On Nonlinear Geostrophic Adjustment

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ABSTRACT

Nonlinear features of the geostrophic adjustment process in a one-dimensional barotropic atmosphere are investigated by means of a perturbation expansion in the Froude number. The initial unbalanced velocity field is a continuous (nonconstant) even function of the spatial coordinate. The steady-state solution shows the southward shift of the axes of maximum geostrophic velocity and zero pressure, first found by Rossby. In addition, the geostrophic fields are asymmetric about their respective axes.

The nonlinear oscillation of the whole current system approaches the inertial period and decays like $t^{-\frac{1}{2}}$ as time $t\rightarrow\infty$. However, this oscillation continues for a significantly longer time, before approximate geostrophic balance is reached, than the "adjustment time" determined from a linear analysis. A possible shortcoming in the quasi-geostrophic approximation, used in some large-scale dynamical models, is indicated by this result.

1 Introduction

The adjustment of an initially unbalanced current, in a homogeneous incompressible fluid of depth H_0, to a state of steady geostrophic balance was first examined by Rossby (1938). The Rossby problem was later reexamined by Mihaljan (1963), who corrected some details in Rossby's solution. In these papers the conservation of potential vorticity was expressed by the nonlinear second-order differential equation

$$\frac{d^2(y_i-y)}{dy^2}-\lambda^{-2}(y_i-y)=-(f\lambda^2)^{-1}u_i(y_i), \qquad (1.1)$$

where y_i-y is the displacement of a particle from its initial position y_i, $u_i(y_i)$ is the velocity of the particle when it is at its initial position, $\lambda=(gH_0)^{\frac{1}{2}}/f$ is Rossby's *radius of deformation*, g is the acceleration of gravity, and f is the Coriolis parameter. Rossby considered $u_i(y_i)$ to be a uniform current into the paper (x direction) of width $2a$. This special initial condition implies that $u_i(y_i)=u_i(y)$ and (1.1) becomes a linear equation. However Rossby's steady geostrophic solution of (1.1) represents a solution to a nonlinear system of equations, since the nonlinear terms in the equations of motion and mass continuity remain intact in the Eulerian formulation of this problem.

Cahn (1945) investigated the transient features of the Rossby problem using a linearized system of equations. His results show how a portion of the initial energy is imparted to gravity-inertia oscillations which disperse this energy away from the initial source region, while a state of geostrophic balance is approached asymptotically with time in an ever-increasing region centered at the source. Washington (1964) performed numerical time integrations of the linearized equations using initial conditions with different characteristic widths. He showed how the adjustment to geostrophic balance is most rapid when the scale of the initial velocity field is small compared with λ. In the adjustment of relatively small-scale motion the mass field essentially readjusts to the initial velocity field. In the opposite case, when the width of the initial current is large compared with λ, most of the initial energy goes into gravity-inertia waves which have relatively small group velocities. Consequently, the adjustment to geostrophic balance proceeds slowly.

The adjustment process was independently investigated by Obukhov (1949), who studied the adjustment of an initially unbalanced symmetric vortex in a barotropic atmosphere. Obukhov's results, which have been summarized by Phillips *et al.* (1960), are essentially the same as those obtained by Cahn (1945). However, Obukhov used the linearized Eulerian form of the potential vorticity equation to determine the steady geostrophic solution so that he, as well as Cahn and Washington, was unable to reproduce the shift of the current southward of its initial position.

In the present paper the nonlinear process of geostrophic adjustment is investigated by means of a perturbation expansion in the Froude number $F\equiv U/(gH_0)^{\frac{1}{2}}$, where U is a characteristic amplitude of the initial velocity field. Specifically, two aspects of the adjustment problem will be emphasized, 1) the modifications to Rossby's solution which occur when the initial unbalanced velocity field is a continuous (nonconstant)

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function of y, the crosswind direction, and 2) the time dependent features associated with the shift of the current from its initial position.

2. Basic equations

The basic system of equations used will be those derived by Obukhov (1949) for motions in a barotropic atmosphere. For simplicity no variation in the x direction ($\partial/\partial x \equiv 0$) is permitted. Nondimensionalization of the system of equations will be accomplished by introducing a characteristic time f^{-1}, length $\lambda \equiv (gH_0)^{\frac{1}{2}} f^{-1}$ and velocity U, chosen to make the time and space derivatives of the velocity field of order unity. If the barotropic atmosphere is adiabatic, H_0 represents the height of a homogeneous atmosphere. $H_0 \sim 8$ km for standard values of pressure and density. The surface pressure field $p(y,0,t)$ is represented by the variable $\chi \equiv [p(y,0,t)-\bar{p}]/\bar{p}$, where $\bar{p}\sim 1000$ mb is a standard surface pressure. Next, we introduce the nondimensional variable π defined by $fU\lambda\pi = gH_0\chi$. Then, Obukhov's nondimensional system of equations takes the form[2]

$$\frac{\partial u}{\partial t} + \mathrm{F}v\frac{\partial u}{\partial y} - v = 0, \tag{2.1}$$

$$\frac{\partial v}{\partial t} + \mathrm{F}v\frac{\partial v}{\partial y} + u + \frac{\partial \pi}{\partial y} = 0, \tag{2.2}$$

$$\frac{\partial \pi}{\partial t} + \mathrm{F}v\frac{\partial \pi}{\partial y} + \frac{\partial v}{\partial y} = 0, \tag{2.3}$$

where (u,v) are density weighted, vertically averaged, horizontal velocities in the (x,y) directions, and $\mathrm{F} \equiv U(gH_0)^{-\frac{1}{2}}$ is the Froude number. If we define U as a characteristic amplitude of the initial velocity field, then $\mathrm{F} \sim O(10^{-1})$ for most atmospheric motions.

