Effects of Diffusion on Baroclinic Instability of the Zonal Wind[1]

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ABSTRACT

The baroclinic instability problem is reformulated to include diffusion of both heat and momentum through conduction and viscosity. *A priori* arguments suggest that the effects of heat conduction should dominate those of viscosity in the critical layer, where local wind speed equals wave speed and the adiabatic model for small disturbances is not uniformly valid. An asymptotic solution of the singular perturbation problem (based on the hypothesis that the Peclet and Reynolds numbers tend to infinity) supports this conjecture but also implies that the effects of diffusion on baroclinic instability are negligible insofar as the critical layer is within the geostrophic regime of the mean flow. This last condition is satisfied for the disturbances of principal meteorological interest.

1. Introduction

We consider here the possible effects of diffusion of heat and momentum on the baroclinic instability of the zonal wind. The adiabatic model for baroclinic instability, as developed originally by Charney (1947), yields solutions that exhibit singularities in consequence of the neglect of diffusion. Both Kuo (1952) and Burger (1962) have considered the effect of viscosity in connection with the singularity at the so-called *critical layer*, where wave speed equals local wind speed, but neither included the effect of heat conduction. Such treatments are basically inconsistent in the sense that the diffusion of heat through conduction introduces terms in the basic equations of motion for a compressible fluid that are at least of the same order of magnitude as the terms introduced by the diffusion of momentum through viscosity. Moreover, this is an understatement for the baroclinic instability problem at large values of the Richardson number, for the unstable motions derive their energy from the potential, rather than the kinetic, energy of the mean motion, in consequence of which the diffusion of heat dominates the diffusion of momentum in the critical layer. To be sure, we might anticipate (on the basis of scale considerations) that neither effect is likely to be substantial for the (meteorologically) more important instabilities, but the question still appears of sufficient interest to warrant a consistent, analytical investigation.

The dominant consideration in estimating the effect of diffusion is the degree of turbulence. There can be no question as to the importance of turbulence in the lower portions of the atmosphere—say in the boundary-layer (or Ekman-Taylor) regime—but the hypothesis of large Richardson number in the geostrophic regime suggests that it is at least reasonable to regard the flow as laminar throughout most of the troposphere. Moreover, the introduction of eddy coefficients of viscosity and heat conduction, in place of their molecular counterparts, is a plausible device insofar as the net effects of diffusion on the instabilities of principal interest remain small. On the other hand, if the introduction of appropriate eddy coefficients leads to effects of the same order of magnitude as those predicted by the adiabatic model, we can only draw the negative conclusion that the adiabatic model may be inadequate; our limited knowledge of turbulence will not permit a stronger statement.

Turning to scale considerations, we begin our formulation by introducing the characteristic temperature T_0 the characteristic density ρ_0, the characteristic wind speed U_1, the gas constant R, the specific heat c_p, the viscosity μ, the thermal conductivity K, the Coriolis parameter f ($f=2\Omega\sin\varphi$, where Ω denotes the angular velocity of the Earth and φ the latitude), the wave number k, the scale height

$$H=RT_0/g, \tag{1.1}$$

the radius of the Earth a, and the dimensionless stability parameter κ [see Eq. (3.5) below]. We anticipate that the final statement of the eigenvalue problem (Sec. 4) involves the latitude φ only as a parameter and hence that we can regard T_0 and ρ_0 as the temperature and pressure at ground level ($z=0$) and local latitude; initially, however, we shall regard T_0 and ρ_0 as constants. A similar statement holds for U_1, except that we shall equate it to the product of H and the vertical

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shear at ground level, such that $U_0'=1$ in the dimensionless notation of the following sections.

