-1} \quad (4.6)$$

with center at $c=1-C$, and that real c must lie on $c\leqq0$ ($0<W\leqq1$). Recalling that the real interval $c=[c_1,1]$ also is forbidden in consequence of (3.2b), invoking the requirement that the trajectory of a complex eigenvalue must be continuous and must lie within the aforementioned circle, and letting

$$\beta_2=\tfrac{1}{4}+[\alpha_1{}^4+(\alpha_1{}^2-\tfrac{1}{4})^2]^{\frac{1}{2}} \quad (4.7)$$

denote that value of β at which $c_1=1-2C$, we infer that admissible values of c along a stability boundary must be restricted according to

$$1-2C<c\leqq0 \quad (\beta<\alpha_1{}^2) \quad (4.8a)$$

or

$$1-2C<c<c_1<0 \quad (\alpha_1{}^2<\beta<\beta_2) \quad (4.8b)$$

and cannot exist for $c_1<1-2C$ or, equivalently, $\beta>\beta_2$. We deduce from these constraints that: $c=1$ can never be an eigenvalue; $c=0$ is a possible eigenvalue only if $\beta<\alpha_1{}^2$; and $c=c_{1-}$ is a possible eigenvalue only if $\beta>\alpha_1{}^2$.

We shall establish in §§5 and 6 that $W=1$ is a zero of Φ if and only if $F(w)$ is a polynomial ($a=-n$, $n=0$, 1, 2,$\cdots$); it then is a double zero by virtue of the factor $(1-w)^2$ in (4.1). We also shall establish that $\Phi(w)$ has exactly two complex zeros if $a\neq-n$. These facts are sufficient to establish instability for all α and β with the possible exception of points on those singular curves defined by $a=-n$. We shall establish in §7 that $c=0$

is only a simple zero on the latter singular curves and that it has a continuation on only one side of each of these curves; accordingly, the aforementioned complex zeros must represent other branches of $c=c(\alpha,\beta)$ and must remain complex for all α and β. Finally, in §8, we shall present approximations to $c_i(\alpha,\beta)$ for $\alpha^2,\beta\ll 1$.

5. Polynomial solutions

Let $a=-n$ or, equivalently,

$$\nu=\mu+(n+1) \quad (n=0, 1, 2, \cdots); \tag{5.1}$$

then

$$F(w)=F(-n,2\mu+n+2; 2\mu+1; w) \tag{5.2}$$

is a polynomial of degree n, as also is $wF'(w)$, and $\Phi(w)$ is a polynomial of degree $n+2$. Substituting (3.2)–(3.5) into (5.1), we find that the corresponding loci of points in an α,β-plane are given by

$$\beta W=(n+1)[(4\alpha^2+9)^{\frac{1}{2}}-3]-n(n-1); \tag{5.3}$$

however, if $n\geqq 1$ the restriction (3.2b) is satisfied only if

$$\alpha>(n-\tfrac{1}{2})^{\frac{1}{2}}(n+\tfrac{5}{2})^{\frac{1}{2}}, \quad \beta W>(n+1)^2-2 \quad (n\geqq 1). \tag{5.4a,b}$$

It is obvious that $\Phi(w)=\lambda(1-w)^2$ for $n=0$, and we therefore need consider further only $n\geqq 1$. We emphasize, however, that the eigensolution corresponding to $n=0$ is not trivial.

Assuming that $n\geqq 1$, we now rewrite (4.1) in the form

$$\Phi(w)=(1-w)^2w^{1-\lambda}G'(w), \quad G(w)=w^\lambda F(w). \tag{5.5a,b}$$

We can establish (HTF* *1*, §2.7.4; *2*, §10.16) that the n zeros of the hypergeometric polynomial (5.2) are distinct and in $w=(0,1)$. It follows that $G(w)$ has n distinct zeros in $w=(0,1)$ and one zero at $w=0$ if $\lambda>0$; Rolle's theorem then implies that $G'(w)$, and hence also $\Phi(w)$, has n simple zeros in $w=(0,1)$. Similarly, $\Phi(w)$ has $n-1$ simple zeros in $w=(0,1)$ and one simple zero in $w<0$ if $-n<\lambda<0$ ($\lambda>-\frac{1}{2}+\kappa$, so that we need not consider $\lambda<-n$, $n\geqq 1$); however, this last zero does not imply an admissible eigenvalue, since $W<0$ implies $\mu>\frac{1}{2}$, which implies $\lambda>0$. We infer from these considerations that, under the constraint (5.1), $\Phi(W;\mu,\nu)$ has n simple zeros in $W=(0,1)$ and one double zero at $W=1$. The modes corresponding to the simple zeros have no bearing on the question of stability, and we shall not consider them further.

