Generalization of the Mixing-Length Argument in Turbulent Diffusion

Gabriel T. Csanady

University of Waterloo, Ontario, Canada[1]

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ABSTRACT

The gradient diffusion hypothesis is replaced by a generalized "mixing-length" type argument, the essence of which is that a drifting particle effectively "originates" some distance from the point where turbulent flux is calculated and carries with itself the mean concentration appropriate to its region of origin. Thus, *turbulent flux is set proportional to a weighted integral of the mean concentration distribution*. On applying the principle of continuity one obtains an integral equation describing turbulent diffusion. To illustrate the use of this equation, various assumptions are made regarding the coordinates of the effective origin and the development of the concentration profile in the wake of a point source calculated numerically. The results appear to be of interest, but accurate experimental data are required before the theory becomes quantitatively useful.

1. Introduction

The simplest problem in the theory of turbulent diffusion seems to be the prediction of the mean concentration field in stationary and homogeneous turbulence. It is a measure of the difficulty of the subject that after many decades of development, a completely satisfactory theory does not yet exist for solving this simple problem, let alone those presented by diffusion in shear flow or concentration fluctuations, or others still more difficult.

In reviewing the available theoretical treatment of turbulent diffusion, Pasquill (1962) observes that the development up to the present has followed two main lines, which may be termed the "transfer-theory" approach and the "statistical" theory approach. In physical terms, the distinction is perhaps better characterized by the fact that the transfer-theory focuses on the *flux* of diffusing material, which is brought about by turbulent movements at a given point in space, while the statistical theory concerns primarily the *dispersion* of a diffusing particle about its initial position. The two approaches are complementary, as Batchelor's (1949) analysis of diffusion in stationary and homogeneous turbulence shows.

In either approach two main principles are applied; one is purely kinematical (the principle of continuity), and the other is what one could perhaps call "dynamical," being, in the language of irreversible thermodynamics, a relationship between a "flux" and a "force." This latter relationship is represented in molecular diffusion by the equivalent of Fourier's law of heat conduction. In turbulent diffusion, the analogous relationship is the gradient diffusion hypothesis, which is known to give, at best, a poor representation of reality, and yet it is used in the transfer theory. The equivalent hypothesis of the statistical theory is that the mean concentration profile is Gaussian. Neither hypothesis is satisfactory and one may say that the "dynamics" of turbulent diffusion is still *sub judice.*

Though the gradient diffusion hypothesis of transfer theory is known to be very crude, the conventional mixing-length argument on which it is based is essentially not unsound physically. It is, however, usually coupled with an assumption to the effect that the mixing length is small compared to the length-scale of the mean concentration distribution. It is shown below that this secondary assumption of transfer theory may be replaced by other, more realistic hypotheses. Thus, it becomes possible to represent correctly the "initial" as well as the "final" phase of diffusion by using two different hypotheses. At present, there is insufficient experimental evidence to suggest a more general approach, but this defect of the theory should be removable. Regardless of a certain indeterminancy because of this, the "generalized mixing-length theory" yields an integral-differential equation in place of the Fickian equation for turbulent diffusion.

Although many of the definitions and arguments of this mixing-length theory are easily extended to shear flow, the discussion herein applies to stationary and homogeneous turbulence, the problem being the determination of the stochastic-mean concentration distribution.

2. The turbulent flux

Consider the random movement of marked fluid particles, released in some prescribed way into a field of *stationary and homogeneous* turbulence. Let the concentration field in an individual trial be $N(\mathbf{x},t)$. In a large number of trials carried out under identical external

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conditions and identical conditions of release, a field of expected (ensemble-mean) concentration $\bar{N}(\mathbf{x},t)$ may be determined. The connection between this mean concentration field and single-particle displacement probability distributions has been discussed by Batchelor (1949).

Next consider the flux of marked fluid due to turbulent movements, averaged over a large number of trials, i.e.,

$$\mathbf{F}(\mathbf{x},t)=\overline{N(\mathbf{x},t)\mathbf{v}(\mathbf{x},t)}. \quad (1)$$

Without loss of generality, one may take the average velocity in a homogeneous field to be zero, i.e.,

$$\bar{\mathbf{v}}=0. \quad (2)$$

In order to express the flux in terms of probability distributions, let $Q(\mathbf{v})d\mathbf{v}$ be the probability that the endpoint of the velocity vector is contained within the element $d\mathbf{v}$ around the point $\mathbf{v}$ in velocity-space. Because we are interested in expressing the Eulerian quantity $\mathbf{F}(\mathbf{x},t)$, $Q(\mathbf{v})$ refers to the velocity vector as observed at a fixed point $(\mathbf{x},t)$. This distribution is known to be Gaussian to a good approximation.

