NOTES AND CORRESPONDENCE

On a Conjecture of Lettau

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1. Introduction

In a recent paper Lettau (1964) has conjectured that the fluctuating velocity in a turbulent shear flow is related to the gradient of mean velocity by

$$\mathbf{u}=\mathbf{r}'\times[\nabla\times\mathbf{U}] \tag{1}$$

where $\mathbf{r}'$ is the "radius vector of turbulent displacement of fluid particles"; no further definition is given for $\mathbf{r}'$, beyond requiring that

$$\overline{\mathbf{r}'}=0 \quad \nabla\cdot\mathbf{r}'=0 \tag{2}$$

in an incompressible flow. (1) is obtained by making the *ad hoc* postulate that

$$\boldsymbol{\zeta}'=-(\mathbf{r}'\cdot\nabla)\boldsymbol{\zeta}+(\overline{\boldsymbol{\zeta}}\cdot\nabla)\mathbf{r}', \tag{3}$$

where $\overline{\boldsymbol{\zeta}}$, $\boldsymbol{\zeta}'$ are the mean and fluctuating vorticity, respectively; to (3) is applied the identity

$$\nabla\times(\mathbf{r}\times\boldsymbol{\zeta}]=-(\mathbf{r}\cdot\nabla)\boldsymbol{\zeta}+(\boldsymbol{\zeta}\cdot\nabla)\mathbf{r} \tag{4}$$

which is only true if $\nabla\cdot\mathbf{r}=0$. The use of (4) permits the integration of (3) to obtain (1) if it is presumed that the function of integration vanishes.

Eq. (1) may prove, with use, to be of value in providing a framework for the semi-empirical description of the Reynolds stresses generated by some simple shear flows.[1] Thus it may take its place alongside the classical mixing length theories (i.e., that of momentum transfer due to Prandtl and of vorticity transfer due to Taylor). However we wish to show that the new theory does not penetrate, in any fundamental way, more deeply into the physics of turbulent shear flows than do these classical theories. In particular we wish to demonstrate the following two points:

(1) Eq. (3) is not complete; as written it describes only the generation of fluctuating vorticity by convection of mean vorticity, and neglects the generation of fluctuating vorticity of stretching of existing vorticity fluctuations by the mean strain-rate field.

(2) Eq. (1) is not a correct integration of (3) because of the non-vanishing of the function of integration and because (4) is not an identity due to the non-vanishing of $\nabla\cdot\mathbf{r}$.

2. The transfer of vorticity

Lettau offers (3) as pure *a priori* conjecture, supported only by a brief verbal argument which offers no support for the precise form chosen. In fact the appearance of the two terms in (3) will be shown below to be a natural consequence of the assumption that for short times neither momentum nor vorticity is conserved, but that *angular momentum* is. The physical idea is appealing. It is embodied in Cauchy's equation, which is a first integral of the inviscid equation of motion, relating an initial vorticity distribution to a final one; it can be written as

$$\boldsymbol{\zeta}(\mathbf{x},t)-\boldsymbol{\zeta}(\mathbf{x},0)=[\boldsymbol{\zeta}(\mathbf{x},0)\cdot\nabla]\mathbf{r}-[\mathbf{r}\cdot\nabla]\boldsymbol{\zeta}(\mathbf{x},0), \tag{5}$$

where we have assumed (a) that the curvature of the vorticity distribution is negligible, and (b) that the difference between the initial and final vorticity distributions $|\boldsymbol{\zeta}(\mathbf{x},t)-\boldsymbol{\zeta}(\mathbf{x},0)|$ is small compared to the initial vorticity $|\boldsymbol{\zeta}(\mathbf{x},0)|$. Eq. (5) is similar to (3), but in (5) $\mathbf{r}$ is defined as the total Eulerian displacement; that is, $\mathbf{r}=\mathbf{x}-\mathbf{a}(\mathbf{x},t)$ where $\mathbf{a}(\mathbf{x},t)$ is the location at $t=0$ of the moving point which is found at $\mathbf{x}$ at time t. Both $\mathbf{r}$ and $\boldsymbol{\zeta}(\mathbf{x},0)$ consist of means and fluctuations—

$$\mathbf{r}=\overline{\mathbf{r}}+\mathbf{r}' \quad \boldsymbol{\zeta}=\overline{\boldsymbol{\zeta}}+\boldsymbol{\zeta}'. \tag{6}$$

