Heat Exchange Ratios of Hailstones in a Model Cloud and their Simulation in a Laboratory

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ABSTRACT

Calculations are presented of the relative contributions of heat exchange by conduction and convection, by evaporation, and by the supercooling of accreted drops to the total heat exchange of growing spherical hailstones. They reveal the regions of dominance of the different ratios for various icing conditions in a model cloud and in laboratory experiments. As an additional result, it can be shown that transfer ratios occurring in hail clouds can only partly be simulated in experiments at constant pressure, but a restricted imitation at constant pressure is possible. These ratios are also considered to represent new parameters for future experiments about the relationship between icing conditions and resulting hailstone shells.

1. Introduction

Theoretical calculations by Schuman (1938) and Ludlam (1958) assumed, and experiments by List (1960) confirmed in principle, that only three components have to be considered for the description of the total heat exchange of growing hailstones. These are a) conduction and convection, b) evaporation or sublimation/condensation, and c) the degree of supercooling of the accreted water droplets relative to the temperature of the collecting surface. Heat transfers by radiation, friction, etc., are in the order of a few per cent or less and can be neglected.

The present investigation on spherical hailstones is concerned with the relative contribution of the individual components to the total heat exchange. In particular, the dominant transfer ratios are determined as a function of height in a model cloud, of the icing conditions, and of the particle diameter. A later part of this paper will be concerned with the possibility of simulation of heat exchange ratios in laboratory experiments without pressure control.

The use of constant transfer ratios as parameters of experiments about the growth of hailstones will also be discussed.

2. Basic equations

Let us consider a spherical hailstone with a diameter D and a temperature t_D, falling at a terminal speed v in a cloud of a temperature t_A, a pressure p, and a free water content w_f. The heat exchange is then represented by the following equations (in cgs-units), taken from List (1963):

Heat exchange by conduction and convection.

$$Q_{CC}=1.68k\theta\nu^{-\frac{1}{2}}v^{\frac{1}{2}}D^{\frac{3}{2}}(t_D-t_A) \tag{1}$$

with k, the thermal conductivity of air, θ the ratio of heat transfer through a natural (rough) surface to heat transfer through a smooth surface, and ν the kinematic viscosity.

Heat exchange by evaporation or sublimation.

$$Q_{ES}=C_{1,2}D_{wa}T_A^{-1}\theta\nu^{-\frac{1}{2}}v^{\frac{1}{2}}D^{\frac{3}{2}}(e_{sh}-e_{sv}), \tag{2}$$

where $C_1=0.207$ cal deg cm^{-3} mb^{-1} for phase transitions liquid-gas, $C_2=0.235$ cal deg cm^{-3} mb^{-1} for solid-gas, D_{wa}, the diffusivity of water vapor in air, T_A, the absolute air temperature, e_{sh}, the saturation vapor pressure over hailstone, and e_{sv}, the saturation vapor pressure over water.

Since evaporation is governed by similar processes as heat transfer, by convection and conduction, the surface roughness is assumed to have the same influence on Q_{ES} as upon Q_{CC}. Hence we use the same factor θ. As sublimation does not play any role in the following considerations, Q_{ES} will be replaced by Q_E.

Heat exchange due to accretion of supercooled water.

$$Q_{CP}=0.785D^2vEw_f\overline{c_w}(t_D-t_A) \tag{3}$$

with c_w, the specific heat of water averaged over the temperature range (t_D,t_A), and E the total efficiency of catch.

The total efficiency of catch E is meant to be the ratio of the difference between the mass of the accreted drops and the H_2O lost to the surroundings in a non gaseous form to the mass of water drops belonging to the volume of air swept out by the falling hailstone. Losses of water can occur, for example, at icing conditions leading to spongy ice with a very high water content. As this deflecting water may not have the temperature of the accreted water, it may also contribute to the heat exchange Q_{CP}. This effect, however, is not considered here.

The sum of the components, Q_{CC}, Q_E, and Q_{CP}, represents the total heat Q_{tot} transferred from the hailstone per time unit. Under quasi-equilibrium conditions, this value equals the latent heat Q_F released by the partial or complete freezing of the accreted drops, given by

$$Q_F = 0.785 D^2 v E w_f L_f I \tag{4}$$

with L_f, the latent heat of fusion at temperature t_D, and I, the ratio between the amount of the freezing water and the total accreted water.

Under *equilibrium conditions* we get:

$$Q_{CC} + Q_E + Q_{CP} = Q_{tot} = Q_F. \tag{5}$$

Dividing by Q_F leads to:

$$\frac{Q_{CC}}{Q_F} + \frac{Q_E}{Q_F} + \frac{Q_{CP}}{Q_F} = 1 \tag{6}$$

or

$$R_{CC/F} + R_{E/F} + R_{CP/F} = 1, \tag{7}$$

where $R_{CC/F}$ stands for Q_{CC}/Q_F, etc. These three ratios represent the different contributions to the total heat exchange. Their values are:

$$R_{CC/F} = 2.14 k\theta (t_D - t_A)(E w_f L_f I)^{-1} (\nu v D)^{-\frac{1}{2}} \tag{8}$$

$$R_{E/F} = 1.274 C_{1,2} D_{wa} \theta (e_{sh} - e_{sv}) \times (T_A E w_f L_f I)^{-1} (\nu v D)^{-\frac{1}{2}} \tag{9}$$

$$R_{CP/F} = \overline{c_w} (t_D - t_A)(L_f I)^{-1}. \tag{10}$$

Only two of these ratios can be assumed to be independent, the third always being determined by Eq. (7). A pair of numerical values for two of these ratios always corresponds to a particular set of icing conditions.

These equations are set up under the assumption of a thermal equilibrium between the growing ice deposit and the intermediary hailstone or, in other words, the hailstone always has the temperature of the growing deposit which in itself is in quasi-equilibrium with its surroundings; 'quasi' takes accounts of the slowly changing size.