We next define the dimensionless parameters:

$$\mathrm{Ri}=\kappa gH/U_1^2 \qquad \text{(Richardson number)}, \quad (1.2)$$

$$\mathrm{Ro}=U_1k/f \qquad \text{(Rossby number)}, \quad (1.3)$$

$$\alpha=(\kappa gH)^{\frac{1}{2}}k/f \qquad \text{(modified wave number)}, \quad (1.4)$$

$$\beta=(\kappa gH/fU_1a)\cot\varphi, \quad (1.5)$$

$$1/\epsilon=kH^2U_1\rho_0c_p/K \qquad \text{(modified Peclet number)}, \quad (1.6)$$

$$1/\sigma\epsilon=kH^2U_1\rho_0/\mu \qquad \text{(modified Reynolds number)}, \quad (1.7)$$

and

$$\sigma=\mu c_p/K \qquad \text{(Prandtl number)}. \quad (1.8)$$

We may regard β either as an inverse measure of wind shear or as a measure of the rate of change of Coriolis force with latitude.

We shall base our analysis on the following assumptions: (i) a perfect gas with constant values of c_p, μ, and K; (ii) $\epsilon\ll1$; (iii) a uniform gravitational field (geopotential $=gz$); (iv) $\mathrm{Ri}\gg1$; (v) $\mathrm{Ro}\ll1$; (vi) $\kappa\ll1$; (vii) Rossby's β-plane approximation. We emphasize that assumptions (i)–(vi) suffice for the final statement of the eigenvalue problem in the sense that a formulation in Mercator coordinates, subject to assumptions (i)–(vi), yields the same problem (cf., Phillips, 1963, and Miles, 1964).

We shall proceed by: presenting, in Sec. 2, approximations to the equations of motion based on (i)–(vii); posing a geostrophic mean flow in Sec. 3; considering travelling-wave perturbations with respect to this mean flow in Sec. 4; formulating an asymptotic (as $\epsilon\to0$) solution of the resulting, fourth-order differential equation in Sec. 5; and calculating the order of magnitude (in ϵ) of the contributions of diffusion to the complex wave speeds in Sec. 6.

The end results of our analysis can be simply stated as follows: let c denote the dimensionless, complex wave speed of a travelling-wave disturbance and c_a the limiting (adiabatic) value of c for $\epsilon=0$; then either

$$c=c_a+O(\epsilon) \quad (|c_a|\gg\epsilon^{\frac{1}{3}}, \quad \epsilon\to0), \quad (1.9a)$$

or

$$c=c_a+O(\epsilon^{\frac{2}{3}}) \quad [c_a=O(\epsilon^{\frac{1}{3}}), \quad \epsilon\to0], \quad (1.9b)$$

provided that c_a is a simple eigenvalue (i.e., c_a is a simple zero of the adiabatic eigenvalue equation). A typical value of ϵ on the hypothesis of molecular diffusion is 10^{-8}; accordingly, we may conclude that $c-c_a$ is negligible insofar as the critical layer is within the geostrophic regime. On the other hand, $c_a=O(\epsilon^{\frac{1}{3}})$ is possible only for a critical layer well within the boundary-layer regime of the Ekman spiral, where our basic model is invalid.

If we overlook the fact that our model does not comprise the Ekman spiral and attempt to allow for turbulence near the ground by introducing the eddy conductivity (Priestley, 1959; we approximate the vertical shear for our problem by U_1/H)

$$K/\rho_0c_p=(U_1/H)(0.4z)^2, \quad (1.10)$$

in (1.6) and eliminating k through (1.4), we obtain

$$\epsilon\alpha=\frac{0.16}{f}\left(\frac{\kappa g}{H}\right)^{\frac{1}{2}}\left(\frac{z}{H}\right)^2. \quad (1.11)$$

Choosing $f=10^{-4}$, $g=10^3$, $H=10^6$ (cgs units), and $\kappa=10^{-1}$, we obtain $\epsilon\alpha\simeq(4z/H)^2$, which is not likely to exceed 10^{-3} in the surface boundary layer. We emphasize, however, that (1.10) is at best conjectural, and we have introduced it only to obtain a rough estimate of the possible order of magnitude for ϵ.