The second quantity needed is the *conditional* expectation, $\bar{N}_c(\mathbf{x},t|\mathbf{v})$, of the concentration at $(\mathbf{x},t)$, *given* that the velocity there is $\mathbf{v}$. The qualification is important as one easily realizes. Suppose, for example, that the velocity vector is so directed that a particle is just arriving from a region where the probability of finding marked particles is higher than in the immediate neighborhood of $(\mathbf{x},t)$. Therefore, the conditional expectation $\bar{N}_c(\mathbf{x},t|\mathbf{v})$, given such a velocity $\mathbf{v}$, is also higher than the overall local expectation $\bar{N}(\mathbf{x},t)$. This intuitively obvious fact lies at the heart of the traditional mixing-length argument and seems to be essential to characterize the "dynamics" of turbulent diffusion, as opposed to its mere kinematics (continuity).

Because, by hypothesis, the field is stationary and homogeneous, $Q(\mathbf{v})$ is not a function of either $\mathbf{x}$ or t. Further, as is usual in the theory of diffusion, we shall suppose the diffusing substance to be dynamically neutral so that $Q(\mathbf{v})$ is independent of whether there is marked fluid at $(\mathbf{x},t)$ or not. Under such conditions, the flux may be expressed as

$$\mathbf{F}(\mathbf{x},t)=\int \mathbf{v}Q(\mathbf{v})\bar{N}_c(\mathbf{x},t|\mathbf{v})d\mathbf{v}. \quad (3)$$

Roberts (1960) has written an equation in which he effectively defines flux in a way similar to Eq. (3), except that he uses a single joint-probability distribution, $V(\mathbf{x},\mathbf{v},t)$, in place of our $Q(\mathbf{v})\bar{N}_c(\mathbf{x},t|\mathbf{v})$ (in other words, he does not incorporate the dynamic neutrality hypothesis into his formulation, although he uses it when dealing with diffusion at very short times t). However, he makes little further use of this formulation.

The principle of the conservation of mass imposes the condition

$$\frac{\partial \bar{N}}{\partial t}=-\operatorname{div}\mathbf{F}. \quad (4)$$

The central problem in the theory of diffusion is to find a second relationship between flux and mean concentration in order that Eq. (4) may be solved. While a gradient hypothesis may be used in molecular diffusion, Eq. (3) suggests a rather more complex interdependence in turbulent diffusion. However, as it stands, Eq. (3) is not yet a complete second relationship because it must be supplemented by an expression connecting $\bar{N}_c$ and $\bar{N}$. An appropriate hypothesis may be formulated on the basis of the classical mixing-length argument to provide the required connection.

3. The mixing-length argument

In conventional mixing length theory it is assumed that:

1) A diffusing particle effectively "originates" some directed distance $\mathbf{l}$ to one side of the point considered and brings with it the mean concentration appropriate to its origin. This would mean in our formulation that

$$\bar{N}_c(\mathbf{x},t|\mathbf{v})=\bar{N}(\mathbf{x}-\mathbf{l},\,t); \quad (5)$$

2) The distance $\mathbf{l}$ is small enough so that, with satisfactory accuracy,

$$\bar{N}(\mathbf{x}-\mathbf{l},\,t)=\bar{N}(\mathbf{x},t)-l_i\frac{\partial \bar{N}}{\partial x_i}. \quad (6)$$

The second assumption can only be true if the length-scale of eddies is small compared to a characteristic length of the distribution $\bar{N}(\mathbf{x},t)$, a highly restrictive condition. This assumption may be dispensed with without abandoning the essential physical idea behind the mixing-length argument, that of an "effective origin." We generalize Eq. (5) into the postulate

$$\bar{N}_c(\mathbf{x},t|\mathbf{v})=\bar{N}(\mathbf{x}^*,t^*), \quad (7)$$

where t^* and $\mathbf{x}^*$ are the time and place of the "effective origin" of particles having velocity $\mathbf{v}$ at $(\mathbf{x},t)$. Because $\bar{N}_c$ cannot be higher than the maximum concentration at release (from the second law of thermodynamics), Eq. (7) is strictly true for some $(\mathbf{x}^*,t^*)$, presumably for a finite space-time domain. The problem, however, is how to obtain practicable functional relationships $\mathbf{x}^*=\mathbf{x}^*(\mathbf{x},t,\mathbf{v})$ and $t^*=t^*(\mathbf{x},t,\mathbf{v})$.