The surface temperature considered in these equations is principally set equal to the temperature of the deposit. This is only an approximation since, in a very thin outer skin, temperature gradients may occur. We know also, as is explained by List (1965), that the outer surface of a growing particle will always be partially wet, even if the deposit has a temperature below 0C. Therefore, the assumption that the saturation vapor pressure for solid deposits has to be taken over ice is not completely adequate. Hence the values of the constant $C_{1,2}$, introduced in Equation 2, also have to be reconsidered. Passing through the Schuman-Ludlam limit, the boundary which separates icing conditions leading to spongy ice from those producing ice deposits with temperatures below 0C, a discontinuity occurs in the value of Q_E. Considering the partially wet surface over ice deposits below 0C, we may therefore introduce a $C_{1,2}$ which takes a value of 0.207 cal deg cm^{-3} mb^{-1} at 0C and increases linearly to 0.235 cal deg cm^{-3} mb^{-1} at -20C, the temperature at which a completely dry surface is assumed. While this assumption is arbitrary, we cannot be far wrong; such a procedure permits us to get rid of the unrealistic discontinuity for Q_E at $t_D = 0$C. One might argue now, that the water vapor pressure should also be adjusted to such partially wet surfaces. We did not, because there is no unrealistic discontinuity. Such an additional adjustment can well be delayed up to the moment at which we have a better physical insight into the the conditions in a water skin. This is justified, since it can be calculated that at temperature differences $t_D - t_A$ of 5C and more, the maximum changes of the ratio $R_{E/F}$ are less than 10 per cent if we assume vapor pressure conditions adjusting linearly from saturation over water at 0C to saturation over ice at -20C.

The validity of the given formulas covers a range of Reynolds number of $10^3 < \mathrm{Re} < 10^5$. Their accuracy is better than ± 15 per cent.

3. Cloud model

Since we wish to obtain a picture of the part the different components of heat exchange play in a cloud, we have to assume a certain cloud model. Similar to Das (1962) we base it upon the radiosonde observations taken in Denver for 18 hail days selected at random during 1955–1958 (Beckwith, 1960). The values for

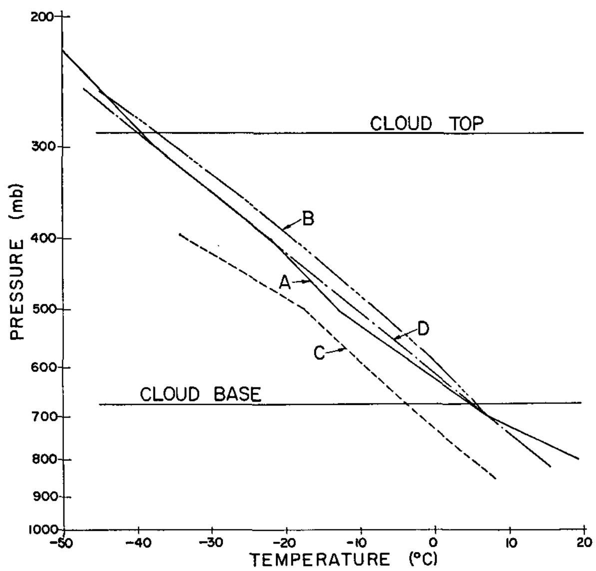

FIG. 1. Mean sounding for 18 hail days: *A* temperature distribution, *B* parcel adiabat, *C* dew point, *D* temperature distribution for cloud model (*A*, *B*, and *C* from Beckwith, 1960).

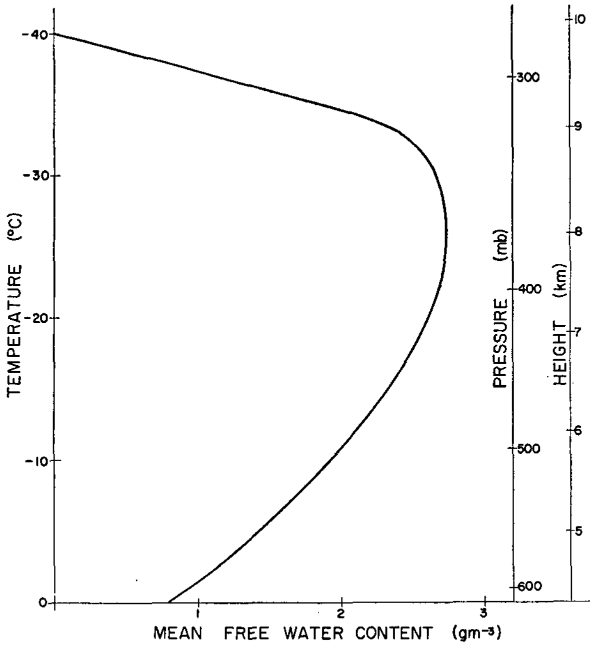

FIG. 2. Distribution of liquid water content of cloud model.

temperature, parcel adiabat, and dew point as function of pressure are displayed in Fig. 1. The freezing level is at 590 mb, the wet-bulb freezing level at 637 mb, the temperature at the cloud base 5C, whereas the height of the tropopause is at 134 mb.

In order to have a mathematical relationship between pressure and temperature, a straight line is introduced in Fig. 1 which represents the sounding as well as possible. The corresponding integrated equation reads as follows:

$$T = 53 \ln(0.284p) \tag{11}$$

with T (°K) and p (mb).

Assuming that the atmosphere obeys the static or hydrostatic equation of meteorology and the perfect gas law, an equation can be deduced which relates absolute temperature T (°K) with height z (km):

$$z = 2.76\times10^{-4}(9.0066\times10^{4} - T^{2}). \tag{12}$$

The pressure, p, as a function of the height can be found by combining Eqs. (11) and (12).