2. Equations of motion

Let x, y, z be right-handed Cartesian coordinates based in a β-plane (such that the x and y axes point east and north, respectively); u, v, w be the corresponding components of velocity; and p, ρ, T, S be the pressure, density, absolute temperature, and entropy. We shall suppose that the mean flow is geostrophic (the zonal wind) and that disturbances with respect to this mean flow travel along parallels. We remark that obliquely travelling disturbances could be accommodated by a simple transformation, but that the most unstable disturbances for a given wave number do travel along parallels (Miles, 1964).

Invoking assumptions (i)–(vii) above, we find that the components of Euler's equation can be approximated by

$$\rho fv=p_x, \quad \rho\left(\frac{Dv}{Dt}+fu\right)=-p_y+\mu v_{zz}, \quad \rho g=-p_z \quad (2.1\text{a,b,c})$$

and the energy equation by

$$\rho T\frac{DS}{Dt}=KT_{zz}, \quad (2.2)$$

where subscripts imply partial differentiation and

$$\frac{D}{Dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}. \quad (2.3)$$

We also have (without approximations, other than that of a continuum) the equation of continuity

$$(\rho u)_x+(\rho v)_y+(\rho w)_z+\rho_t=0 \quad (2.4)$$

and the perfect-gas equations

$$p=R\rho T \quad (2.5)$$

and

$$S=c_p \log T-R \log p+\text{const.} \tag{2.6}$$

We remark that Ri≫1 justifies our neglect of the friction term in (2.1a) vis-a-vis (2.1b) and of the heat generated by friction (i.e., the Stokes dissipation function) in (2.2). We also have approximated $\nabla^2 v$ by v_{zz} and $\nabla^2 T$ by T_{zz} by virtue of the vertical-scale considerations ($H \ll a$ and $kH \ll 1$) implicit in assumptions (i)–(vi).

We shall seek a solution to (2.1)–(2.6) in the form of a mean (geostrophic) flow, which is independent of both x and t, plus a travelling-wave perturbation that exhibits the x,t-dependence $\exp[ik(x-U_1ct)]$, where c is a dimensionless, complex wave speed. The adiabatic model for baroclinic instability yields a second-order, homogeneous differential equation for the amplitude of the perturbation, *qua* function of z. Invoking the boundary condition

$$w=0 \qquad (z=0), \tag{2.7}$$

together with an appropriate null condition at $z=\infty$, then yields an eigenvalue problem for $c(\alpha,\beta,\kappa)$. We anticipate that the present model will yield a fourth-order differential equation, namely (4.5) below, and we shall invoke the additional boundary condition (where $\bar{T}$ is the mean temperature)[3]

$$T-\bar{T}=0 \qquad (z=0), \tag{2.8}$$

together with an additional null condition at $z=\infty$, in order to obtain an eigenvalue problem for $c(\alpha,\beta,\kappa,\epsilon)$.

3. Mean motion

We shall suppose the mean motion to be prescribed by the dimensionless wind speed $U(\zeta)$ and the dimensionless temperature $\tau(\zeta)$, such that

$$\bar{u}=U_1U(\zeta), \qquad \bar{v}=\bar{w}=0, \tag{3.1}$$

and

$$\bar{T}=T_0\tau(\zeta), \quad \bar{\rho}=\rho_0\eta(\zeta), \quad \bar{p}=\rho_0 gH\eta(\zeta)\tau(\zeta), \tag{3.2}$$

where

$$\zeta=z/H, \quad \eta(\zeta)=\frac{1}{\tau}\exp\left(-\int_0^\zeta \frac{d\zeta}{\tau}\right). \tag{3.3}$$

We remark that U and τ also depend on y, as well as ζ, but that we can regard y as a parameter, rather than as an independent variable, in all that follows. With this in mind, we shall denote partial differentiations with respect to ζ by primes.