Mihaljan (1963) found that the maximum shift of the current in the Rossby problem is F/2. Since the shift of the current is a nonlinear phenomenon, but only represents a deviation of order F from the linear solution, we shall look for solutions of (2.1) through (2.3) by expanding the dependent variables in a power series in F. If the coefficients of each power of F are equated to zero, the zero- and first-order systems of equations become

$$\frac{\partial u_0}{\partial t} - v_0 = 0, \tag{2.4}$$

$$\frac{\partial v_0}{\partial t} + u_0 + \frac{\partial \pi_0}{\partial y} = 0, \tag{2.5}$$

$$\frac{\partial \pi_0}{\partial t} + \frac{\partial v_0}{\partial y} = 0, \tag{2.6}$$

and

$$\frac{\partial u_1}{\partial t} - v_1 = -v_0\frac{\partial u_0}{\partial y}, \tag{2.7}$$

$$\frac{\partial v_1}{\partial t} + u_1 + \frac{\partial \pi_1}{\partial y} = -v_0\frac{\partial v_0}{\partial y}, \tag{2.8}$$

$$\frac{\partial \pi_1}{\partial t} + \frac{\partial v_1}{\partial y} = -v_0\frac{\partial \pi_0}{\partial y}. \tag{2.9}$$

Initially, there is an unbalanced velocity field $u_i(y)$ directed along the x axis. The initial conditions for the zero- and first-order equations may be expressed as

$$\left.\begin{array}{ll} u_0 = u_i(y); & \partial u_0/\partial t = 0 \\ v_0 = 0 \quad ; & \partial v_0/\partial t = -u_i(y) \\ \pi_0 = 0 \quad ; & \partial \pi_0/\partial t = 0 \end{array}\right\} t = 0, \tag{2.10}$$

and

all first-order variables and derivatives are zero$\}\, t=0$. (2.11)

The initial velocity field, prescribed in the region $-\infty \leq y \leq \infty$, is given by

$$u_i(y) = [(1+r^2) - (y/r)^2]e^{-\frac{1}{2}(y/r)^2}, \tag{2.12}$$

and is illustrated in Fig. 1 for a half-width $r=0.6$.

3. Steady-state solution

Obukhov (1949) has shown that in the linear (or zero-order) system of equations the potential vorticity, $\Omega_0 \equiv \partial v_0/\partial x - \partial u_0/\partial y - \pi_0$, is conserved. In the present one-dimensional system this conservation principle may be expressed as

$$\partial u_0/\partial y + \pi_0 = -\Omega_i \equiv du_i(y)/dy, \tag{3.1}$$

if (2.10) is taken into account. If the flow becomes steady it must also become geostrophic, $u_{0g} + \partial \pi_{0g}/\partial y = 0$, so that the potential vorticity equation may be written

$$d^2\pi_{0g}/dy^2 - \pi_{0g} = -du_i(y)/dy. \tag{3.2}$$

It is shown in the following section, and also in the Appendix, that the flow actually does approach a steady state.

The solution of (3.2) is

$$\pi_{0g} = \int_{-\infty}^{\infty} G(y;\xi)\frac{du_i(\xi)}{d\xi}d\xi, \tag{3.3}$$

where

$$G(y;\xi) = \tfrac{1}{2}e^{-|y-\xi|} \tag{3.4}$$

is the Green's function, in an infinite domain, for the operator appearing on the left-hand side of (3.2) (Goertzel and Tralli, 1960), and u_i is given by (2.12).

The integral in (3.3) may be evaluated by introducing the new variable $\eta \equiv (\xi + r^2)/\sqrt{2}r$ and then performing a

[2] The system of Eqs. (2.1) through (2.3) also apply to motions in a homogeneous incompressible fluid of mean depth H_0 if we redefine $\chi \equiv [h(y,t)-H_0]/H_0$, where h is the depth of the fluid.

series of integrations by parts. The geostrophic pressure field is found to be

$$\pi_{0g} = -ye^{-\frac{1}{2}(y/r)^2}, \tag{3.5}$$

and the geostrophic velocity is given by

$$u_{0g} = -d\pi_{0g}/dy = [1-(y/r)^2]e^{-\frac{1}{2}(y/r)^2}. \tag{3.6}$$

This geostrophic solution agrees with a similar case examined by Obukhov (1949). When $r \ll 1$, the change in the velocity field from its initial state is small while the pressure field adjusts to the velocity field. As r increases, the difference between the initial and final states increases appreciably. However, we shall not consider $r>1$ since $u_i \sim O(1)$ has been assumed in the nondimensionalization.