The liquid water content was calculated up to a cloud level of −32C using the adiabatic parcel method. In agreement with Das (1962), the free water content is assumed to decrease linearly from the value calculated for −32C to zero at a level of −40C. The distribution of the free water content is shown in Fig. 2. Its maximum is in the order of 3×10^{-6} gm cm^{-3}. This value might be regarded as quite low, but it seems to be representative for storms over the midwestern plains of the United States.

4. Heat transfer ratios in clouds

The three heat transfer ratios, $R_{CC/F}$, $R_{E/F}$ and $R_{CP/F}$ can be displayed for a cloud model by considering simultaneously the equilibrium condition given by Eq. (7). This latter formula inter-relates pressure, p,

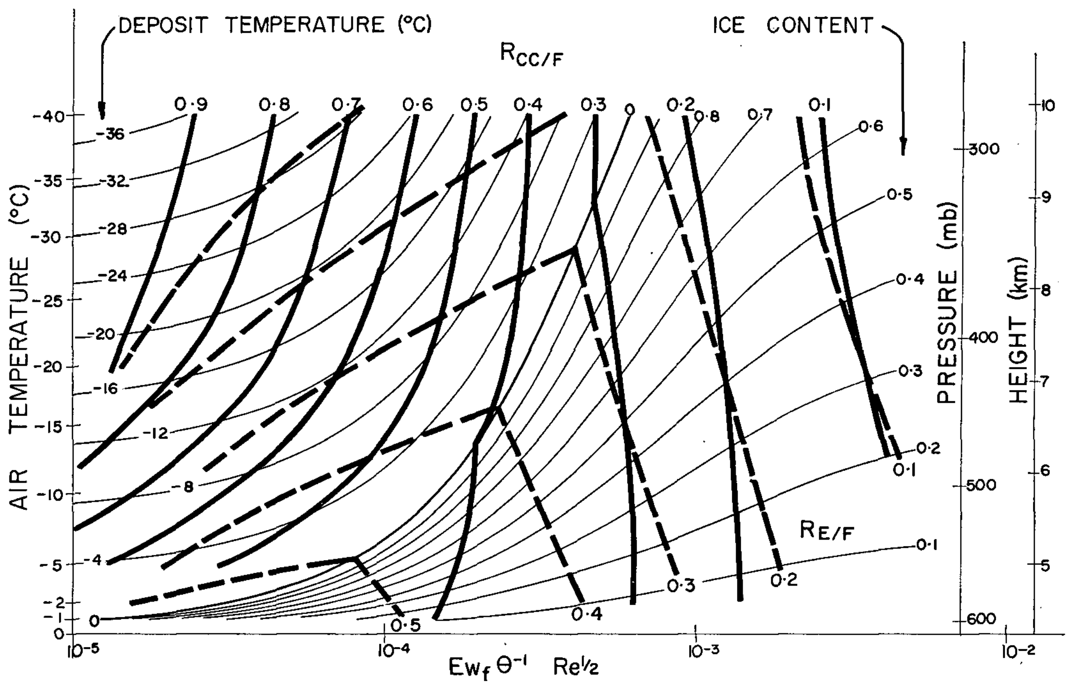

FIG. 3. Distribution of the ratios $R_{CC/F}$ and $R_{E/F}$ as functions of the product of free water content and Reynolds number, with fine background lines representing temperature of ice deposits or ice content in spongy deposits as a function of the same product and height in the model cloud.

air temperature, t_A, free water content, w_f, deposit temperature, t_D, or ice content of the spongy deposit, I, (if $t_D=0C$) and Reynolds number, Re, of the spherical hailstone under consideration, and this relation is shown by fine background lines in Fig. 3. It is divided into two parts by a deposit temperature of 0C and an ice content of $I=1.0$. Spongy ice is growing to the right of this line, whereas ice deposits are formed by conditions on the left side of this boundary. Superposed are the ratios $R_{CC/F}$ and $R_{E/F}$ where the values of $R_{CP/F}$ can be obtained for each point by subtracting the sum of the two others from one. In Fig. 3 the free water content is not a function of height; it can be chosen arbitrarily. Four observations are obvious: (a) the contribution of the heat exchange by conduction and convection steadily decreases with increasing free water content or increasing Reynolds number; (b) the influence of evaporation is highest at temperatures near the zero degree level for conditions leading to non-spongy deposits at 0C (Schuman-Ludlam limit); (c) the contribution of the supercooling of the accreted drops increases with increasing values of free water content and Reynolds number; (d) all curves of equal ratios change their slope by passing through the Schuman-Ludlam limit.

To get a picture of the variation of the Reynolds number within our cloud model, we refer to Fig. 4. It shows that Re remains practically unchanged by moving from the 0C level to the −40C level. In other words, the Reynolds number of a free falling spherical hailstone of a given size is roughly independent of the cloud level of our model from which it falls. Re could principally be replaced by a power function of the diameter D.

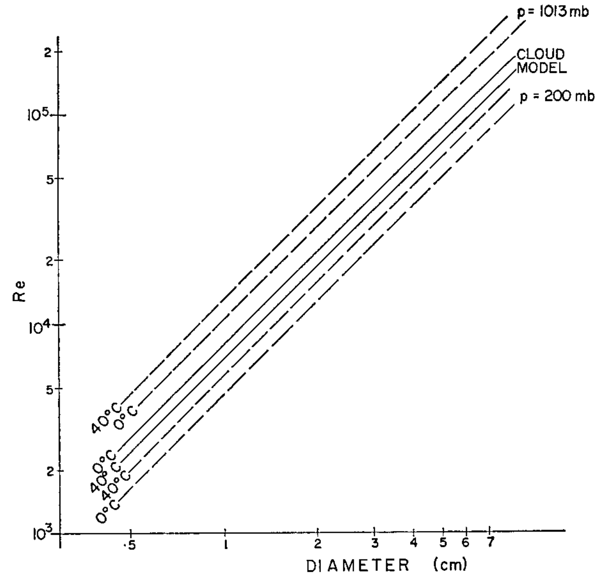

FIG. 4. Reynolds number of free falling spherical hailstones (density 0.917 gm cm^{-3}, drag coefficient 0.5) as a function of diameter; for cloud model or constant pressure range. Lines labeled 40°C should be labeled −40C.

