Convection Flows Due to Local Heating of a Horizontal Surface

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ABSTRACT

Local boundary layer approximations of first order are computed for the steady laminar convection flow of a semi-infinite medium, which is produced by nonuniform axisymmetric heating or cooling of an infinite horizontal surface. Velocity, pressure, and temperature profiles as well as Nusselt numbers are presented for the Prandtl numbers 0.7 and 7, and for various Grashof and Eckert numbers. Critical Grashof numbers for a heated surface exist beyond which the flow field becomes unstable. A cooled surface always yields stable motions.

1. Introduction

For some years attention has been drawn to natural and artificial convection flows over a locally heated surface. Examples are the papers by Malkus, Stern and Bunker (1952, 1953a and b, 1963) who studied the updrafts over small islands exposed to solar radiation. Black and Tarmy (1963) examined the artificial increase of rainfall by asphalt ground coatings, and Landsberg (1947) investigated fire storms over burning areas. Theoretical studies involved in these problems are based either on the assumption of ideal fluid motions [for instance, Smith (1955)] or on a linearized flow theory [Stommel and Veronis (1957), and Trubnikov (1961)]. For turbulent flow an approach was achieved by Priestley and Ball (1955) with a certain similarity hypothesis.

It is the subject of this paper to investigate the steady laminar convection of a semi-infinite medium which is caused by nonuniform axisymmetric heating or cooling of an infinite horizontal surface. This study is based on a local series expansion around the axis of symmetry together with an adjustable boundary layer assumption introduced by Schwiderski and Lugt (1964). Comparisons of such first-order approximations with experimental data (Lugt and Schwiderski, 1965) yielded satisfactory results.

For the present problem numerical calculations are performed with various Prandtl, Grashof (Rayleigh) and Eckert numbers. Critical Grashof numbers for a heated surface are obtained beyond which the flow field becomes unstable. A cooled surface causes stable motions for all Grashof numbers.

2. Flow model and basic equations

A surface of infinite dimension in the $z=0$ plane of a cylindrical coordinate system (r, φ, z) is locally heated or cooled in an axisymmetric manner around $r=0$. The resulting buoyant convection is assumed to vanish far away from the origin so that the fluid motion yields streamlines as sketched in Fig. 1. Since the Coriolis force is neglected, no tangential velocity component occurs. Furthermore, the motion shall be steady and laminar, and the fluid incompressible in the sense of the Boussinesq simplification. With these assumptions the Navier-Stokes equations and the energy equation appropriate to this problem are in dimensionless form

$$uu_r+wu_z=-p_r+u_{rr}+(u/r)_r+u_{zz}, \tag{1}$$

$$uw_r+ww_z=-p_z+w_{rr}+(1/r)w_r+w_{zz}+\mathrm{G}t, \tag{2}$$

$$u_r+u/r+w_z=0, \tag{3}$$

$$\mathrm{P_r}(ut_r+wt_z)=t_{rr}+(1/r)t_r+t_{zz} +\mathrm{P_r E}[2u_r^2+2(u/r)^2+2w_z^2+(u_z+w_r)^2], \tag{4}$$

where u, w, t, and p denote the dimensionless radial and axial velocity components, the temperature difference between flow field and infinity, and the pressure, respectively. The corresponding reference length, velocity, pressure, and temperature are L, ν/L, $\nu^2\rho/L^2$, and τ_0. Here, L is a characteristic length, which will be defined later, ν, ρ and τ_0 are the kinematic viscosity, the density, and the temperature difference between origin and infinity. The dimensionless parameters are identified as

$$\text{Grashof number:}\quad \mathrm{G}=\frac{1}{\nu^2}g\beta\tau_0L^3, \tag{5}$$

$$\text{Prandtl number:}\quad \mathrm{P_r}=\frac{\nu\rho c}{k}, \tag{6}$$

$$\text{Eckert number:}\quad \mathrm{E}=\frac{\nu^2}{c\tau_0L^2}, \tag{7}$$

(Rayleigh number: $\mathrm{R}=\mathrm{GP_r}$).

The quantities g, β, c, and k designate the acceleration due to gravity, the thermal expansion coefficient, the specific heat, and the (thermal) conductivity, respectively.