There remains the singular zero $W=1$ ($c=0$). As we have anticipated in §4, this zero does not imply a double zero of $\Psi(c)$ for prescribed α and β, but it does imply a singular neutral mode in consequence of the branch point of $\Phi(w)$ at $w=1$. Supposing β to be prescribed in the range (0,2), there is only one such mode, corresponding to

$$\alpha^2=\tfrac{1}{4}\beta(\beta+6) \quad (m=0); \tag{5.6}$$

but if $(n+1)^2-2<\beta<(n+2)^2-2$ there exist $n+1$ such modes, corresponding to that specified by (5.6) and the n modes specified by

$$\alpha^2=\tfrac{1}{4}(m+1)^{-2}[\beta+m(m-1)][\beta+(m+2)(m+3)] \quad (m=1, 2, \cdots, n; \quad n\geqq 1). \tag{5.7}$$

We shall consider these singular neutral modes further in §7.

6. Transcendental solutions

Now let us consider the possible real zeros of $\Phi(w)$ for prescribed values of μ and ν, such that

$$n<\nu-\mu<n+1 \tag{6.1a}$$

or, equivalently,

$$-n<a<1-n \quad (n=0, 1, \cdots). \tag{6.1b}$$

We emphasize that (6.1a) implies that μ, as well as ν, must be real and that $\nu>\mu$. The latter restriction is implied by the restriction $W>0$ ($c<1$), already established in §4.

We rule out $w=1$ as a possible zero by starting from the known representation [HTF *1*, 2.3.1.(4)]

$$\begin{aligned}[\Gamma(a)\Gamma(b)/\Gamma(a+b-1)]F(w) &= (1-w)^{-1}+\sum_{m=0}^{\infty}[\Gamma(a+m)\Gamma(b+m)/\Gamma(a-1)\\ &\quad\times\Gamma(b-1)m!(m+1)!][\log(1-w)+\psi(a+m)\\ &\quad+\psi(b+m)-\psi(m+1)-\psi(m+2)](1-w)^m\\ &\qquad (|1-w|<1).\end{aligned} \tag{6.2}$$

Substituting (6.2) into (4.1) and setting $w=1$, we obtain

$$\Phi(1)=\Gamma(a+b-1)/\Gamma(a)\Gamma(b) \tag{6.3a}$$

$$=\Gamma(1+2\mu)/\Gamma(1+\mu-\nu)\Gamma(1+\mu+\nu), \tag{6.3b}$$

which cannot vanish under the joint restrictions of (3.2b) and (6.1). We therefore can rewrite (4.3) in the form

$$H(w)\equiv wF'(w)/F(w)=-\lambda. \tag{6.4}$$

Having already established (in §4) that real eigenvalues must lie in $W=(0,1)$ we seek the roots of (6.4) in $w=(0,1)$.

We can establish, from the known results (Van Vleck, 1902) for the hypergeometric function $F(a,b;a+b-1;w)$ under the restriction (6.1), that $F(w)$ has exactly n simple zeros in $w=(0,1)$, say $w=u_m$ ($m=1$, 2, $\cdots$, n), and no other zeros in the w-plane cut along $w=(1,\infty)$. It follows that $H(w)$ has n simple poles in $w=(0,1)$ and, from Rolle's theorem, $n-1$ simple zeros in $w=(u_1,u_n)$. Differentiating (6.4) twice and making

* The abbreviation HTF stands for Erdélyi, Magnus, Oberhettinger and Tricomi (1953).