Fig. 4 is based on a constant drag coefficient ($c_w=0.5$) and a constant hailstone density ($\rho_{hs}=0.915$ gm cm^{-3}), which leads to a relationship:

$$\mathrm{Re}=4.74\times10^5 p^{\frac{1}{2}} T^{-\frac{1}{3}} D^{\frac{3}{2}} \tag{13}$$

with p (mb), T (°C) and D (cm).

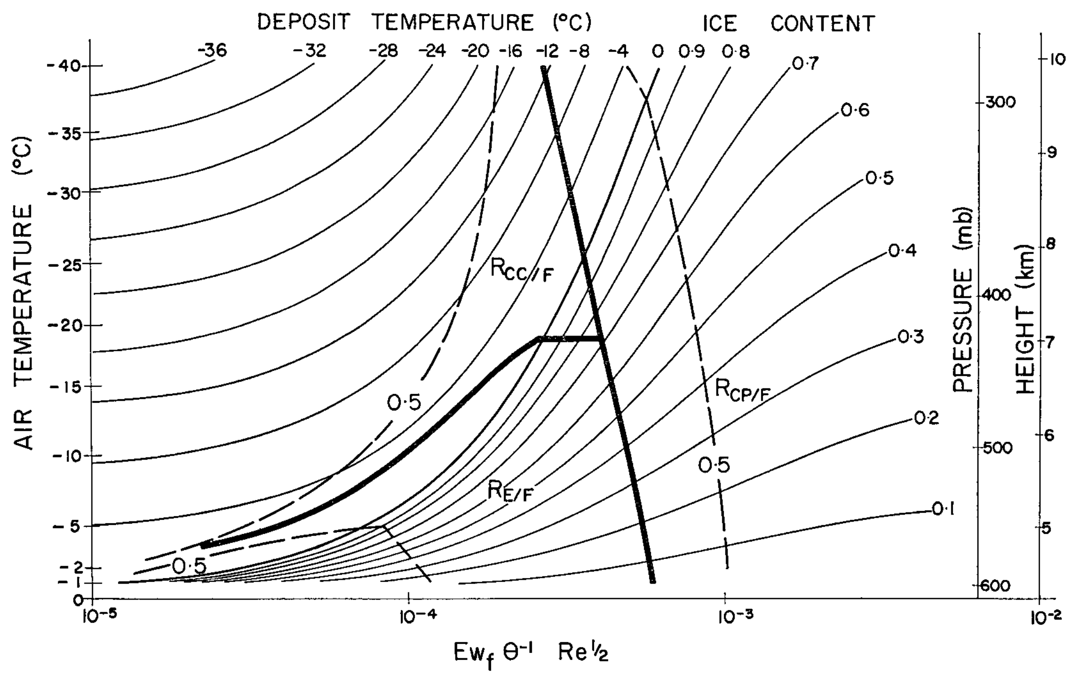

FIG. 5. Distribution of relatively and absolutely dominating heat exchanges in a model cloud, as function of a product of free water content and Reynolds number.

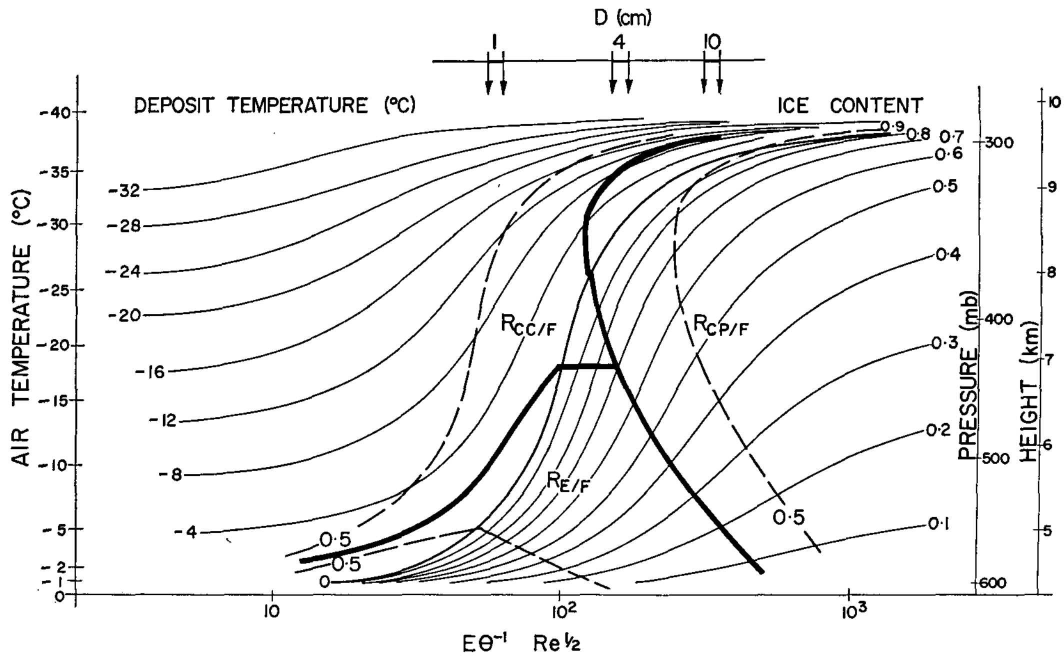

FIG. 6. Distribution of relatively and absolutely dominating heat exchanges in a cloud as function of Reynolds number; the distribution of the free water content within the model cloud is assumed according to Fig. 2.