A solution of the elliptic partial differential equations (1) through (4) may be specified by the boundary data

$$z=0,\quad r\leq\infty:\qquad u=0,\quad w=0,\quad t=\tau(r)/\tau_0, \tag{8}$$

$$\left.\begin{array}{ll} z=\infty, & r<\infty \\ z>0, & r=\infty \end{array}\right\}:\quad u=0,\quad w=0,\quad t=0. \tag{9}$$

The function $\tau(r)$ is the surface temperature at one's disposal with the restriction that it tends monotonically and sufficiently fast to zero with increasing distance from the origin in order to prevent any secondary circulation or other disturbances and to ensure a finite energy supply. The first two conditions of Eq. (8) express the nonslip assumption at the rigid surface. The conditions (9) state the fact that a local heat source of finite strength at the origin leaves the motion at rest at infinity.

For the following solution of the boundary value problem formulated above the function $\tau(r)$ is assumed analytic and may be expressed in the series expansion

$$\tau(r)=\tau_0-\tau_2(Lr)^2\pm\cdots, \tag{10}$$

which is convergent around $r=0$ at least for some finite r. The dimensionless form of Eq. (10) is

$$(t)_{z=0}=t_s=1-\left(\frac{\tau_2}{\tau_0}L^2\right)r^2\pm\cdots, \tag{11}$$

t_0 always being positive, no matter whether the surface is heated (τ_0, τ_2 positive) or cooled (τ_0, τ_2 negative). Eq. (11) suggests a definition of the characteristic length L as $L^2=\tau_0/\tau_2$, which simplifies the expansion (11) to

$$t_s=1-r^2\pm\cdots. \tag{12}$$

The characteristic length L is now directly connected with the properties of the surface temperature. Indeed, L represents the geometric mean of τ_0 and twice the radius of curvature $1/\tau_2$ of the surface temperature at $r=0$. Therefore, the characteristic length increases with increasing intensity of the heat source.

According to the ideas of adjustable local boundary layer approximations (Schwiderski and Lugt, 1964), the region which is affected by friction forces may be confined by the surface $z=0$ and the "limiting line" defined by the ϵ-condition (see Fig. 1)

$$z=\delta(r):|w|=\epsilon. \tag{13}$$

The analytic and even function δ can be expanded in the form

$$\delta(r)=a-br^2\pm\cdots. \tag{14}$$

Following the procedure of the local boundary layer approximations, the transformation

$$(r,z)\longrightarrow(r,\eta)\text{ with }\eta=za/\delta \tag{15}$$

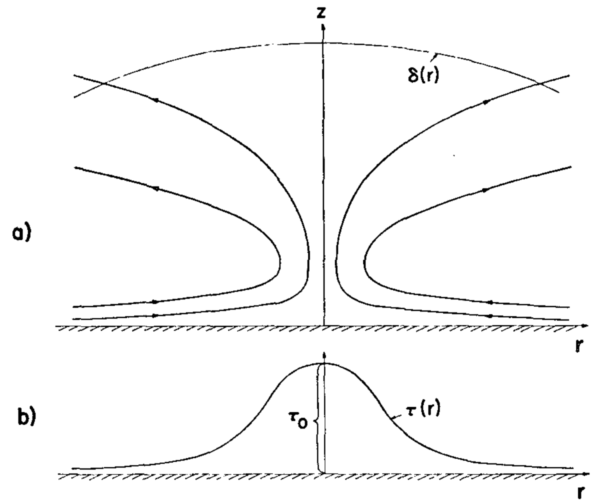

FIG. 1. Convection flow caused by local heating of a surface around the origin, (a), and temperature distribution of the surface, (b). The direction of motion is reversed for a cooled surface.

is proposed, and the series expansion around $r=0$,

$$f=t_s[F(\eta)+F_2(\eta)r^2+\cdots] \tag{16}$$

is introduced, where f stands for the functions wu/r, p, and t, and F for W, U, P, and T. At this point numerical calculations of the authors are anticipated which show that the coefficient b vanishes for all cases. The demonstration of this result is omitted here because the following successful construction of approximate solutions confirms, *a posteriori*, the assumption $b=0$. This and the adjusted local boundary layer assumption of first order

$$F_2(\eta)\equiv\frac{\partial^2}{\partial r^2}[f(r,\eta)/t_s(r)]_{r=0}\approx 0$$