use of the differential equation (3.7), we obtain

$$wH'(w)=abw(1-w)^{-1}+2[w(1-w)^{-1}-\mu]H-H^2 \quad (6.5a)$$

and

$$wH''(w)=(ab+2H)(1-w)^{-2}+2[w(1-w)^{-1}-\mu-\tfrac{1}{2}-H]H'. \quad (6.5b)$$

Setting $H'=0$ in (6.5b), we infer that H may have a minimum, but not a maximum, in $H>-\frac{1}{2}ab$ and conversely in $H<-\frac{1}{2}ab$. We also note the limiting behaviors

$$H(w)=ab(a+b-1)^{-1}w+O(w^2) \quad (w\to 0) \quad (6.6)$$

and

$$H(w)=(1-w)^{-1}+O[\log(1-w)] \quad (w\to 1). \quad (6.7)$$

A typical case ($n=3$) is sketched in Fig. 1.

We consider first the atypical case $n=0$, for which $ab>0$ and $F(w)$ has no zeros. We then deduce from (6.6) and (6.7) that $H(w)$ is a positive, monotonically increasing function in $w=(0,1)$. Remarking that $n=0$ implies $\mu>\nu-1$, and hence that

$$\lambda>\nu+\kappa-\tfrac{3}{2}>0 \quad (n=0), \quad (6.8)$$

we conclude that (6.4) has no roots in $w=(0,1)$ if $0<\nu-\mu<1$.

Now let us suppose that $n\geqq 1$; then $ab<0$, and $H(w)$ is a monotonically decreasing function in $w=(0,u_1)$, (u_1,u_2), $\cdots$, (u_{n-1},u_n) that jumps from $-\infty$ to $+\infty$ as w increases through u_m, whilst in $w=(u_n,1)$ it has a single, positive minimum, beyond which it increases monotonically to $+\infty$ in accordance with (6.7). Double roots of (6.4) are impossible in $w=(0,u_n)$ by virtue of the monotone behavior of $H(w)$. We can rule out the possibility of roots in $w=(u_n,1)$ through the inequality $H>-\frac{1}{2}ab$, which follows from (6.5); accordingly,

$$H+\lambda>-\tfrac{1}{2}ab+\lambda=\tfrac{1}{2}(\nu^2-\mu^2)-1+\kappa=\tfrac{1}{2}\beta W+\kappa>0 \quad (u_n\leqq w\leqq 1). \quad (6.9)$$

We conclude that (6.4) has n simple roots in $w=(0,u_n)$ if $\lambda>0$ or $n-1$ simple roots in $w=(u_1,u_n)$ and one root in $w<0$ if $\lambda<0$ (see the third paragraph in §5 in regard to the root in $w<0$ for $\lambda<0$). The additional zero of Φ that accrues as $\nu-\mu$ increases through $n+1$ must appear at the branch point $w=1$ (since we already have ruled out the other singular point, $w=\infty$, as a possible zero); in fact, as we have shown in the preceding section, $w=1$ is a double zero, and $\Phi(w)$ has $n+2$ real zeros, for $\nu-\mu=n+1$.

Having established that $\Phi(w;\mu,\nu)$ has n simple zeros in $w<1$ and no other real zeros if μ and ν are real and restricted according to (6.1), we can determine the number of complex zeros by applying Cauchy's principle of the argument. Let Γ be a closed curve comprising a large circular arc (with radius tending to infinity) extending from $\arg w=0+$ to $2\pi-$, radii from $w=1+$ along $\arg w=0+$ and $2\pi-$, and a small circle (with radius tending to zero) around the branch point at $w=1$. Starting from the asymptotic approximation

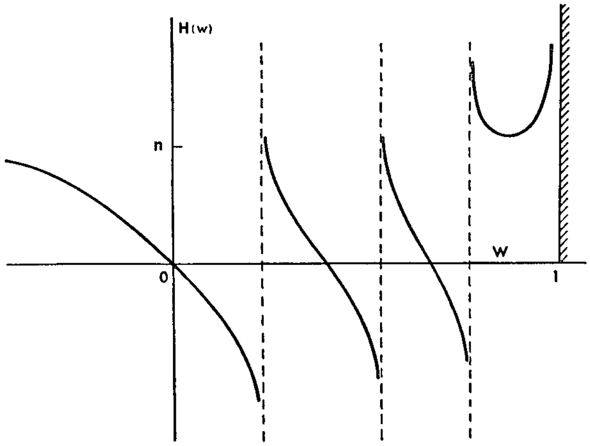

FIG. 1. A sketch of $H(w)$, as defined by (6.4), for $n=3$.