[1] Although there may be some doubt on this point. For example, it is instructive to consider the velocity field generated from an initial field by (1). If we imagine that we have initially an isotropic field, generating $\mathbf{r}'$; that this produces, through (1) a new field, being the sum of the original and (1); that this new field produces, through (1) (assuming that velocity and displacement are proportional, which will be true for small time—this is then essentially a time expansion) still another field, and so on iteratively; then it is not difficult to show that the field so generated can *at no step* have other than *diagonal Reynolds stresses*. That is, no Reynolds stresses in the usual sense of the term will appear. This failure to produce Reynolds stresses in a reasonably well-defined initial value problem is a fairly serious inadequacy. It is also worth noting that according to (1) no *fluctuations* in velocity can be developed in an irrotational mean velocity field. This seems a curious limitation.

The division of the vorticity thus into mean plus fluctuating contributions is a clearcut procedure of a kind ubiquitously used in turbulence theory, and presents no special problems. The corresponding division of **r**, however, is not trivial, especially that component in the direction of the mean flow. In the classical momentum and vorticity transfer theories only displacements normal to the direction of mean flow are considered relevant, but it is one of the special features of Lettau's theory that displacements in the direction of the mean flow are of importance. The difficulty is that in general one cannot rely upon the formal average of the Eulerian displacement, divided by the time interval, being equal to the local mean velocity. If the velocity gradient is uniform and the turbulence therefore homogeneous, this will be the case (Lumley, 1962); otherwise, we shall assume that this difficulty is minimized in general by restricting ourselves to short time intervals and small rate of strain gradients, consistent with the assumptions leading to (5).

With this caveat, we substitute (6) in (5) and get

$$\boldsymbol{\zeta}'(\mathbf{x},t)-\boldsymbol{\zeta}'(\mathbf{x},0)=(\bar{\boldsymbol{\zeta}}\cdot\nabla)\bar{\mathbf{r}}-(\bar{\mathbf{r}}\cdot\nabla)\bar{\boldsymbol{\zeta}}+(\boldsymbol{\zeta}'\cdot\nabla)\bar{\mathbf{r}} +(\bar{\boldsymbol{\zeta}}\cdot\nabla)\mathbf{r}'-(\bar{\mathbf{r}}\cdot\nabla)\boldsymbol{\zeta}'-(\mathbf{r}'\cdot\nabla)\bar{\boldsymbol{\zeta}}, \quad (7)$$

where we have neglected terms of second degree in fluctuating quantities. All vorticity terms on the right hand side are evaluated at the initial time. The first two terms on the right describe, respectively, the stretching and the advection of the mean vorticity field by the mean velocity field. In a steady state flow, since each contains only averaged terms, they cannot contribute to the fluctuations and so must cancel in (7). Finally, then

$$\boldsymbol{\zeta}'(\mathbf{x},t)-\boldsymbol{\zeta}'(\mathbf{x},0)=(\boldsymbol{\zeta}'\cdot\nabla)\bar{\mathbf{r}}+(\bar{\boldsymbol{\zeta}}\cdot\nabla)\mathbf{r}'-(\bar{\mathbf{r}}\cdot\nabla)\boldsymbol{\zeta}'-(\mathbf{r}'\cdot\nabla)\bar{\boldsymbol{\zeta}}. \quad (8)$$

The second and fourth terms are those suggested by Lettau; they describe generation of new vorticity fluctuations in mixing of mean vorticity by velocity fluctuation. The first and third describe the generation of new vorticity fluctuation by stretching and advection (by the mean strain field) of fluctuations already present.