Since the center of the geostrophic current has not been displaced from the origin, we shall explore the effect of the nonlinear terms by seeking solutions of the first-order system (2.7)–(2.9). In order to find the steady first-order geostrophic solution, with given initial conditions, the potential vorticity equation for the first-order motion will be derived. Eq. (2.7) is differentiated with respect to y, and (2.9) is used to obtain

$$\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial y}+\pi_1\right) = -\frac{\partial}{\partial y}\left(v_0\frac{\partial u_0}{\partial y}\right) - v_0\frac{\partial \pi_0}{\partial y}. \tag{3.7}$$

If (3.1) is used to eliminate $\partial u_0/\partial y$ and use is made of (2.4) and (2.6), noting that Ω_i is independent of time, we obtain an equation which expresses the conservation of first-order potential vorticity, i.e.,

$$\frac{\partial}{\partial t}\left(\frac{\partial u_1}{\partial y}+\pi_1+\frac{\pi_0^2}{2}+\frac{\partial}{\partial y}(u_0\Omega_i)\right)=0. \tag{3.8}$$

Integration of (3.8) and application of the initial conditions (2.10) and (2.11) yields

$$\frac{du_{1g}}{dy}+\pi_{1g}+\frac{\pi_{0g}^2}{2}+\frac{d(u_{0g}\Omega_i)}{dy}=\frac{d(u_i\Omega_i)}{dy}, \tag{3.9}$$

since the time-dependent terms must sum to zero. Since the first-order motion is ultimately geostrophic, $u_{1g}+\partial\pi_{1g}/\partial y=0$, and (3.9) becomes

$$\frac{d^2\pi_{1g}}{dy^2}-\pi_{1g}=\frac{\pi_{0g}^2}{2}+\frac{d}{dy}[(u_{0g}-u_i)\Omega_i]. \tag{3.10}$$

This equation may be integrated since the right-hand side of (3.10) is known. However, if use is made of (3.2) and $u_{0g}+\partial\pi_{0g}/\partial y=0$, we obtain a simpler expression of (3.10), i.e.,

$$\frac{d^2\phi}{dy^2}-\phi=\frac{u_{0g}^2-u_i^2}{2}, \tag{3.11}$$

where

$$\phi \equiv \pi_{1g}+\tfrac{1}{2}(\pi_{0g}^2+u_i^2-u_{0g}^2). \tag{3.12}$$

Since (3.11) has the same form as (3.2) it may be integrated similarly. The solution is

$$\phi=\frac{r^4}{4}e^{-(y/r)^2}+\frac{r^3}{8}\left(1+\frac{r^2}{2}\right)e^{r^2/4} \times\pi^{\frac{1}{2}}\left[e^{-y}\operatorname{erfc}\left(\frac{r}{2}-\frac{y}{r}\right)+e^{y}\operatorname{erfc}\left(\frac{r}{2}+\frac{y}{r}\right)\right], \tag{3.13}$$

where

$$\operatorname{erfc}(z)=2\pi^{-\frac{1}{2}}\int_z^\infty e^{-\xi^2}d\xi \tag{3.14}$$

denotes the complementary error function (Gautschi, 1964). The pressure field, determined from (3.12) and (3.13), is

$$\pi_{1g}=\frac{r^2}{2}\left[\left(\frac{y}{r}\right)^2-2\left(1+\frac{r^2}{4}\right)\right]e^{-(y/r)^2}+\frac{r^3}{8}\left(1+\frac{r^2}{2}\right)e^{r^2/4} \times\pi^{\frac{1}{2}}\left[e^{-y}\operatorname{erfc}\left(\frac{r}{2}-\frac{y}{r}\right)+e^{y}\operatorname{erfc}\left(\frac{r}{2}+\frac{y}{r}\right)\right], \tag{3.15}$$

and the geostrophic velocity is

$$u_{1g}=y\left[\left(\frac{y}{r}\right)^2-3\left(1+\frac{r^2}{6}\right)\right]e^{-(y/r)^2}+\frac{r^3}{8}\left(1+\frac{r^2}{2}\right)e^{r^2/4} \times\pi^{\frac{1}{2}}\left[e^{-y}\operatorname{erfc}\left(\frac{r}{2}-\frac{y}{r}\right)-e^{y}\operatorname{erfc}\left(\frac{r}{2}+\frac{y}{r}\right)\right]. \tag{3.16}$$