Having $\bar{T}$, we can calculate the stability according to

$$s=c_p^{-1}\bar{S}_z=\kappa/H\tau(\zeta), \tag{3.4}$$

where

$$\kappa=\tau'(\zeta)+\gamma^{-1}(\gamma-1)\ll 1. \tag{3.5}$$

We also note that

$$c_p^{-1}\bar{S}_y=(\bar{p}_y/\gamma\bar{p})-(\bar{\rho}_y/\bar{\rho}) \tag{3.6a}$$

$$=(f/g)(s\bar{u}-\bar{u}_z) \tag{3.6b}$$

by virtue of the equilibrium forms of (2.1b,c). We shall, in the interests of simplicity, regard κ as a constant throughout the subsequent development; however, the qualitative character of our results is independent of this approximation insofar as κ is positive definite.[4] We also remark that $\kappa \ll 1$ permits us to choose

$$U_0=0 \tag{3.7}$$

without loss of generality provided that $U_1U(\zeta)$ and the wave speed U_1c in the subsequent analysis are defined relative to the wind speed at ground level.

We emphasize that the mean motion of (3.1) is geostrophic and does not comprise the Ekman spiral, which is a consequence of the shear stresses at the surface. We could have included an Ekman spiral in our basic flow by including the term $-\mu u_{zz}$ on the RHS of (2.1a), but this would have led to a much more complicated description of the perturbation flow than that developed in the following section.[5] As we already have remarked (in Sec. 1), our present knowledge of turbulent flow does not appear to justify this more complicated description.

4. Perturbation motion

Let ψ be a dimensionless function of the form

$$\psi=\exp[ik(x-U_1ct)]\times\text{func.}(\zeta,y), \tag{4.1}$$

such that the perturbation pressure is given by

$$p-\bar{p}=\rho_0 gH\eta\psi \quad (|\psi|\ll 1). \tag{4.2}$$

Substituting (4.1) and (4.2), together with the mean motion of Sec. 3, into the equations of motion of Sec. 2, we obtain (after a fairly lengthy elimination procedure) the required perturbations

$$T-\bar{T}=T_0(\tau\psi'-\tau'\psi) \tag{4.3}$$

and

$$w=-(ikU_1H/\kappa)\{i\epsilon\eta^{-1}(\tau\psi'')' + \tau[(U-c)\psi'-U'\psi]+\kappa c\psi\} \tag{4.4}$$

and the linearized differential equation

$$i\epsilon[(\tau\psi'')''-\sigma\alpha^2\psi'']+L\psi=0, \tag{4.5}$$

where

$$L\psi=(U-c)(\tau\eta\psi')'+[\lambda-\alpha^2(U-c)]\eta\psi \tag{4.6}$$

and

$$\lambda=\beta-\eta^{-1}(\tau\eta U')'. \tag{4.7}$$

[3] It might be closer to reality to replace $T-\bar{T}=0$ by $A(T-\bar{T})_z+(T-\bar{T})=0$, but setting $A=0$ does not affect the qualitative character of our results, and it would appear to be rather difficult to arrive at a rational estimate of A.

[4] This approximation is tantamount to the assumption that τ is linear in ζ, which implies that the mean flow is adiabatic.

[5] Barcilon (1964) has shown that the Ekman layer(s) may play an important role in respect to baroclinic instability in a rotating annulus.