lead to $\eta=z$ and, hence, for Eqs. (1) through (4), to the following system of ordinary differential equations:

$$U''-WU'-(U+8)U+2P=0, \tag{17}$$

$$W''-WW'-4W-P'+\mathrm{G}T=0, \tag{18}$$

$$2U+W'=0, \tag{19}$$

$$T''-\mathrm{P_r}WT'-4T+2\mathrm{P_r}\mathrm{E}(2U^2+W'^2)=0. \tag{20}$$

The primes denote derivatives with respect to z. These equations are simplified by introducing the combining function $g=\frac{1}{2}W$, $g'=-U$ which satisfies Eq. (19):

$$g'''-2gg''+(g'-8)g'-2P=0, \tag{21}$$

$$g''-2gg'-4g-\tfrac{1}{2}P'+\tfrac{1}{2}\mathrm{G}T=0, \tag{22}$$

$$T''-2\mathrm{P_r}gT'-4T+12\mathrm{P_r}\mathrm{E}g'^2=0. \tag{23}$$

The corresponding boundary conditions are

$$z=0:\quad g=0,\quad g'=0,\quad T=1, \tag{24}$$

$$z=\infty:\quad g=0,\quad T=0. \tag{25}$$

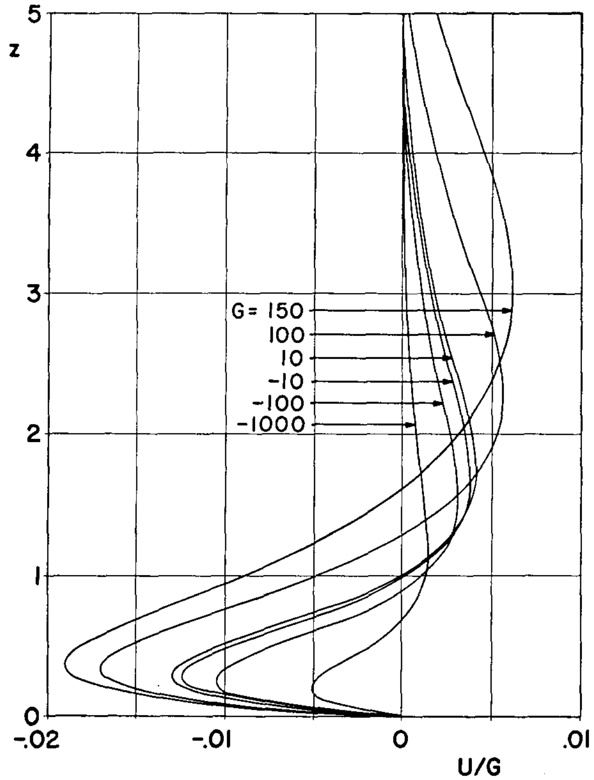

FIG. 2. The dimensionless radial velocity U vs. the dimensionless height z for $P_r=0.7$, $E=0$, and for different G.

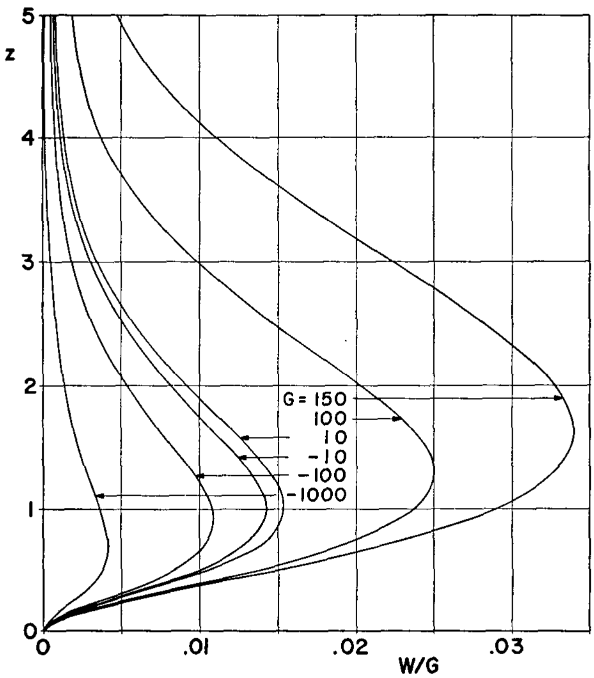

FIG. 3. The dimensionless axial velocity W vs. the dimensionless height z for $P_r=0.7$, $E=0$, and for different G.