The total geostrophic pressure and velocity fields are

$$\pi_g=\pi_{0g}+F\pi_{1g}+O(F^2), \tag{3.17}$$

and

$$u_g=u_{0g}+Fu_{1g}+O(F^2), \tag{3.18}$$

where π_{0g}, π_{1g}, u_{0g} and u_{1g} are given by (3.5), (3.6), (3.15), and (3.16), respectively. Fig. 1 illustrates the solution expressed by (3.17) and (3.18) for $r=0.6$. Three principle nonlinear features emerge from this solution: 1) the axis of the velocity maximum has shifted to the right (southward), 2) the axis of $\pi_g=0$ is also south of the origin but is not coincident with the axis of the velocity maximum, and 3) u_g and π_g do not have even and odd symmetry, respectively, about their new axes. Features 2) and 3) are not apparent in Fig. 1 due to the scale at which the figure was drawn. Fig. 2 shows the shift of the velocity maximum (solid line) and axis of zero pressure (dashed line) as a function of the half-width r for F=0.1 and 0.2. If $H_0=9$ km and $f=10^{-4}$ sec^{-1}, then $\lambda=3000$ km. An initial current of 1200 km width ($2r=0.4$) will shift about 12 and 24 km

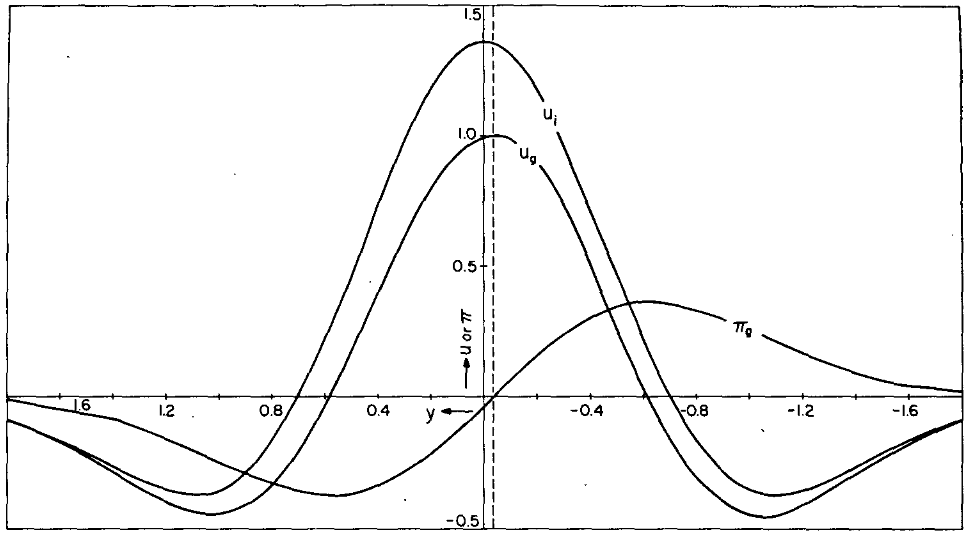

FIG. 1. The initial velocity u_i, geostrophic velocity u_g, and geostrophic pressure π_g as a function of y for $r=0.6$ and $F=0.1$. The dashed vertical line represents the axis of the geostrophic velocity maximum. A positive velocity indicates a west wind.

for F=0.1 and 0.2, respectively. The corresponding shifts in the axes of zero pressure are approximately 10.8 and 21.6 km, respectively. Since the axes of zero pressure and maximum geostrophic velocity do not coincide, the steepest gradient of pressure is slightly southward of $\pi_g=0$.

The asymmetry in the geostrophic current about the axis of the velocity maximum is expressed by the parameter δ, defined as

$$\delta=d^{+}-d^{-}, \tag{3.19}$$

where d^{+} is the magnitude of the distance between the axis of maximum velocity and $u_g=0$ on the positive (northward) side, and d^{-} is the corresponding distance on the negative (southward) side. δ as a function of r for F=0.1 and 0.2 appears in Fig. 3. Since $\delta>0$, the steepest gradient of positive geostrophic velocity occurs on the southward side.

Rossby (1938) showed that it is possible to derive an expression for the magnitude of the shift of the current from its original position by using the principle of conservation of absolute angular momentum following a parcel of fluid or its analog in the present model, $u-fy$. Integration of the dimensional form of (2.1) yields

$$u(y)-fy=u_i(y_i)-fy_i, \tag{3.20}$$

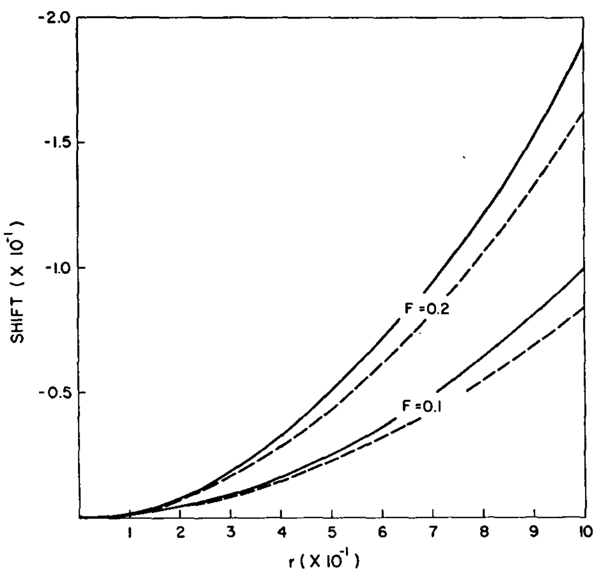

FIG. 2. Shift of the axes of the geostrophic velocity maximum (solid) and zero geostrophic pressure (dashed) as a function of the half-width r. Negative values represent a southward shift.