The boundary conditions (2.7) and (2.8) require w and $T-\bar{T}$ to vanish at $\zeta=0$. We shall find it convenient, in formulating these boundary conditions, to introduce the dimensionless vertical velocity ω according to

$$w=(ikU_1H/\kappa)\omega(\zeta) \quad (4.8)$$

and to invoke the inequalities

$$\kappa\ll 1, \quad |\tau'|\ll 1. \quad (4.9\text{a,b})$$

We then can pose the boundary conditions in the form

$$\omega=0, \quad \psi'=0 \quad (\zeta=0). \quad (4.10\text{a,b})$$

Comparing (4.4) and (4.8), invoking (4.9a,b), and recalling that $\tau=\eta=U_0'=1$ and $U_0=0$ at $\zeta=0$, we obtain

$$\omega_0=\psi+c\psi'-i\epsilon\psi''' \quad (4.11\text{a})$$

$$=-\int_0^\infty[\beta-\alpha^2(U-c)]\eta\psi d\zeta-i\epsilon\sigma\alpha^2\psi', \quad (4.11\text{b})$$

where (4.11b) follows from (4.11a) with the aid of (4.5) and the null conditions at $\zeta=\infty$. All quantities in (4.11a,b) are implicitly evaluated at $\zeta=0$.

5. Asymptotic solution

Comparing (4.5) with the Orr-Sommerfeld equation (Lin, 1955), we seek an asymptotic solution (as $\epsilon\to 0$) in the form

$$\psi=C_1\phi(\zeta)+\delta C_2\chi(\xi), \quad \xi=(\zeta-\zeta_c)/\delta, \quad (5.1\text{a,b})$$

where ϕ and χ are perturbation functions for which appropriate length scales are H and δH, respectively, and ζ_c is defined by $U(\zeta_c)=c$. We may assume, without loss of generality, that both ϕ and χ are $O(1)$ as $\epsilon\to 0$. Substituting $\psi=C_1\phi(\zeta)$ into (4.5), we obtain

$$L\phi(\zeta)+O(\epsilon)=0 \quad (\epsilon\to 0). \quad (5.2)$$

We remark that the error estimate in (5.2) is based on the hypothesis that H is an appropriate length scale. This hypothesis fails in the neighborhoods of: (a) $\zeta=\zeta_c$, where the second-order differential equation $L\phi=0$ has a regular singularity (we assume that $U-c$ has only simple zeros); and (b) $\zeta=\infty$, which is both a singularity of $L\phi=0$ and a point at which two boundary conditions must be invoked. In fact, our model does not seek to give an accurate representation of the motion in $\zeta\gg H$, and we shall rest content with the remark that the asymptotic solution developed here can be made to satisfy adequate null conditions as $\zeta\to\infty$.

Substituting $\psi=\delta C_2\chi(\xi)$ into (4.5), expanding $\tau(\zeta)$, $\eta(\zeta)$, and $U(\zeta)$ about $\zeta=\zeta_c$, and choosing

$$\delta=(\epsilon/\eta_cU_c')^{\frac{1}{3}}, \quad (5.3)$$

we obtain

$$\chi^{iv}(\xi)-i\xi\chi''(\xi)+O(\delta)=0 \; [\zeta-\zeta_c=O(\delta),\ \delta\to 0]. \quad (5.4)$$

We emphasize that the primes on χ imply differentiation with respect to ξ, whereas elsewhere they imply differentiation with respect to ζ. The subscript c denotes evaluation at $\zeta=\zeta_c$ (thus, $U'\equiv dU/d\zeta$ at $\zeta=\zeta_c$).

We shall designate (5.2) and (5.4) as (the asymptotic forms of the) *adiabatic* and *diffusion* equations, respectively, and those solutions for ϕ and χ that satisfy null conditions at $\zeta=\infty$ as the *adiabatic solution* and the *diffusion solution*, respectively (cf. the *inviscid* and *viscous* solutions of the Orr-Sommerfeld equation). The adiabatic equation (5.2) gives an asymptotically correct representation of (4.5) as $\epsilon\to 0$ with fixed $\zeta\neq\zeta_c$; the diffusion equation (5.4) gives an asymptotically correct representation of (4.5) as $\delta\to 0$ for fixed ζ. Moreover, the diffusion of heat and the diffusion of momentum introduce errors of the same order of magnitude, namely $O(\epsilon)$, in the former limit, whereas diffusion of heat dominates diffusion of momentum by a factor of $O(\epsilon^{-\frac{2}{3}})$ in the latter limit [the viscous term $-i\epsilon\sigma\alpha^2\psi''$ in (4.5) contributes a term proportional to $\delta^2\chi''$ to (5.4)].