3. Transformation to Volterra integral equations

An efficient integration of the ordinary boundary value problem (21) through (25) can be achieved by a transformation into an equivalent integral equation. This was successfully performed by the authors in a similar problem (Schwiderski and Lugt, 1963). Assuming the knowledge of the functions

$$A(z)=2P+2gg''-g'^2 \tag{26}$$

$$B(z)=2P_r(gT'-6Eg'^2), \tag{27}$$

the linear differential equations

$$g'''-8g'=A \tag{28}$$

$$T''-4T=B \tag{29}$$

are integrable by quadrature. Their solutions are the solutions of the following Volterra integral equations:

$$g=\frac{1}{16}e^{\sqrt{8}z}\left[\int_0^z A(x)e^{-\sqrt{8}x}dx-\int_0^\infty A(z)e^{-\sqrt{8}z}dz\right] +\frac{1}{16}e^{-\sqrt{8}z}\left[\int_0^z A(x)e^{\sqrt{8}x}dx-\int_0^\infty A(z)e^{-\sqrt{8}z}dz\right] +\frac{1}{8}\left[\int_0^\infty A(z)e^{-\sqrt{8}z}dz-\int_0^z A(x)dx\right], \tag{30}$$

and

$$T=e^{-2z}+\tfrac{1}{4}e^{2z}\left[\int_0^z B(x)e^{-2x}dx-\int_0^\infty B(z)e^{-2z}dz\right] +\tfrac{1}{4}e^{-2z}\left[\int_0^\infty B(z)e^{-2z}dz-\int_0^z B(x)e^{2x}dx\right]. \tag{31}$$

The pressure can be calculated from Eq. (22), if g and T are assumed known functions. The solutions g, T, and P may be obtained by an iteration process which starts for small Grashof numbers with

$$g\equiv 0,\quad T=e^{-2z},\quad P'=-\frac{G}{2}e^{-2z}. \tag{32}$$

For larger Grashof numbers the iterations may be initiated by solutions which have been computed for smaller Grashof numbers.

4. Discussion of the numerical results

The convection flows and associated temperature fields have been calculated with the IBM 7030 computer for various Prandtl, Grashof (Rayleigh), and Eckert numbers. A selection of velocity and temperature profiles is displayed in Figs. 2 through 9. For the special case of vanishing dissipation (E=0), Nusselt numbers are presented as a function of the Rayleigh and Prandtl numbers in Fig. 10 and in Table 1.