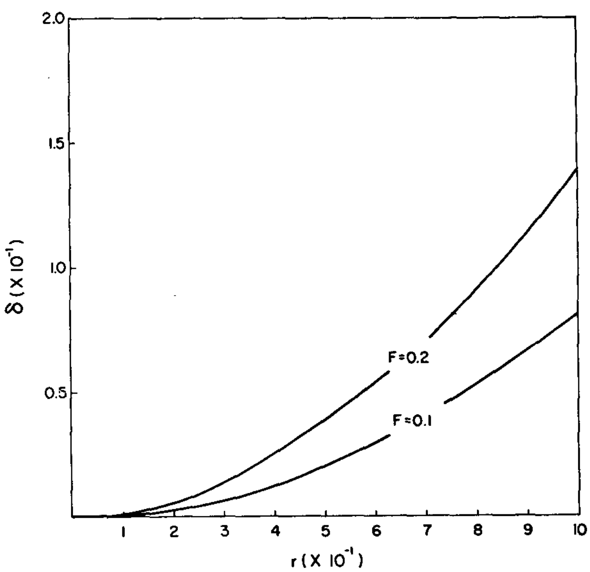

FIG. 3. The parameter δ, defined in the text, as a function of the half-width r.

where terms on the right-hand side of (3.20) have been defined following (1.1). Since the geostrophic state is eventually reached, the nondimensional form of (3.20) becomes

$$\mathrm{F}[u_i(y_i)-u_g(y)]=y_i-y. \qquad (3.21)$$

Our previous results have shown that the displacement of the velocity maximum is not greater than O(F), therefore we shall assume that

$$\left.\begin{array}{l} y_i-y=\mathrm{F}(y_i-y)^{(1)}+\mathrm{F}^2(y_i-y)^{(2)}+\ldots \\ u_i(y_i)=u_i(y)+\mathrm{F}(y_i-y)^{(1)}du_i(y)/dy+\ldots \end{array}\right\}. \qquad (3.22)$$

If (3.22) is substituted into (3.21) and only terms of O(F) are retained, we obtain

$$(y_i-y)^{(1)}=u_i(y)-u_{0g}(y). \qquad (3.23)$$

Thus

$$y_i-y\approx \mathrm{F}r^2e^{-\frac{1}{2}(y/r)^2}, \qquad (3.24)$$

using (2.12) and (3.6). The displacement of the velocity maximum, $\mathrm{F}r^2$, is determined by setting $y=0$ in (3.24). This expression describes the solid curve in Fig. 2 when $\mathrm{F}=0.1$, while the curve for $\mathrm{F}=0.2$ deviates slightly from $\mathrm{F}r^2$.

4. Time-dependent solution

A time-dependent solution of the zero-order equations (2.4)–(2.6), applied to a homogeneous incompressible ocean of finite depth, has been found by Cahn (1945). Cahn's linear solution for the transverse velocity component is

$$v_0(y,t)=-\frac{1}{2}\int_{-t}^{t} u_i(y-\alpha)J_0(\sqrt{t^2-\alpha^2})d\alpha, \qquad (4.1)$$

where J_0 denotes the zero-order Bessel function, and the initial velocity $u_i(y)$ must have finite total energy,

$$\int_{-\infty}^{\infty} |u_i(y)|^2dy<0, \qquad (4.2)$$

but is otherwise arbitrary. Cahn solved (4.1) with Rossby's initial condition and determined u_0 and π_0 from (2.4) and (2.6), respectively. Cahn also found that a parcel of fluid, originally at $y=0$, is displaced southward. However, an examination of Cahn's solution for $u_0(y,t)$ as $t\rightarrow\infty$ shows that the steady geostrophic current remains centered about the central axis of the initial current ($y=0$). The current itself is not displaced southward because absolute momentum, $u-fy$, following the parcel, is not conserved. As shown in the preceeding section the nonlinear terms must be taken into account in order to describe the shift of the current.

In this section nonlinear features associated with the adjustment process have been investigated by solving the zero- and first-order systems of Eqs. (2.4) through (2.9). These systems of equations yield the partial differential equations