$$F(w)\sim\text{const.}\times w^{-a} \quad (|w|\to\infty, 0<\arg w<2\pi), \quad (6.10)$$

we find that

$$\Phi(w)\sim\text{const.}\times w^{2-a} \quad (|w-1|\to\infty) \quad (6.11)$$

on the large circle. Invoking (6.1) and (6.3), we find that $\arg\Phi$ increases by $2\pi(n+2)$ in one circuit around Γ [we omit the detailed analysis for this step, as it is essentially similar to that presented by Burger (1962) for Charney's model insofar as μ is real]. Invoking the fact that $\Phi(w)$ has n real zeros and no poles [which follows from the fact that $F(w)$ has no poles] in Γ, we conclude that it has exactly two complex zeros, say $W_c(\mu,\nu)$ and $W_c^*(\mu,\nu)$, in the cut w-plane under the restriction (6.1).

The complex zeros W_c and W_c^* are not eigenvalues for our problem for real α and β (since complex W implies complex c, which implies complex μ for real α and β), but the fact that $\Phi(w;\mu,\nu)/\Gamma(2\mu+1)$ is an entire function of μ for $w\neq 1$ or ∞ implies that $W_c(\mu,\nu)$ has an analytic continuation for complex μ if $W_c\neq 1$ or ∞. Having ruled out both $W=1$ (for $a\neq -n$) and $W=\infty$ as possible zeros of Φ, we conclude that Φ admits exactly two complex zeros for any complex μ and hence that $\Psi(c;\alpha,\beta)$ admits exactly two complex zeros for all α and β with the possible exception of points on the singular curves of §5.

7. Nearly singular modes

We now consider small perturbations of the singular neutral modes of §5. As we have shown, these modes are defined by $W=1$ and $a=-n$ and correspond to points on the singular curves

$$\mu=\hat{\mu}(\nu)=\nu-(n+1)>0 \quad (n=0,1,2,\cdots) \quad (7.1a)$$

or, equivalently,

$$\beta=\hat{\beta}(\nu)=(n+1)(2\nu-3)-n(n-1)>(n+1)^2-2. \quad (7.1b)$$

The lowest member of this family, $\hat{\beta}=2\nu-3$ ($n=0$), increases monotonically from the origin in an α,β-plane to the asymptote $\beta=2\alpha$. The higher members terminate on the line $\beta=\alpha_1^2$ ($\mu=c=c_1=0$) in consequence of the restriction (3.2b) and increase monotonically to the asymptotes $\beta=2(n+1)\alpha$. As we have remarked in §4, these singular curves comprise all possible singular neutral modes in $\beta<\alpha_1^2$.

We can deduce a suitable representation of $F(w)$ in the neighborhood of $a=-n$ and $w=1$ by dividing (6.2) through by $\Gamma(a)\Gamma(1-a)$ and invoking the formulae [HTF *1*, §1.2(6), §1.7.1(11)]

$$\Gamma(a)\Gamma(1-a)=\pi\csc(\pi a) \quad (7.2)$$

and

$$\psi(m+a)=\psi(1-a-m)-\pi\cot(\pi a). \quad (7.3)$$

The result is

$$F(w)=C\epsilon\{(1-w)^{-1} + \sum_{m=0}^{\infty}[\Gamma(a+m)\Gamma(b+m)/\Gamma(a-1)\Gamma(b-1) \times m!(m+1)!][\epsilon^{-1}+\log(1-w) + \psi(1-a-m)+\psi(b+m)-\psi(m+2) - \psi(m+1)](1-w)^m\}$$
$$(|1-w|<1,\ |\arg(1-w)|<\pi), \quad (7.4)$$