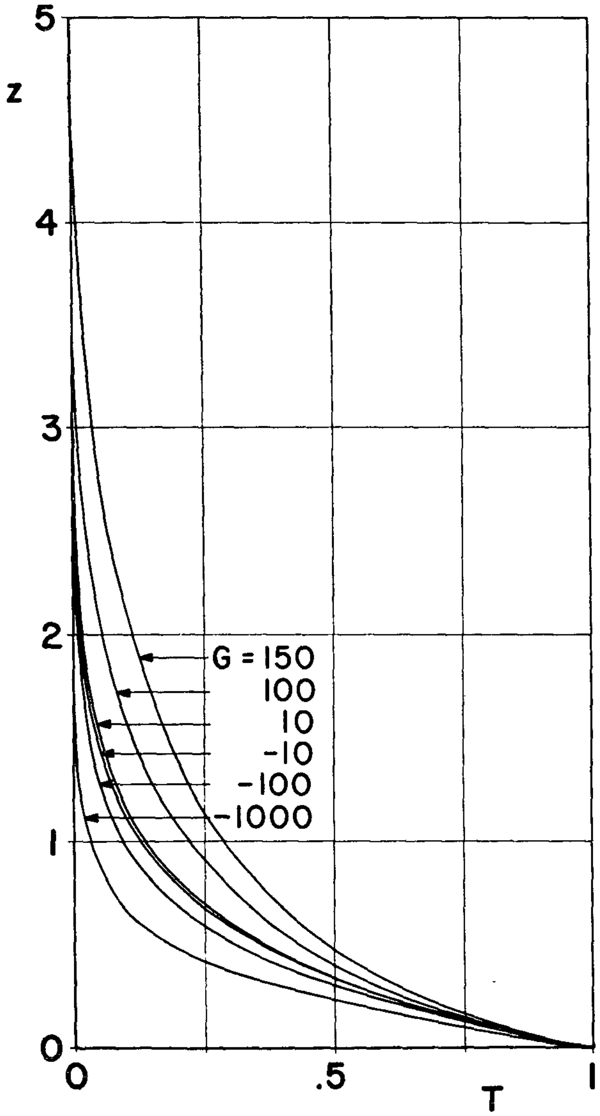

FIG. 4. The dimensionless temperature T vs. the dimensionless height z for $P_r=0.7$, $E=0$, and for different G.

Before beginning the discussion of the results a brief note may be added concerning the simultaneous use of the Grashof and the Rayleigh numbers. Although both parameters differ only in a formal way from each other ($R=GP_r$), it is practical to compare the velocity profiles by means of the Grashof number whereas instability and heat transfer effects are best described by the Rayleigh number.

According to the definitions of G, R, and L in section 2, positive and negative Grash of (or Rayleigh) numbers refer to a heated and a cooled surface, respectively. A heated surface causes a convection flow of wake type (the flow near the axis of symmetry is directed away from the surface), whereas a cooled surface generates a stagnation type flow (the motion near the axis of symmetry is directed toward the surface). In the latter case

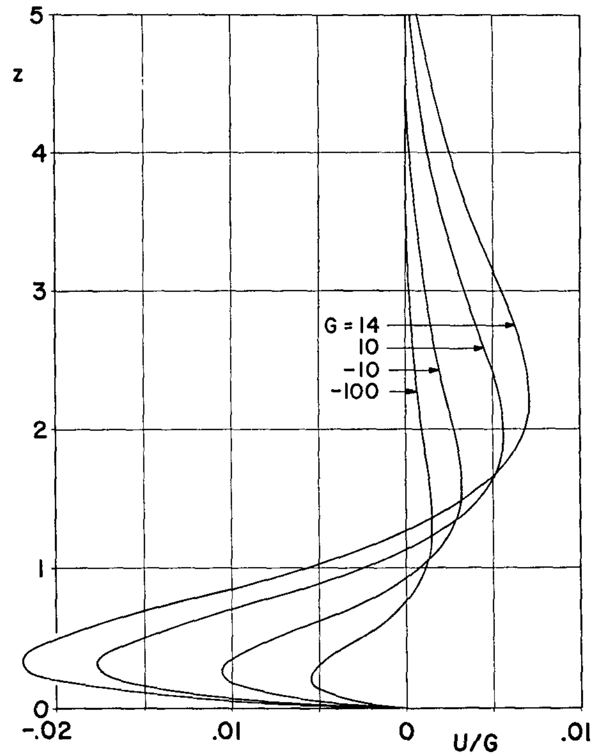

FIG. 5. The dimensionless radial velocity U vs. the dimensionless height z for $P_r=7$, $E=0$, and for different G.

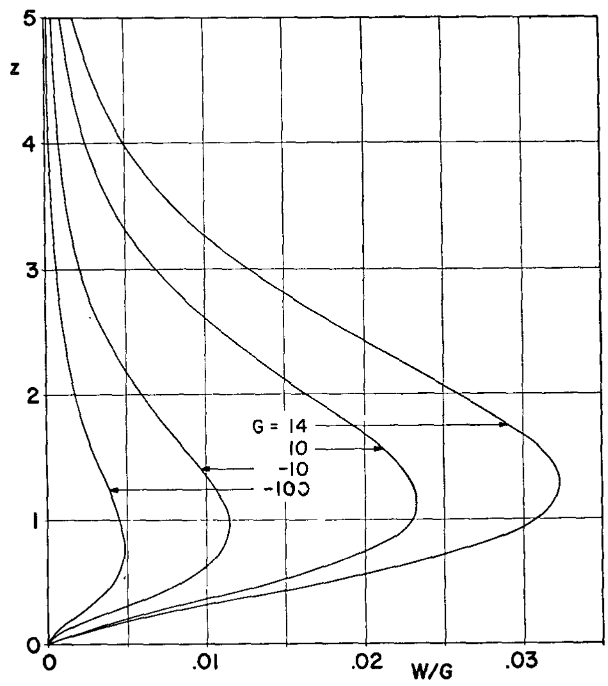

FIG. 6. The dimensionless axial velocity W vs. the dimensionless height z for $P_r=7$, $E=0$, and for different G.