$$\mathfrak{L}v_0=0, \qquad (4.3)$$

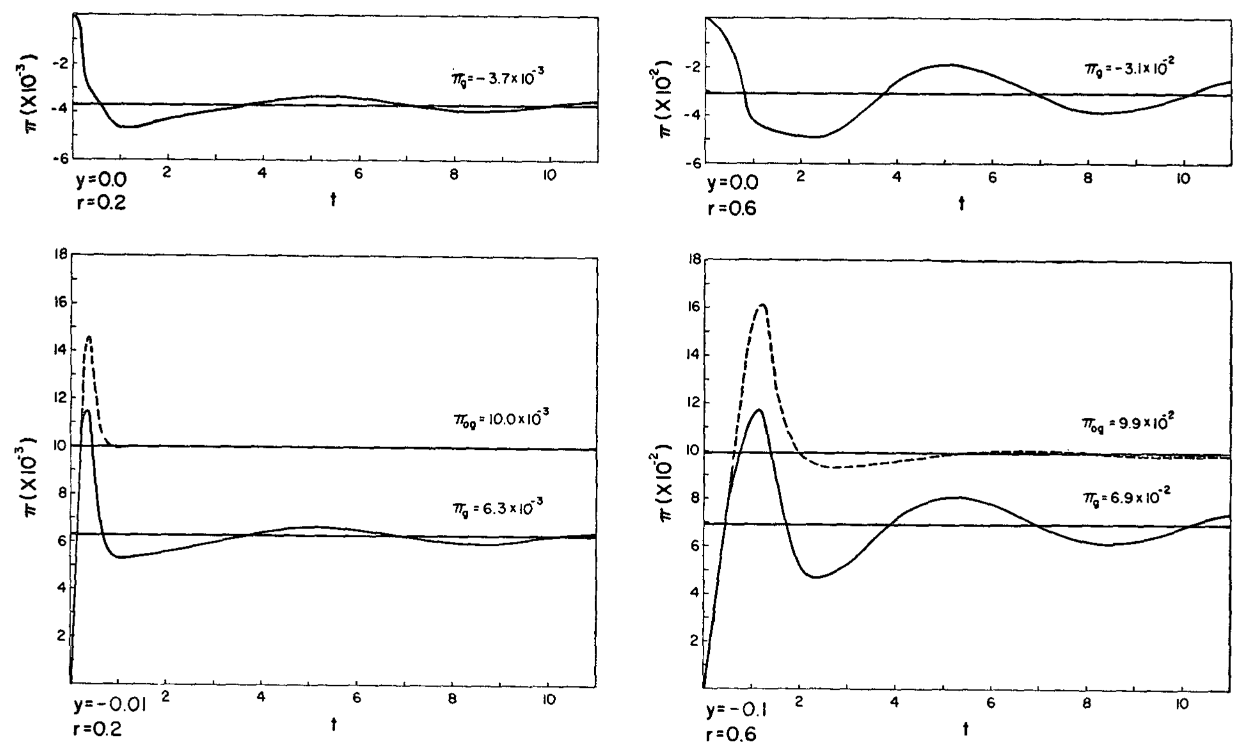

FIG. 4. The pressures π_0 (dashed) and $\pi_0+\mathrm{F}\pi_1$ (solid) as functions of time t for $\mathrm{F}=0.1$. The positions y and half-widths r appear below each diagram. The solid horizontal lines represent the indicated geostrophic pressures.

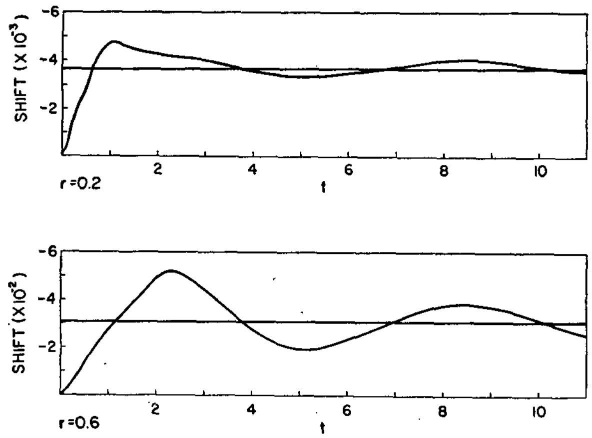

FIG. 5. Shift of the axis of $\pi(y,t)=0$ about its equilibrium position as a function of time t for $r=0.2$ and $r=0.6$.

and

$$\mathcal{L}v_1=-\left(\frac{\partial^2}{\partial t\partial y}v_0^2+u_0\frac{\partial v_0}{\partial y}\right), \qquad (4.4)$$

where $\mathcal{L}\equiv\partial^2/\partial t^2-\partial^2/\partial y^2+1$. For a given initial velocity field $u_i(y)$, the solution of (4.3) is given by (4.1). This solution may then be used to evaluate the right-hand of (4.4) and $v_1(y,t)$ may be found by standard analytical methods. However, for computational purposes it was found that the most practical method of solving (4.3) and (4.4) was by numerical integration. Centered time and space increments were used in the region $-13.5\leq y\leq 13.5$. The initial conditions for (4.3) and (4.4) were (2.10) and (2.11), respectively. The initial velocity $u_i(y)$ of (2.12) was specified in the region $-2.5\leq y\leq 2.5$ and the computations were terminated when the front of the wave, emanating from the initial region and moving at the maximum nondimensional group velocity $c=1$, reached the boundary. This was done to preclude energy reflection back into the region, i.e., to simulate an infinite domain. The variables $u_j(y,t)$ and $\pi_j(y,t)$ $(j=0,1)$ were then determined from the original system of zero- and first-order equations.