On the basis of the ratios given in Fig. 3, we can divide the icing conditions of hailstones, characterized by their Reynolds numbers, into three regions (Fig. 5). In each of those, one of the individual heat transfers is absolutely and/or relatively dominant. At the triple point (air temperature $T_A \sim -18C$, $(Ew_f\theta^{-1} \times Re^{\frac{1}{2}} \sim 4.2 \times 10^{-4})$ all the three contributions are equal. Absolutely dominant means the region where a given component is bigger than 0.5, i.e., bigger than the sum of the two other components.

Superposing the profile of the free water content, according to Fig. 2, to the pressure, we end up with a division of the icing conditions as shown in Fig. 6. First we notice a shift of the background representing

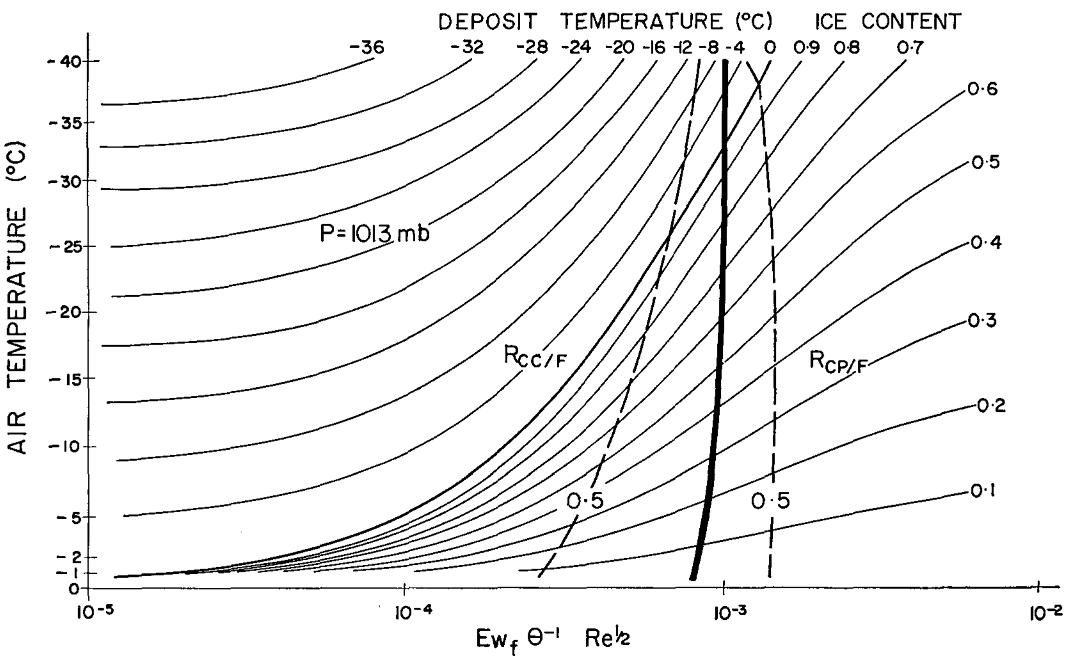

FIG. 7. Distribution of relatively and absolutely dominating heat exchanges for laboratory conditions without pressure control, as function of a product of free water content and Reynolds number.

the equilibrium conditions. Because of the drop of the free water content to zero for a cloud temperature of −40C, the curves for equal deposit temperature and ice content reach this level asymptotically for Re tending to infinite values.

Within the accuracy of the drawing, the triple point occurs at the same temperature (−18C) as in Fig. 5. Icing conditions leading to ice deposits with temperatures below −2C are governed through heat exchange by conduction and convection. Heat transfer due to evaporation is important for the formation of spongy ice if the cloud temperature is above −18C and if the factor $E\theta^{-1}\,\mathrm{Re}^{\frac{1}{2}}$ is less than about 2×10^2. Above this latter value the supercooling of the accreted drops is relatively dominant; at cloud temperatures below −18C, the same ratio $R_{CP/F}$ is absolutely dominant except for the temperature range close to −40C, where the liquid water content drops to zero.

In earlier experiments in a wind tunnel at a pressure of $p=733$ mb., List (1960) found a value of $E\theta^{-1}=0.675$. Assuming that the reduction of pressure has no great effect on this factor, we find for similar icing conditions at the triple point a value of the Reynolds number of $\mathrm{Re}\sim6\times10^4$. This corresponds to a hailstone diameter of $D\sim4$ cm. If the free water content at this level was twice as high, which means 5×10^{-6} gm cm^{-3} instead of 2.5×10^{-6} gm cm^{-3}, the diameter corresponding to this singular point would be $D\sim1.6$ cm. Hereby the importance of the factor $R_{CP/F}$ would increase.

5. Heat exchange ratios at sea level

Having a picture of the heat exchange ratios and their distribution as a function of the icing conditions within a cloud, we want to compare these data with conditions which would occur during icing experiments in a laboratory at atmospheric pressure. Calculation shows that the triple point does not exist anymore: there are no conditions where the contribution of the heat transfer by evaporation equals the other individual exchanges. The results are displayed in Fig. 7 as functions of the air temperature and the product $E\theta^{-1}w_f\,\mathrm{Re}^{\frac{1}{2}}$. The dominance of the contribution by conduction and convection to the exchange occurs at values of $E\theta^{-1}w_f\,\mathrm{Re}^{\frac{1}{2}}<10^{-3}$, whereas supercooling is the main factor beyond this boundary, which is practically independent of air temperature. Assuming a free water content of 5×10^{-6} gm cm^{-3} and a $E\theta^{-1}=0.675$, the boundary would lie at $\mathrm{Re}\sim8.8\times10^4$.