In order to formulate a satisfactory general hypothesis of the form of Eq. (7), a considerable amount of detailed experimental evidence on conditional expectations, $\bar{N}_c(\mathbf{x},t|\mathbf{v})$, would have to be analyzed. Although such evidence does not exist at present, some extreme cases and some speculative models are explored below.

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4. Diffusion close to a concentrated source

One asymptotic condition of interest is the diffusion in the neighborhood of an instantaneous point-source at short times, $t\rightarrow 0$. The initial distribution is described by

$$\bar{N}(\mathbf{x},0)=M\delta(\mathbf{x}). \tag{8}$$

Individual marked particles retain their velocities for a short period (short compared to the Lagrangian time-scale T) so that when $t\ll T$, any such particle moves through a distance $\mathbf{v}t$. Therefore, the mean concentration in a travelling element of given velocity $\mathbf{v}$ will be (within a short time interval from release)

$$\bar{N}_c(\mathbf{x},t|\,\mathbf{v})=\bar{N}(\mathbf{x}-\mathbf{v}t,\,0). \tag{9}$$

In other words, for the "effective origin," one may set $\mathbf{x}^*=\mathbf{x}-\mathbf{v}t$, $t^*=0$, which amounts to equating the effective origin with the physical origin. For such short times, therefore, by Eq. (8),

$$\bar{N}_c(\mathbf{x},t|\,\mathbf{v})=M\delta(\mathbf{x}-\mathbf{v}t). \tag{10}$$

Substitution into (3) yields, at $t\ll T$

$$\mathbf{F}(\mathbf{x},t)=\frac{M}{t^4}\mathbf{x}Q\left(\frac{\mathbf{x}}{t}\right). \tag{11}$$

Eq. (4) is easily solved with this value of $\mathbf{F}$ to give

$$\bar{N}(\mathbf{x},t)=\frac{M}{t^3}Q\left(\frac{\mathbf{x}}{t}\right)\quad(t\ll T). \tag{12}$$

This relationship also follows from Batchelor's (1952) analysis, its meaning being simply that the concentration distribution is proportional to the probability distribution of velocities. Therefore, the result shows that the effective-origin assumption embodied in Eq. (9) is strictly correct at short times t.

5. Diffusion a long time after release

At the opposite extreme, when $t\rightarrow\infty$ (counted from release), the velocity of any individual marked particle is uncorrelated with its velocity at or near its time of release. The average displacement of a particle having given velocity components v_j at $(\mathbf{x},t)$, just prior to t, is

$$\bar{x}_i(t)-\bar{x}_i(t_1)=\frac{1}{v_j(t)}\int_{t_1}^{t}\overline{v_i(t')v_j(t)}dt'. \tag{13}$$

As the velocity correlation reduces to zero, the integral tends to a constant value. In a homogeneous field, where $\overline{v_iv_j}=0$ (except if $i=j$), it is reasonable to set

$$\bar{x}_i(t)-\bar{x}_i(t_1)=v_i(t)T, \tag{14}$$

where T is the Lagrangian time-scale and t_1 is sufficiently far into the past so that the velocities at t and t_1 are uncorrelated (i.e., the particle is "lost in the crowd").

Consider all particles that were in the neighborhood of $\mathbf{x}-\mathbf{v}T$ at t_1. At time t the majority is still in the neighborhood of $\mathbf{x}-\mathbf{v}T$, because this is their mean position if $\bar{\mathbf{v}}=0$, their mean concentration there being $\bar{N}(\mathbf{x}-\mathbf{v}T,\,t)$. It is reasonable to assume that particles of the same diffused group carry the *same* mean concentration. In particular, those that are at $\mathbf{x}$ at t, having velocity $\mathbf{v}$ so that they are part of the group from $\mathbf{x}-\mathbf{v}T$, may be assumed to have the conditional expectation of concentration

$$\bar{N}_c(\mathbf{x},t|\,\mathbf{v})=\bar{N}(\mathbf{x}-\mathbf{v}T,\,t). \tag{15}$$

In other words, the coordinates of the effective origin are assumed to be $\mathbf{x}^*=\mathbf{x}-\mathbf{v}T$, $t^*=t$, with $T=$constant. This is akin to assumption 1) of the conventional mixing-length theory described in Section 3.