We show in the Appendix that the required solution to (5.4), subject to null conditions at $\xi=\infty$, has the form

$$\chi(\xi)=\int_\infty^w \mathrm{Ai}(w)dw-\mathrm{Ai}'(w) \quad (5.5)$$

where

$$w=e^{i\pi/6}\xi, \quad (5.6)$$

and $\mathrm{Ai}(w)$ is Airy's function. We observe in passing that (5.4) also admits the linearly independent solutions $\chi=1$ and $\chi=\xi$, which correspond to the two linearly independent solutions of $L\phi=0$ in the neighborhood of $\zeta=\zeta_c$, namely

$$\phi=\zeta-\zeta_c+O(\zeta-\zeta_c)^2 \quad (5.7)$$

and

$$\phi=1+O[(\zeta-\zeta_c)\log(\zeta-\zeta_c)]. \quad (5.8)$$

6. Eigenvalue equation

Substituting (5.1) into (4.10a,b) and requiring the determinant of the resulting equations for C_1 and C_2 to vanish, we obtain the eigenvalue equation

$$\Delta(c)=\frac{\omega(\phi)}{\phi_0'}-\frac{\delta\omega(\chi)}{\chi_0'}=0, \quad (6.1)$$

where $\omega(\phi)$ and $\omega(\chi)$ denote the values of ω obtained by substituting ϕ and χ, respectively, into either (4.11a) or (4.11b), and the subscript zero implies evaluation at $\zeta=0$ (thus, $\phi_0'\equiv d\phi/d\zeta$ at $\zeta=0$ and $\chi_0'=d\chi/d\xi$ at $\xi=0$).

Referring to (4.11a) and (5.2), we obtain

$$\omega(\phi)/\phi_0'=\Delta_a(c)+O(\epsilon), \quad (6.2)$$

where

$$\Delta_a(c)=c+(\phi/\phi')_0 \quad (6.3)$$

is based on the adiabatic equation $L\phi=0$. We are interested in those zeros of $\Delta(c)$ that tend to the zeros of $\Delta_a(c)$, say $c=c_a$, as $\epsilon\rightarrow 0$. [$\Delta(c)$ also may have zeros that do not tend to the zeros of $\Delta_a(c)$ as $\epsilon\rightarrow 0$, but the corresponding solutions would be strongly damped and of little interest for the baroclinic stability problem.]

We show in the Appendix that the diffusion solution (4.5) yields

$$\frac{\delta\omega(\chi)}{\chi_0'}=\tfrac{1}{2}\delta^2\beta D(z)[1+O(\delta)] \quad [z=O(1),\, \delta\rightarrow 0] \tag{6.4a}$$

$$=O(\epsilon) \qquad [|c|\gg\epsilon^{\frac{1}{3}},\, \epsilon\rightarrow 0], \tag{6.4b}$$

where

$$z=\zeta_c/\delta=\epsilon^{-\frac{1}{3}}c\,[1+O(\delta)] \tag{6.5}$$

and

$$D(z)=z^2-e^{-i\pi/3}\left\{[w\,\mathrm{Ai}(w)]'\Big/\int_\infty^w \mathrm{Ai}(w)dw\right\},$$

$$w=e^{-5i\pi/6}z. \tag{6.6}$$

Substituting (6.2) and (6.4b) into (6.1) and supposing that c_a is a simple zero of $\Delta_a(c)$, we infer that

$$c=c_a+O(\epsilon) \quad [|c|\gg\epsilon^{\frac{1}{3}},\, \epsilon\rightarrow 0]. \tag{6.7}$$

This result covers the *strong* instabilities that are of principal meteorological interest (for which the eigenvalues *are* simple zeros of Δ_a). We also find, after a more detailed investigation, that the imaginary part of $c-c_a$ is negative if c_a-U_0 is real and negative. This result, together with the known result that real c_a implies $c_a<0$ if $\lambda(\zeta)$ and $U'(\zeta)$ are positive definite (Miles, 1964), establishes the dissipative effect of diffusion for the (non-singular) normal modes of an important class of configurations. On the other hand, diffusion may be destabilizing if $c_a>0$, although the effect is $O(\epsilon)$ unless $c_a\ll 1$.