In experiments where it is reasonable to investigate only the icing of suspended particles, the relative velocity and the particle diameter D are not coupled anymore by the condition of free fall. These values can be varied independently as is true also for the free water content. But even with this additional degree of freedom it is not possible to produce icing conditions where evaporation is dominant.

We may also notice that at laboratory pressure, heat exchange by conduction and convection is not only dominant for the growth of ice deposits with tempera-

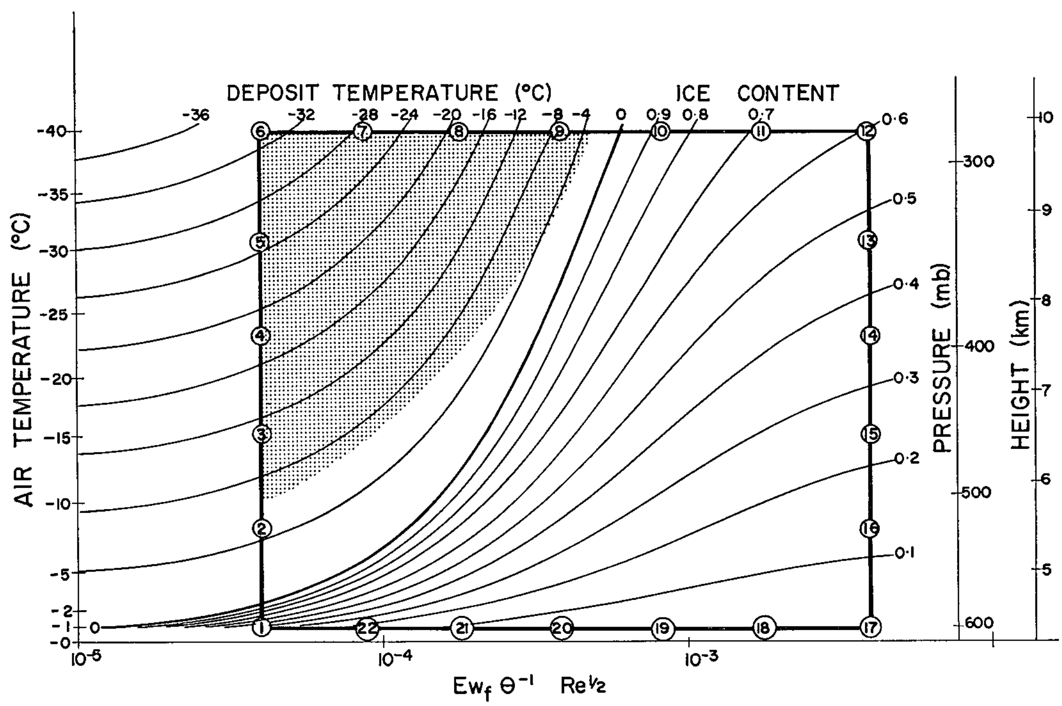

FIG. 8. Equilibrium deposit temperatures or ice content for spongy deposits, as functions of icing conditions in a model cloud and Reynolds number of particles under consideration. Rectangle representing boundary of an area which is considered for simulation at laboratory conditions; dotted area is region whose individual heat exchange ratios can be imitated at a pressure level $p=1013$ mb.

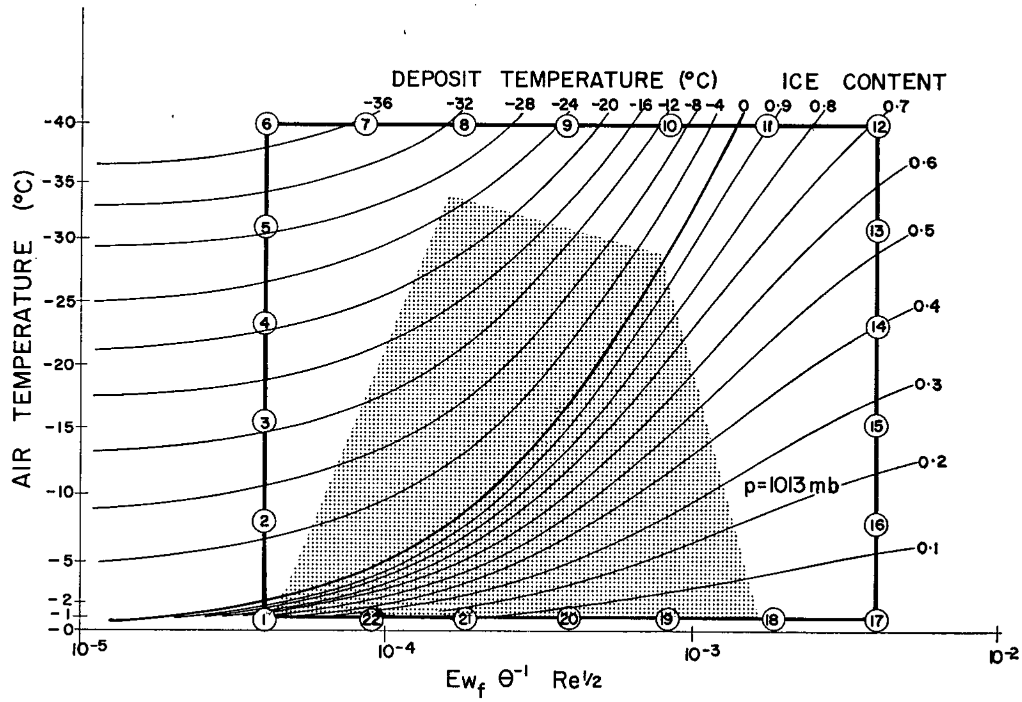

FIG. 9. Equilibrium icing conditions for the laboratory case; rectangular area considered for simulation but only dotted area corresponds to ratios which also occur in a cloud.

tures below 0C, but also governs a considerable part of the icing conditions leading to spongy deposits. This is particularly so at higher temperatures.