The central limit theorem suggests, and experimental evidence shows, that the mean concentration field of a point source is asymptotically Gaussian as $t\rightarrow\infty$. Thus, we examine here under what conditions such a distribution of $\bar{N}(\mathbf{x},t)$ is consistent with assumption (15), as well as with the fundamental Eqs. (3) and (4). For simplicity, we consider one-dimensional diffusion, the mean concentration field being given by

$$\bar{N}(x,t)=\frac{M}{\sqrt{2\pi}S}e^{-x^2/2S^2}, \tag{16}$$

where $S=S(t)$ is the standard deviation of dispersion. For the velocity distribution we also assume a Gaussian form

$$Q(v)=\frac{1}{\sqrt{2\pi}\sigma}e^{-v^2/2\sigma^2}, \tag{17}$$

so that σ is the standard deviation of turbulent velocities. Now, using (15), Eq. (3) yields

$$F=\frac{M}{2\pi S\sigma}\int v\exp\left[-\frac{v^2}{2\sigma^2}-\frac{(x-vT)^2}{2S^2}\right]dv. \tag{18}$$

The integration is straightforward and results in:

$$F=\frac{Qx\sigma^2T}{\sqrt{2\pi}(S^2+\sigma^2T^2)^{\frac{3}{2}}}\exp\left[-\frac{x^2}{2(S^2+\sigma^2T^2)}\right]. \tag{19}$$

This result and Eq. (16) are consistent with the continuity equation (4) provided that

$$\sigma T\ll S \tag{20}$$

and

$$S\frac{dS}{dt}=\sigma^2T. \tag{21}$$

Because a diffusing cloud grows continuously, condition (20) will be satisfied for sufficiently large t. On the other hand, Eq. (21) is Taylor's (1921) well-known

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theorem for large t, which upon integration gives

$$S^2 = 2\sigma^2 T(t - t_0), \tag{22}$$

where t_0 = constant. We may conclude that the effective-origin assumption of Eq. (15) is accurate at large t, confirming our intuitive reasoning.

6. Intermediate models

Unfortunately, it is not easy to suggest a sensible interpolation formula between the effective-origin hypotheses of Eqs. (9) and (15). There seems to be no physical argument to assist in the interpolation. A formula consistent with both Eqs. (9) and (15) for the *space* coordinates of the effective origin at intermediate times is

$$\mathbf{x}^* = \mathbf{x} - \mathbf{v}\int_0^t R dt, \tag{23}$$

where R is the Lagrangian velocity-autocorrelation coefficient, which may be taken to be $R = \exp(-t/T)$.

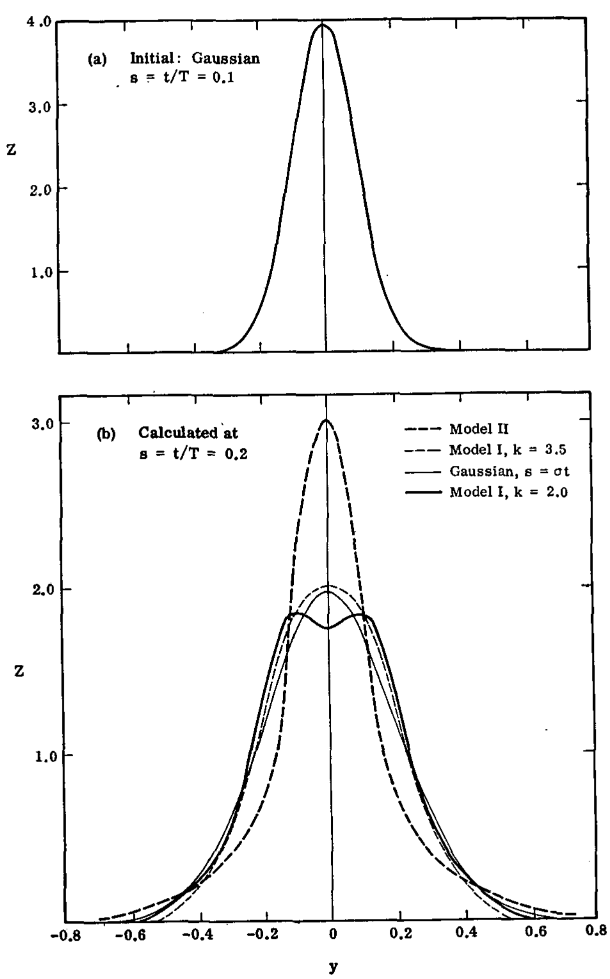

FIG. 1. Numerically calculated concentration distributions at $t/T = 0.2$. The Gaussian model label in (b) should read $S = \sigma t$.