Now let us suppose that $|c|\ll 1$. Substituting (5.3), (6.2), and (6.4a) into (6.1), we obtain

$$\Delta(c)=\Delta_a(c)-\tfrac{1}{2}\epsilon^{\frac{2}{3}}\beta D(\epsilon^{-\frac{1}{3}}c)+O(\epsilon)$$

$$[|c|\ll 1,\, \epsilon\rightarrow 0]. \tag{6.8}$$

If we also suppose that $\Delta_a(c)$ has a simple zero at $c=c_a$, such that

$$\Delta_a(c)=\Delta_a'\cdot(c-c_a), \quad \Delta_a'=(d\Delta_a/dc)_{c=c_a} \quad (c\rightarrow c_a), \tag{6.9}$$

and set $\Delta(c)=0$, we obtain

$$c=c_a+\tfrac{1}{2}\epsilon^{\frac{2}{3}}(\beta/\Delta_a')D(\epsilon^{-\frac{1}{3}}c_a)+O(\epsilon). \tag{6.10}$$

We conclude that $c_a=O(\epsilon^{\frac{1}{3}})$ implies $c-c_a=O(\epsilon^{\frac{2}{3}})$.

Finally, let us suppose that $\Delta_a(c)$ has a double zero at $c=c_a$, such that

$$\Delta_a(c)=\tfrac{1}{2}\Delta_a''\cdot(c-c_a)^2, \quad \Delta_a''=(d^2\Delta_a/dc^2)_{c=c_a}$$

$$(\Delta_a'=0,\, c\rightarrow c_a). \tag{6.11}$$

Substituting (6.11) into (6.8), we obtain

$$\Delta(c)=\tfrac{1}{2}\Delta_a'\cdot(c-c_a)^2-\tfrac{1}{2}\epsilon^{\frac{2}{3}}\beta D(\epsilon^{-\frac{1}{3}}c)+O(\epsilon), \tag{6.12}$$

which implies that $c-c_a=O(\epsilon^{\frac{1}{3}})$ if $c_a=O(\epsilon^{\frac{1}{3}})$, in consequence of which an explicit solution of the form (6.10) is not possible. We emphasize, however, that (6.11) is atypical.

APPENDIX

The Diffusion Solution

We can pose a solution to (5.4) that exhibits the required behavior as $\zeta\rightarrow\infty$ in the form

$$\chi''(\xi)=\mathrm{Ai}(w), \quad w=e^{i\pi/6}\xi, \tag{A1a,b}$$

where $\mathrm{Ai}(w)$ denotes Airy's function and satisfies

$$\mathrm{Ai}''(w)-w\mathrm{Ai}(w)=0. \tag{A2}$$

We can establish that $\chi(\xi)$ and its derivatives vanish exponentially as $|\xi|\rightarrow\infty$ in $-\frac{1}{2}\pi<\arg\xi<\frac{1}{6}\pi$. Integrating in from $\xi=\infty$ and requiring χ and χ' to vanish there, we obtain

$$\chi'=e^{-i\pi/6}\int_\infty^w \mathrm{Ai}(u)du \tag{A3}$$

and

$$\chi(\xi)=e^{-i\pi/3}\left[w\int_\infty^w \mathrm{Ai}(u)du-\mathrm{Ai}'(w)\right], \tag{A4}$$

where (A4) follows from (A3) after integration by parts with the aid of (A2).