3. Integration of the equation

Because of the similarity of form of (4) and (5) or (4) and each part of (8), it is appealing to try to use (4) to integrate one of them. If both $\mathbf{r}'$ and $\bar{\mathbf{r}}$ were divergence free, (4) would be valid, and (8) could be integrated, following Lettau, by writing

$$\boldsymbol{\zeta}'(\mathbf{x},t)-\boldsymbol{\zeta}'(\mathbf{x},0)=\nabla\times[\bar{\mathbf{r}}\times\boldsymbol{\zeta}'(\mathbf{x},0)+\mathbf{r}'\times\bar{\boldsymbol{\zeta}}(\mathbf{x},0)] \quad (9)$$
$$=\nabla\times[u'(\mathbf{x},t)-u'(\mathbf{x},0)]$$

or

$$\mathbf{u}'(\mathbf{x},t)-\mathbf{u}'(\mathbf{x},0)=\mathbf{r}'\times\bar{\boldsymbol{\zeta}}(\mathbf{x},0)+\bar{\mathbf{r}}\times\boldsymbol{\zeta}'(\mathbf{x},0)+\nabla\phi, \quad (10)$$

where $\nabla\phi$ has been added as a function of integration, of unknown form, which must satisfy only the requirement that it be irrotational, which is satisfied for arbitrary ϕ. Lettau sets $\phi=0$. In (10) the first term on the right is Lettau's, and the second once more represents the effect of the advection and stretching of existing fluctuations by the mean velocity field. This procedure can give us no information about the nature of ϕ. Let us examine the validity of (4). Since we now have a definition for **r**, we can test the validity of the assumption $\nabla\cdot\mathbf{r}=0$. Beginning from the definition

$$\mathbf{a}(\mathbf{x},t;t_0)=\mathbf{x}+\int_t^{t_0}\mathbf{u}(\mathbf{a}(\mathbf{x},t;t'),t')dt', \quad (11)$$

where $\mathbf{a}(\mathbf{x},t;t_0)$ is the position at t_0 of the point which will be at **x** at t), then forming the successive derivatives with respect to t_0 evaluated at $t_0=t$ and finally setting $t_0=0$ it is not difficult to show that

$$\nabla\cdot\mathbf{r}=-\sum_{i,P=1}^{3}\frac{\partial u_i}{\partial x_p}\frac{\partial u_p}{\partial x_i}\frac{t^2}{2}+O(t^3). \quad (12)$$

That is: $\nabla\cdot\mathbf{r}$ vanishes only to first order in time. Hence, the integration (10) is valid only to first order in time.

Since that is the case, it is easier to use the equation of motion directly to give the first derivative in a Taylor series expression of the velocity field. Using the identity

$$(\mathbf{u}\cdot\nabla)\mathbf{u}=\frac{\nabla q^2}{2}-\mathbf{u}\times\zeta, \quad \mathbf{u}\cdot\mathbf{u}=q^2, \quad (13)$$

we obtain exactly (neglecting viscous terms)

$$\mathbf{u}'(\mathbf{x},t)-\mathbf{u}'(\mathbf{x},0)=t\left[\mathbf{u}\times\zeta-\nabla\left(\frac{P}{p}+\frac{q^2}{2}\right)\right]+O(t^2). \quad (14)$$

The second term contains the irrotational velocity field that we needed. (Its mean value must exactly cancel the contribution from the mean velocity and vorticity field.) The fluctuating value of $\nabla(P/p+q^2/z)$ certainly cannot be neglected, although it may be satisfactory to neglect the fluctuating static pressure; the latter approximation would give $-t(\mathbf{u}\cdot\nabla)\mathbf{u}$, which is not only mathematically, but physically quite different from Lettau's suggestion.

If the velocity **u** is also broken into mean and fluctuating parts

$$\mathbf{u}=\bar{\mathbf{u}}+\mathbf{u}'$$

then it can be seen that the fluctuating part of $\mathbf{u}^2$ is

$$2\mathbf{u}'\bar{\mathbf{u}}+\mathbf{u}'^2.$$

Normally $\mathbf{u}'^2$ will be of the order of the static head fluctuation and may be much smaller than $2\mathbf{u}'\bar{\mathbf{u}}$. In that case we see that the $\nabla\phi$ in (10) term is also essentially an advection term.

4. Conclusions

A form can be obtained for the vorticity fluctuations by the assumption of conservation of angular momentum. Lettau's suggested expression is similar to this, but differs from it by neglecting one of the two physical mechanisms present. The vorticity expression can only be integrated to first order in time; the integration of Lettau differs from the exact form even to this order by the neglect of the fluctuations in total head. Inclusion of the neglected term permits the conclusion that Lettau's form is physically misleading as to the mechanism involved, in addition to being not an approximation in any sense.

REFERENCES

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