6. Complete similarity of exchange ratios

We can ask now about the combinations of ratios occurring in a cloud which can be truly simulated in a laboratory without pressure variation. For this purpose we consider two rectangular areas in our representations of the growth and exchange conditions within the model cloud, Fig. 8, and for the laboratory case, Fig. 9. The combinations of transfer ratios which are common to both areas under consideration can best be found if we attach a pair of ratios $R_{CC/F}$ and $R_{E/F}$ to a number of points on the border of the rectangles,

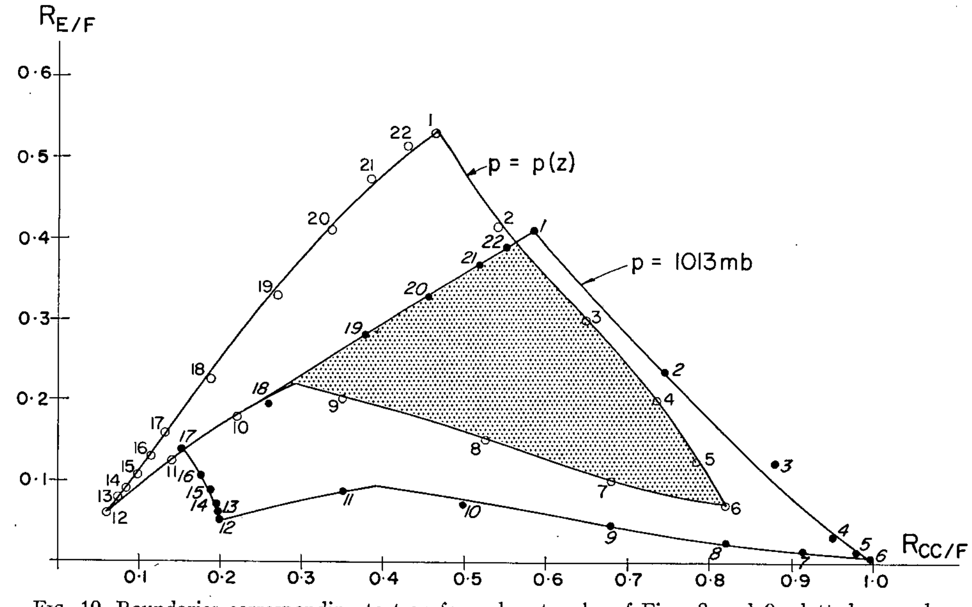

FIG. 10. Boundaries corresponding to transformed rectangles of Figs. 8 and 9; dotted area shows pairs of heat exchange ratios which occur in the model cloud (not including the free water content distribution) and in the laboratory case.

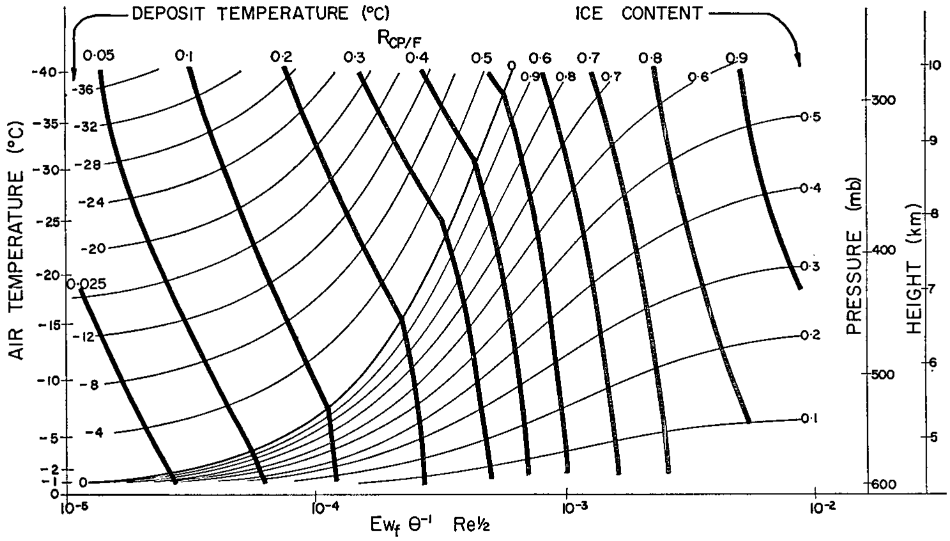

Fig. 11. Contribution of heat exchange by accreted drops, $R_{CP/F}$, as function of equilibrium icing conditions and Reynolds number for model cloud.

and transform the rectangles into a plane having these ratios as coordinates. Fig. 10 shows which grouping of ratios occur for both of these two principal cases. The common region can then be transformed back to the model cloud (Fig. 8) and the laboratory case (Fig. 9) and shows which growth conditions (within the rectangles) can be truly simulated in terms of exchange ratios. These areas are dotted. We learn from Fig. 8 that only such icing conditions of our model cloud as lead to non-spongy deposits with a temperature of ~−4C and less can be imitated.

Comparison of the transformable areas in Figs. 8 and 9 shows that the heat exchange of non-spongy deposits can sometimes be imitated by the transfer from growing spongy deposits. The temperature of deposits does not represent a quantity which remains unchanged by transformations. Carrying out experiments at other pressures would shift the transformable areas.