We can calculate $\omega(\chi)$ from (4.11b) [(4.11a) yields only the degenerate result $\omega(\chi)=O(\delta)$] according to

$$\omega(\chi)=-\int_0^\infty [\beta-\alpha^2(U-c)]\eta\chi d\zeta-i\epsilon\sigma\alpha^2\delta^{-1}\chi_0' \tag{A5a}$$

$$=-\delta\beta\eta_c\int_{\xi_0}^\infty \chi(\xi)d\xi[1+O(\delta)]-i\epsilon\sigma\alpha^2\delta^{-1}\chi_0'. \tag{A5b}$$

Integrating (5.4) twice by parts, we obtain

$$\chi''-i\xi\chi+2i\int_\infty^\xi \chi d\xi=0. \tag{A6}$$

Substituting (A1) and (A4) into (A6) and then eliminating the integral between (A5b) and (A6), we obtain

$$\frac{\delta\omega(\chi)}{\chi_0'}=\tfrac{1}{2}\delta^2\beta\eta_c e^{-i\pi/3}\left\{w^2-\left[\frac{w\,\mathrm{Ai}'(w)+\mathrm{Ai}(w)}{\int_\infty^w \mathrm{Ai}(u)du}\right]\right\}$$

$$\times[1+O(\delta)]-i\epsilon\sigma\alpha^2. \tag{A7}$$

We can express this last result in terms of the function [cf., Lin, 1955, and Miles, 1960]

$$\mathfrak{F}(z)=\frac{w}{\mathrm{Ai}'(w)}\int_{\infty}^{w}\mathrm{Ai}(u)du \tag{A8a}$$

$$=1.288e^{-5i\pi/6}z+0.686e^{-2i\pi/3}z^2+O(z^3) \tag{A8b}$$

$$\sim 1+e^{i\pi/4}z^{-\frac{3}{2}}+(9/4)iz^{-3}+O(z^{-9/2})\quad (z\to\infty), \tag{A8c}$$

where

$$z=e^{5i\pi/6}w=e^{i\pi}\xi_0=\zeta_c/\delta. \tag{A9}$$

Differentiating (A8a), we obtain

$$\mathfrak{F}'(z)=z^{-1}\mathfrak{F}(z)+e^{i\pi/3}z[1-\mathfrak{F}(z)][\mathrm{Ai}(w)/\mathrm{Ai}'(w)]. \tag{A10}$$

Substituting (A8)–(A10) into (A7), we can place the result in the form

$$\frac{\delta\omega(\chi)}{\chi_0'}=\tfrac{1}{2}\delta^2\beta\eta_c D(z)[1+O(\delta)]-i\epsilon\sigma\alpha^2, \tag{A11}$$

where

$$D(z)=z^2+[\mathfrak{F}(z)]^{-1}\{-z^2+i[\mathfrak{F}'(z)-z^{-1}\mathfrak{F}(z)][\mathfrak{F}(z)-1]^{-1}\} \tag{A12a}$$

$$\sim 2iz^{-1}+O(z^{-5/2})\quad (z\to\infty) \tag{A12b}$$

$$=0.533e^{-i\pi/3}+O(z)\quad (z\to 0). \tag{A12c}$$

Substituting (A12b) into (A11), we obtain

$$\delta\omega(\chi)/\chi_0'\sim i\delta^2\beta\eta_c z^{-1}-i\epsilon\sigma\alpha^2 =i\epsilon[(\beta/U_c'\zeta_c)-\sigma\alpha^2]\quad (z\to\infty). \tag{A13}$$

If, on the other hand, $z=O(1)$, we can reduce (A11) to

$$\delta\omega(\chi)/\chi_0'=\tfrac{1}{2}\delta^2\beta D(z)[1+O(\delta)]\quad [z=O(1),\quad \delta\to 0]. \tag{A14}$$

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