While Eq. (23) is an arbitrary assumption, it makes sense physically in view of Eq. (13). On the other hand, any assumption regarding t^* has to be entirely arbitrary. For example, one may try

$$\text{(Model I)} \quad t^* = t(1 - e^{-kt/T}), \tag{24}$$

where k = constant of order unity. Alternatively, one may use the "initial-phase" hypotheses up to some t/T, followed by Eq. (23) supplemented with the asymptotic time relationship

$$\text{(Model II)} \quad t^* = t. \tag{25}$$

The development of the concentration distribution may now be followed using an electronic computer. To carry out the calculations, we combine Eqs. (3), (4) and (7) to give

$$\frac{\partial \bar{N}(\mathbf{x},t)}{\partial t} = -\frac{\partial}{\partial x_i}\int v_i Q(\mathbf{v}) \bar{N}(\mathbf{x}^*, t^*) d\mathbf{v}. \tag{26}$$

We specialize again to one-dimensional diffusion and use Eq. (17) for $Q(v)$. Eq. (26) may then be solved numerically, given an initial distribution. The latter is provided by Eq. (12), for a suitably small t, say $t = 0.1T$. The calculations are best carried out in terms of the non-dimensional variables,

$$\left.\begin{aligned} Z &= \sigma T \bar{N}/M \\ s &= t/T \\ p &= v/\sigma \\ y &= x/\sigma T \end{aligned}\right\}. \tag{27}$$

Eq. (26) then becomes

$$\frac{\partial Z}{\partial s} = -\frac{\partial}{\partial y}\int \frac{p}{\sqrt{2\pi}} e^{-p^2/2} [Z] dp, \tag{28}$$

where $[Z]$ is to be evaluated at y^*, s^*. The initial distribution, at $s = 0.1$ is

$$Z = \frac{10}{\sqrt{2\pi}} \exp(-50y^2).$$

Some results obtained in this manner on a 7090 com-

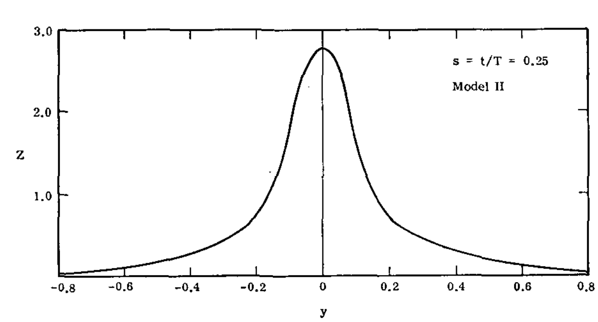

FIG. 2. Model II profile at $t/T = 0.25$.

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puter are shown in Fig. 1. It is seen that the profile development is quite sensitive to the assumption made regarding t^*. Considerable differences already exist at the short time $t/T=0.2$ between the Gaussian model (using the initial phase hypothesis, $S=\sigma t$) and what is designated here as Model I and Model II [Eqs. (24) and (25)].

If the constant k is small in Model I, a "saddle" appears at the middle of the distribution. The opposite is true at large k, in the limit of which one arrives at Model II. Coupled with an excessively high peak at the center, Model II shows secondary peaks further out as t increases (Fig. 2. shows a Model II profile at relatively low t/T).

Despite this sensitivity to the assumptions regarding t^*, it is evident that for some choice of the coordinates of the effective origin, the Gaussian profile may be approximated closely. Experimentally, this is all we know, i.e., that the distribution is nearly Gaussian. From the present investigations we may then reasonably conclude that such a distribution *can* be produced by a reasonable distribution of the conditional expectation $\bar{N}_c$. In other words, to explain the observed near-Gaussian concentration profiles, it is not necessary to fall back on the philosophically unsatisfactory gradient-diffusion hypothesis, or its equivalent in statistical theory, the random-walk model. Experimental evidence on conditional expectations $\bar{N}_c$ may one day enable us to fill in the gaps in current theory by suggesting some concrete interpolation model.

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