7. Ratio for reduced similarity

In Section 6 it was shown for spherical hailstones that only a very restricted region of growth conditions can be simulated, with respect to heat exchange ratios, by laboratory experiments without varying the pressure. However, a restricted similarity can be defined if we regard the heat exchange by conduction and convection as somewhat equivalent to that by evaporation. This is justified to a certain extent as they are both in the same way dependent on variations of Reynolds number and free water content. If one assumes that the hailstone itself cannot distinguish whether heat is

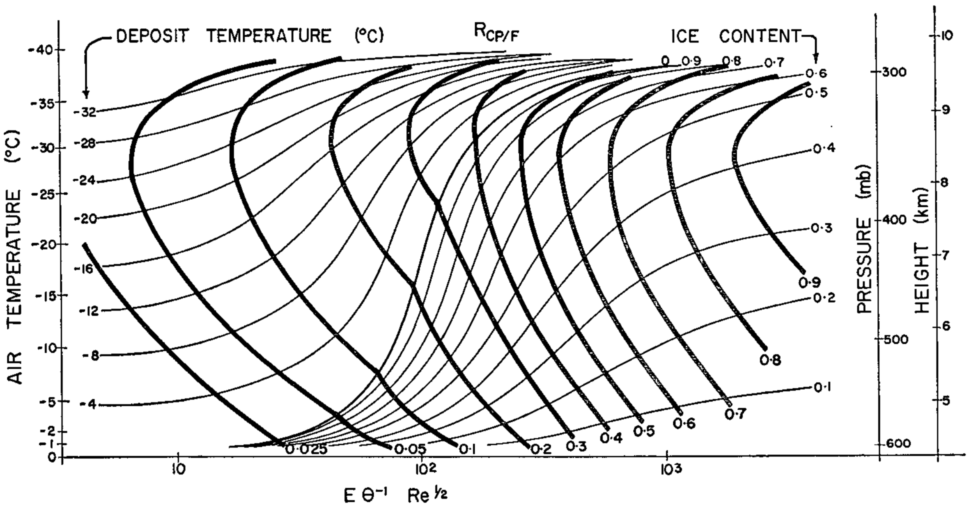

Fig. 12. Heat exchange ratio based on accretion of drops, $R_{CP/F}$, as function of equilibrium icing conditions and Reynolds number for model cloud with a free water content as shown in Fig. 2.

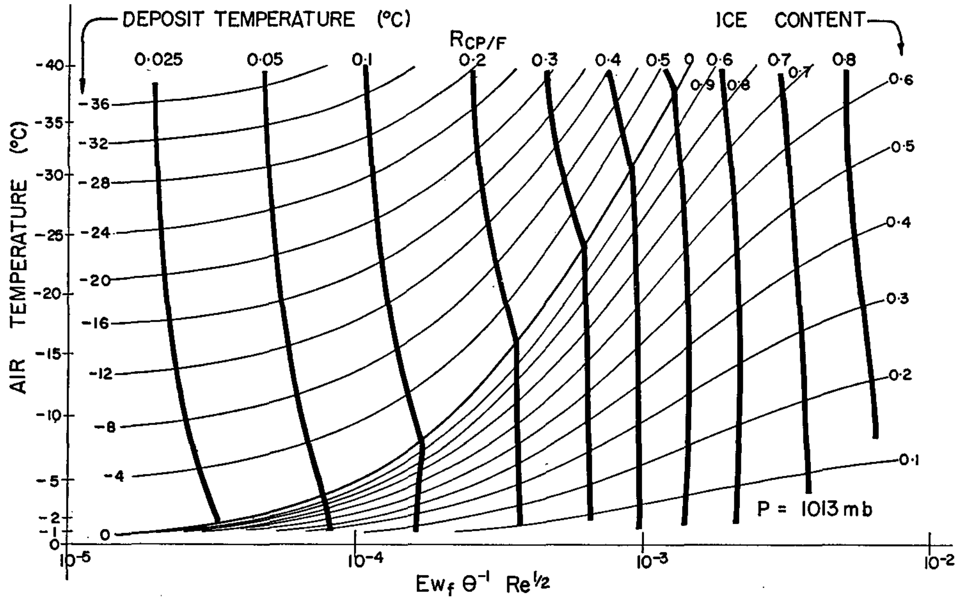

FIG. 13. Heat exchange ratio $R_{CP/F}$ as function of equilibrium icing conditions and Reynolds number for the case of constant pressure, $p=1013$ mb.

transferred by convection, conduction or evaporation so that only the sum of both terms is decisive, the number of independent ratios in Eq. (7) is reduced from two to one.

Such a restricted similarity of ratios is, therefore, just given, for example, by the relative contribution of the accreted supercooled water droplets, $R_{CP}/_F$, which is shown in Figs. 11 to 13. Fig. 11 is valid for the cloud model without assuming a distribution of free water content, whereas the latter is accounted for in Fig. 12. Fig. 13 shows the ratios for laboratory experiments at constant pressure.

Comparing these curves, we immediately see that the range of ratios in the cloud is practically the same as it is in the case of constant pressure. This means that a restricted simulation of heat exchange ratios of a hailstone, growing in a cloud, can easily be achieved by laboratory experiments.

8. Conclusion and final remarks

It has been shown that the introduction of heat exchange ratios is very helpful for the understanding of the physics of hailstone formation. In addition, these factors represent a set of new variables with which the structure of ice deposits might be correlated more satisfactorily than with icing parameters like pressure, temperatures, free water content, particle size, etc.

The calculations throw also some light upon future needs for experiments. Up to now only heat and mass exchanges due to convection/conduction and sublimation/evaporation respectively seemed important; but in the light of icing conditions within a model cloud it could be shown that the heat exchange due to temperature differences between accreted droplets and the collecting hailstone is dominant in the important region where spongy ice is formed upon relatively big particles. This means that we really have to pay greater attention to the collision and collection mechanism and eventual differences of efficiences between mass and heat transfer.

The calculations reported here were carried out on an IBM 7094.

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