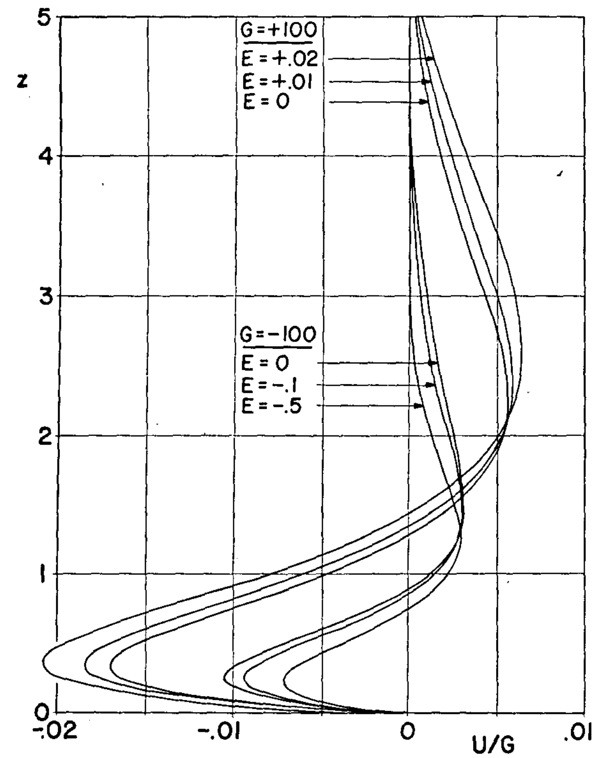

FIG. 7. The dimensionless radial velocity U vs. the dimensionless height z for $P_r=0.7$, G=+100, −100, and for different E.

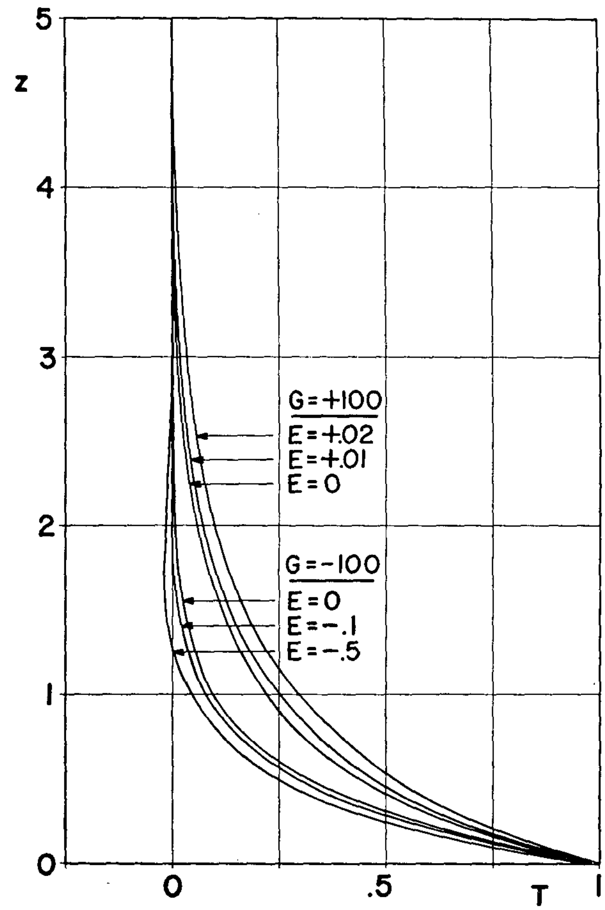

FIG. 9. The dimensionless temperature T vs. the dimensionless height z for $P_r=0.7$, G=+100, −100, and for different E.