where

$$C=-[\Gamma(a+b-1)\Gamma(1-a)/\Gamma(b)]\cos(\pi a) \quad (7.5a)$$
$$=[\Gamma(2\mu+1)\Gamma(\nu-\mu)/\Gamma(\mu+\nu+1)]\cos[\pi(\nu-\mu)] \quad (7.5b)$$

and

$$\epsilon=-\pi^{-1}\tan(\pi a)=\pi^{-1}\tan[\pi(\nu-\mu)]. \quad (7.6a,b)$$

Substituting (7.4) into (4.1); eliminating a and b through (3.10) and (3.2)–(3.5), λ through (4.2), and W through (3.8); and retaining only the dominant terms in c and $\log(-c)$, we obtain

$$\Psi(c)=C\{\epsilon[1-(\mu+\beta+\kappa+\tfrac{1}{2})c+O(c^2)]-(\beta+2) \times(\tfrac{1}{2}\beta+\kappa)c^2[1+\epsilon\log(-c)][1+O(c)]\}, \quad (7.7)$$

where the error terms are real for real c. Invoking the restrictions set forth in §4, we must choose that branch of the logarithm for which

$$\log(-c)=\log|c|+i(\arg c-\pi) \quad (0<\arg c<2\pi). \quad (7.8)$$

Expanding μ and ϵ about $c=0$, we obtain

$$\mu=[\mu_0^2+\beta(1-W)]^{\frac{1}{2}}=\mu_0+O(c) \quad (7.9a,b)$$

and

$$\epsilon=\epsilon_0+\tfrac{1}{2}(\beta/\mu_0)[c+(1+\tfrac{1}{4}\mu_0^{-2}\beta)c^2]+O(\epsilon_0^2c,\epsilon_0c^2,c^3), \quad (7.10)$$

where μ_0 and ϵ_0 are evaluated at $c=0$ and are given by

$$\mu_0=\hat{\mu}+\hat{\mu}^{-1}(\nu\delta\nu-\tfrac{1}{2}\delta\beta) \quad (7.11)$$

and

$$\epsilon_0=\hat{\mu}^{-1}[\tfrac{1}{2}\delta\beta-(n+1)\delta\nu] \quad (7.12)$$

for first-order variations in ν and β with respect to a point on the singular curve of (7.1).

Now let us suppose that $\epsilon_0\to 0$ and explore the possibility that $\Psi(c)$ has a zero in the neighborhood of $c=0$ or, equivalently, that $\Phi(W;\mu,\nu)$ has a zero in the neighborhood of $W=1$. We infer from (7.7) that a necessary condition for the existence of such a zero is $\epsilon=O(c^2)$. Invoking this condition in (7.10), we obtain the *provisional* approximation

$$c\to-(2\mu_0/\beta)\epsilon_0, \quad \arg c\to\pi \quad (\epsilon_0\to 0+). \quad (7.13a,b)$$

It then is a straightforward matter to show that $\Psi(c)$, as given by (7.7)–(7.10), does have a negative-real zero that tends to zero in accordance with (7.13).

Now let us suppose that $\epsilon_0\to 0-$. The argument antecedent to (7.13) then suggests that $c\to 0+$ and that either $\arg c\to 0$ or $\arg c\to 2\pi$. We find, however, that the provisional hypothesis $\arg c\to 0+$ in (7.7)–(7.10) implies (after setting $\Psi=0$) the contradictory result $c_i\to 0-$; conversely, $\arg c\to 2\pi-$ implies $c_i\to 0+$. We therefore must conclude that $\Psi(c)$ has no zeros in the neighborhood of $c=0$ for $\epsilon_0\to 0-$.

Summing up, we have shown that each of the singular neutral modes described by $c=\epsilon_0=0$ has a continuation into $\epsilon_0>0$, such that $c<0$, but not into $\epsilon_0<0$. It follows that the complex zeros of $\Psi(c)$ do not tend to zero as the singular curves of (7.1b) are approached and hence that these singular curves are *not* stability boundaries.