The power series solution $\pi\approx\pi_0+F\pi_1$ permits us to examine the linear and nonlinear effects separately. Fig. 4 shows the pressure π as a function of time at specific values of y for $r=0.2$ and 0.6. The point $y=0$ was chosen because $\pi(0,t)\approx F\pi_1(0,t)$, i.e., $\pi_0(0,t)=0$ for all time. The pressure variation occuring at $y=0$ reflects the shift of mass about the steady-state equilibrium pressure $\pi_g(0)$, determined from (3.17) and (3.15). This variation, a strictly nonlinear phenomenon, is brought about by the shift of the whole current system to a new equilibrium position southward of its initial position. Since $\chi\equiv[p(y,0,t)-\bar{p}]/\bar{p}=F\pi$, the maximum amplitudes of these pressure oscillations are about 0.1 and 2 mb for $r=0.2$ and $r=0.6$, respectively.

Rossby (1938) predicted that the current system would asymptotically undergo a damped inertia oscillation ($\tau_f=2\pi$) about its new equilibrium position. This prediction is verified by the computational results illustrated in Fig. 4 and the asymptotic solution presented in the Appendix, where it is also shown that the pressure amplitude decays like $t^{-\frac{1}{2}}$.

The lower diagrams in Fig. 4 illustrate the oscillations associated with the linear solution π_0 (dashed line) and the modification which occurs when the first-order solution $F\pi_1$ is added to it (solid line). The linear solution shows that the adjustment to geostrophic balance occurs in a relatively short time, especially when the initial current width is small compared to λ. However, when the first-order solution is taken into

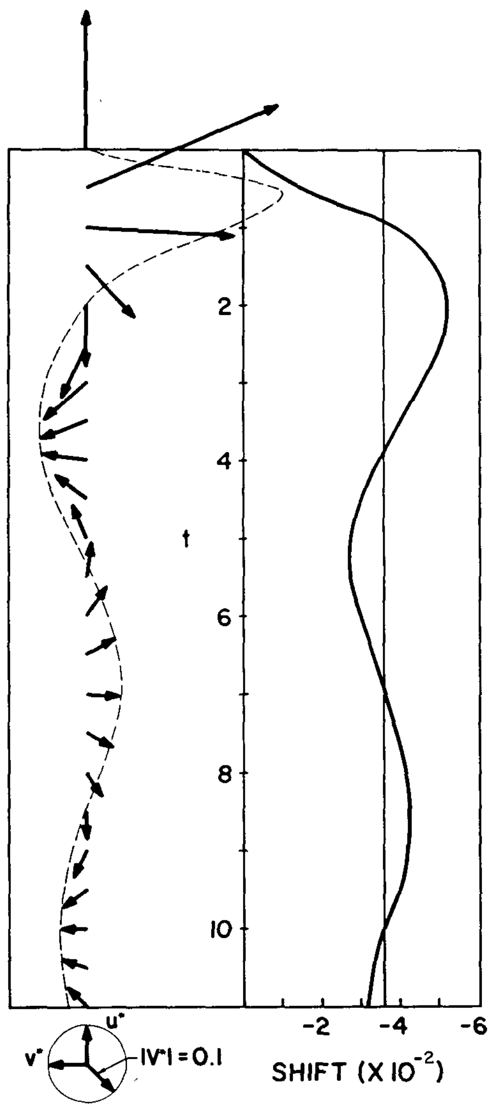

FIG. 6. Shift of the axis of maximum zonal velocity u as a function of time t (right-half) for $r=0.6$ and $F=0.1$. The arrows in the hodograph (left-half) represent the ageostrophic velocity, defined in the text, at the position of the zonal velocity maximum. v^* as a function of time t is represented by the dashed line. The magnitude of the ageostrophic velocity $|V^*|$ is indicated by the scale at the bottom of the diagram.

account, the oscillation of the pressure field is sustained for a relatively long time as the current system oscillates about its new equilibrium position. This effect may be seen quite clearly by comparing the pressure oscillations in Fig. 4 with the shift of the axis of $\pi(y,t)=0$ about its equilibrium position in Fig. 5.

The shift of the zonal velocity maximum about its equilibrium position as a function of time for $r=0.6$ is shown in the right half of Fig. 6. The starred variables appearing on the left side of the diagram are defined as

$$u^*(y',t)=u(y',t)-u_{g\max};\quad v^*(y',t)=v(y',t). \quad (4.5)$$

The variables are evaluated at the position y' of the zonal velocity maximum $u(y',t)$ and $u_{g\max}=1.006$ is the maximum value of the geostrophic velocity. The magnitude of the ageostrophic deviation $V^*=(u^{*2}+v^{*2})^{\frac{1}{2}}$, is indicated by the scale at the bottom of the diagram. The light dashed line shows the transverse velocity component v^* as a function of time. This latter curve has been drawn to enhance the clarity of the hodograph.

The period of the oscillations of both the position of the axis of the zonal velocity maximum and the magnitude of this maximum asymptotically approach the inertial period and decay like $t^{-\frac{1}{2}}$. It is noted that the phases of these oscillations are adjusted so that the absolute momentum $\mathrm{F}u-y'$, following the displacement of the zonal velocity maximum, is conserved; $u(y',t)$ attains its minimum and maximum values when the axis of $u(y',t)$ is displaced farthest southward and northward, respectively.