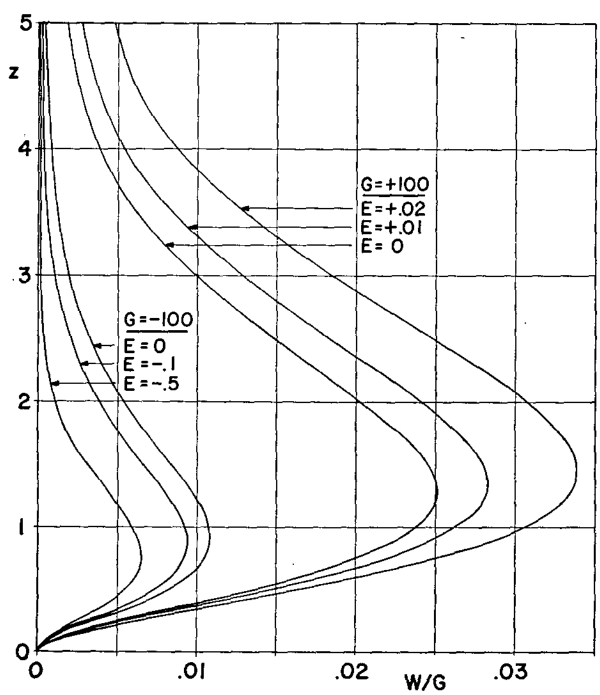

FIG. 8. The dimensionless axial velocity W vs. the dimensionless height z for $P_r=0.7$, G=+100, −100, and for different E.

the convective flow diminishes its spatial extent with increasing absolute values of the Grashof number but enlarges its strength. For increasing positive Grashof numbers the convection grows in strength as well as in extent (Figs. 2, 3, 5, 6). These results are physically plausible because an enhancing stagnation type flow reduces the thickness of the boundary layer whereas an

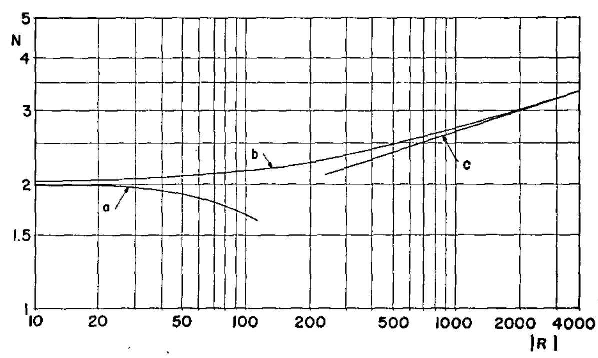

FIG. 10. The Nusselt number N vs. the absolute value of the Rayleigh numbers |R| for a heated surface (a) and for a cooled surface (b).

TABLE 1. Nusselt numbers for various Prandtl and Rayleigh numbers (E=0).

R	P_r 0.7	7	7000
−3500	3.286	3.389	3.399
−1400	2.857	2.913	2.918
−700	2.600	2.632	2.634
−350	2.399	2.415	2.416
−70	2.121	2.123	2.123
−7	2.014	2.014	2.014
0	2.000	2.000	2.000
7	1.985	1.985	1.985
70	1.817	1.812	1.812
98	1.733	1.697	1.695
122	1.67		

increasing wake type flow exerts a magnifying effect. If the Grashof number is kept constant and the Eckert number increases, the motions enhance in strength and extent (Figs. 7 and 8).

Analogous to the Reynolds numbers in similar problems (Schwiderski and Lugt, 1964) there exists for G a noncritical range

$$-\infty < G \leq G_{crit} < \infty, \tag{33}$$

or, in other words, the wake type flow becomes spatially unstable beyond a critical Grashof number. The beginning of this instability is indicated by the observation that far away from the surface the inflection point of the axial velocity profile, which guarantees the fulfillment of the boundary condition $g=0$ at $z=\infty$, moves towards infinity when approaching the critical Rayleigh number (Figs. 3 and 6). It marks the end of a simple circulatory convection flow as sketched in Fig. 1, which can be described by functions of the truncated form (16) ($F_2\equiv 0$). The point of instability lies at about

$$P_r G_{crit} = R_{crit} \approx 120 \text{ for } E=0, \tag{34}$$

which has been estimated from the computed examples for the Prandtl numbers 0.7 (air), 7 (water), and 7000 (engine oil). Thus, the critical Rayleigh number is almost independent of the Prandtl number. If the Rayleigh number is kept constant and the Eckert number increases, the same kind of instability occurs. It is, for instance,

$$E_{crit} \approx 0.02 \text{ for } R=70. \tag{35}$$

The influence of the Prandtl number on convection can be observed by comparing the curves of Figs. 2 and 3 for $P_r=0.7$ with the corresponding graphs of Figs. 5 and 6 for $P_r=7$. An increasing Prandtl number results in an enlarged convection both in strength and extent for a heated surface, whereas the situation is reversed for a cooled surface.