8. Approximate representations

We can obtain an explicit representation of $c(\alpha,\beta)$ in a domain that includes the lower part of the critical curve $\beta=2\nu-3$ ($a=0$) by letting $\alpha^2,\beta\to 0$ ($a\to 0, b\to 3$) and either expanding F and Φ in powers of a and $b-3$ or invoking the general formulation of *II*. Introducing

$$\delta=\max.(\alpha^2,\beta)\ll 1 \quad (8.1)$$

and supposing that

$$\kappa=O(\delta), \quad (8.2)$$

we find (considerable detail is omitted at this point) that Ψ has the representation

$$\Psi(c)=A(c)+B(c)\log(-c), \quad (8.3)$$

where $A(c)$ and $B(c)$ are real for real c and have the limiting approximations

$$A(c)=(1-c)^{-2}[(\alpha^2+\kappa)c^2+(\beta-\alpha^2)c +(\tfrac{1}{3}\alpha^2-\tfrac{1}{2}\beta)]+O(\delta^2) \quad (8.4)$$

and

$$B(c)=2(1-c)^{-1}(\tfrac{1}{3}\alpha^2-\tfrac{1}{2}\beta-\tfrac{1}{3}\alpha^2 c)\times[\tfrac{1}{3}\alpha^2-\tfrac{1}{2}\beta-\tfrac{1}{3}\alpha^2 c-A(c)]+O(\delta^3). \quad (8.5)$$

Let [cf II(5.4)–(5.6) with $U_1=\tfrac{1}{2}$ and $U_2=\tfrac{1}{3}$]

$$c_{\pm}=\tfrac{1}{2}(\alpha^2+\kappa)^{-1}\{(\alpha^2-\beta)\pm[\beta^2-\tfrac{1}{3}\alpha^4+2\kappa(\beta-\tfrac{2}{3}\alpha^2)]^{\frac{1}{2}}\} \quad (8.6)$$

denote the zeros of $A(c)$,

$$\beta'=(\alpha^2+\kappa)^{\frac{1}{2}}(\tfrac{1}{3}\alpha^2+\kappa)^{\frac{1}{2}}-\kappa<\tfrac{2}{3}\alpha^2 \quad (8.7)$$

that value of β at which $c_+=c_-$, and

$$\beta''=\tfrac{2}{3}\alpha^2 \quad (8.8)$$

that value of β at which $c_-=0$. Then: (i) $c_{\pm}$ are complex conjugates with positive-real parts if $0\leqq\beta<\beta'$; (ii) $c_+>c_->0$ if $\beta'<\beta<\beta''$; and (iii) $c_{\pm}\gtrless 0$ if $\beta>\beta''$. We can show that Ψ, as approximated by (8.3)–(8.5), admits the following zeros as $\delta\to 0$:

$$c\to c_{\pm} \quad (0\leqq\beta<\beta'); \quad (8.9a)$$

$$c_r\to c_+, \quad c_i\to\pm 2\pi(1-c_+)[\tfrac{1}{3}\alpha^2(1-c_+)-\tfrac{1}{2}\beta]^2\times[\beta^2-\tfrac{1}{3}\alpha^4+2\kappa(\beta-\tfrac{2}{3}\alpha^2)]^{-\frac{1}{2}},$$

$$\arg c\to\pi\pm\pi \quad (\beta>\beta'); \quad (8.9b)$$

and

$$c\to c_-<0 \quad (\beta>\beta''). \quad (8.9c)$$

The complex-conjugate zeros given by (8.9a,b) are approximations, as $\delta\to 0$, to the two complex-conjugate branches of $c(\alpha,\beta)$ that we have shown must exist for all α^2 and β. The negative-real zero given by (8.9c) corresponds to the zero given by (7.13) as $\epsilon_0\to 0+$, where, from (7.1a) and (7.12),

$$\epsilon_0=\beta-\tfrac{2}{3}\alpha^2+O(\delta^2). \quad (8.10)$$

We remark that $c_i=O(\delta^0)$ for $\beta<\beta'$ and $c_i=O(\delta)$ for $\beta>\beta'$. We also can show that $c_i=O(\delta^{\frac{1}{2}})$ at $\beta=\beta'$ [see II(5.16)], although neither (8.9a) nor (8.9b) provides a uniformly valid approximation to c_i as $\beta\to\beta'$.

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