5. Concluding remarks

The present study has emphasized certain features of the geostrophic adjustment process which are basically nonlinear in character. Perhaps the most significant feature is the longer "adjustment time" necessary to reach an approximate state of geostrophic balance than is indicated by a linear analysis. The quasi-geostrophic approximation, often used in large-scale dynamical models, implies that the wind and pressure fields remain adapted continually. Bolin (1953) found that the adjustment process proceeds relatively slowly in a baroclinic fluid because internal wave dispersion is less rapid than barotropic (or external) wave dispersion. On this basis Bolin conjectured that the quasi-geostrophic approximation may be a poor approximation for some dynamical models. The present results provide another basis for questioning the usefulness of the approximation.

The sharper geostrophic velocity gradient found south of the axis of maximum velocity developed from an initial velocity field which is an even function of y. The significance of this result is not apparent. However, the magnitude of this asymmetry is relatively small for the range of Froude numbers considered here.

In view of the fundamental importance of the geostrophic adjustment process, it would be desirable if numerical integrations of (2.1) through (2.3) for $\mathrm{F}\gtrsim 0.2$ were carried out. A description of the transient features and asymmetries which develop, as the nonlinear terms become increasingly important, would be of definite interest.

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APPENDIX

Asymptotic Evaluation of $\pi_1(y,t)$

We shall assume that the asymptotic value of $\pi_1(y,t)$ is determined by the part of the forcing which is dominant when time t is large.

The initial velocity field $u_i(y)$ may be represented by

$$u_i(y)=\int_{-\infty}^{\infty} U_i(k)e^{iky}dk, \quad (A.1)$$

where $U_i(k)$ is the Fourier transform of $u_i(y)$ given by

$$U_i(k)=(2\pi)^{-\frac{1}{2}}r^3(1+k^2)e^{-r^2k^2/2}.$$

If (A.1) is substituted into (4.1) and use is made of the Fourier transform relation

$$\frac{\sin\sqrt{1+k^2}\,t}{\sqrt{1+k^2}}=\frac{1}{2}\int_{-t}^{t} J_0(\sqrt{t^2-\alpha^2})e^{-ik\alpha}d\alpha,$$

we obtain

$$v_0(y,t)=-\int_{-\infty}^{\infty} U_i(k)\frac{\sin\sqrt{1+k^2}\,t}{\sqrt{1+k^2}}e^{iky}dk. \quad (A.2)$$

The asymptotic value of v_0, determined by the method of stationary phase, is

$$v_0\approx r^3e^{-\frac{1}{2}(ry/t)^2}t^{-\frac{1}{2}}\sin(t+\pi/4). \quad (A.3)$$

Equivalently, if use is made of (2.4) and (2.10),

$$u_0\approx r^3e^{-\frac{1}{2}(ry/t)^2}t^{-\frac{1}{2}}\cos(t+\pi/4)+u_{0_g}(y), \quad (A.4)$$

when $y\ll t\rightarrow\infty$. Then, for large t, the dominant forcing will be due to terms which may be represented as the product of the geostrophic velocity with an ageostrophic velocity component. The product of two ageostrophic components decreases like t^{-1}. On the basis of the above assumption, (2.9) becomes

$$\partial\hat{\pi}_1/\partial t=-\partial\hat{v}_1/\partial y+u_{0_g}v_0, \quad (A.5)$$

where $\hat{v}_1(y,t)$ is determined from the simplified form of (4.4),

$$\mathcal{L}\hat{v}_1=-u_{0_g}\partial v_0/\partial y. \quad (A.6)$$

The capped variables represent the response to the forcing produced by the product of the geostrophic and ageostrophic zero-order velocities. If (A.2) and (3.6) are used to evaluate the right-hand side of (A.6), then it is possible to show that the solution of (A.6) yields $\partial\hat{v}_1/\partial y \sim O(r^2 u_{0g} v_0)$. Then, for small r (A.5) reduces to

$$\partial\hat{\pi}_1/\partial t \approx u_{0g} v_0. \qquad \text{(A.7)}$$

If we use (2.4), and take (2.10) and (2.11) into account, we obtain

$$\hat{\pi}_1 \approx u_{0g}(u_0 - u_i), \qquad \text{(A.8)}$$

where $u_i(y)$ is given by (2.12). When $t \rightarrow \infty$

$$\hat{\pi}_1 \rightarrow u_{0g}[r^3 e^{-\frac{1}{2}(ry/t)^2} t^{-\frac{1}{2}} \cos(t+\pi/4) + u_{0g} - u_i]. \qquad \text{(A.9)}$$

The total first-order pressure is $\pi_1(y,t) = \hat{\pi}_1(y,t)$ plus the error in neglecting the forcing due to terms arising from the product of ageostrophic velocity components and the term of order r^2. The error

$$\epsilon \equiv |(\pi_1 - \hat{\pi}_1)/\pi_1|$$

was evaluated at $y=0$ for $r=0.2$. If $u_0(y,t)$, appearing in (A.8), is evaluated either by (A.4) or by the values computed numerically, then $\epsilon_{max} < 0.1$ for all $t > 1$.

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