The meaning of the Eckert number has been described in Lugt and Schwiderski (1965). Similar to the problems solved in that reference, the Eckert number exhibits the same behavior here. Increasing negative values of E—the surface is in this case cooler than the fluid far away from the surface—enhance the friction effects in such a way, that the flow is heated, and the temperature difference between surface and fluid grows (see the next paragraph on heat flux). From a certain Eckert number on, the temperature within the convection flow exceeds the temperature far away from the surface (Fig. 9). Increasing positive values of E diminish the temperature difference between surface and fluid because of larger frictional heating of the flow.

The dimensionless heat flux at the surface, which is called the Nusselt number N, is here defined as the negative dimensionless temperature gradient at the surface

$$N = -T_0', \tag{36}$$

so that the Nusselt number is positive in the direction of the heat flux. In Fig. 10 the Nusselt number is plotted versus the absolute value of the Rayleigh number for vanishing dissipation (E=0). Table 1 shows that both curves of Fig. 10 are almost independent of the Prantdl number. When the surface is cooled the Nusselt number becomes an increasing function of the Rayleigh number and approaches in the limit $R \to -\infty$ the values of the asymptote

$$N = 0.87 |R|^{\frac{1}{6}}. \tag{37}$$

The small influence of the Prandtl number is here neglected. The occurrence of a power law is not surprising and is well known in literature for boundary layer problems [see Schlichting (1960)].

A heated surface causes a decrease of the Nusselt number. This may be explained by the larger friction layer in flows of wake type. When the Rayleigh number tends to its critical value, the Nusselt number arrives at the minimum

$$N_{min} = N_{crit} \approx 1.6. \tag{38}$$

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REFERENCES

Black, J. F., and B. L. Tarmy, 1963: The use of asphalt coatings to increase rainfall. *J. Appl. Meteor.*, **2**, 557–564.

Landsberg, H., 1947: Firestorms resulting from bombing conflagrations. *Bull. Amer. Meteor. Soc.*, **28**, 72.

Lugt, H. J., and E. W. Schwiderski, 1965: Temperature distribubutions in rotating flows normal to a flat surface. *Quart. Appl. Math.*, **23**, 133–150.

Malkus, J. S., 1963: Tropical rain induced by a small natural heat source. *J. Appl. Meteor.*, **2**, 547–556.

——, and A. F. Bunker, 1952: Observational studies of the air flow over Nantucket Island during the summer of 1950. *Papers in Phys. Ocean. and Meteor., M.I.T. and Woods Hole Ocean. Inst.*, **12**, No. 2.

——, and M. E. Stern, 1953a: The flow of a stable atmosphere over a heated island, Part I. *J. Meteor.*, **10**, 30–41.

Priestley, C. H. B., and F. K. Ball, 1955: Continuous convection from an isolated source of heat. *Quart. J. R. Meteor. Soc.*, **81**, 144–157.

Schlichting, H., 1960: *Boundary Layer Theory*. Fourth Edition, 334 pp.

Schwiderski, E. W., and H. J. Lugt, 1963: Vortex flows normal to a flat surface. *Tellus*, **15**, 241–245.

—— and ——, 1964: Rotating flows of von Kármán and Bödewadt. *Phys. Fluids*, **7**, 867–875.

Smith, R. C., 1955: Theory of air flow over a heated land mass. *Quart. J. R. Meteor. Soc.*, **81**, 382–395.

Stern, M. E., and J. S. Malkus, 1953: The flow of a stable atmosphere over a heated island, Part II. *J. Meteor.*, **10**, 105–120.

Stommel, H., and G. Veronis, 1957: Steady convective motion in a horizontal layer of fluid heated uniformly from above and cooled nonuniformly from below. *Tellus*, **9**, 401–407.

Trubnikov, B. N., 1961: On the theory of free convection above a plane surface of the earth. *Bull. (Izvestiya), Acad. Sci., USSR*, Geophys. Series No. 8, 813–816.