Governing Equations and Spectra for Atmospheric Motion and Transports in Frequency, Wave-Number Space

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ABSTRACT

The governing equations, power and cross spectra for the atmospheric motion, and transports in the frequency, wave-number space are derived. Discussions are made of the contributions of the nonlinear interactions of atmospheric waves in velocity and temperature fields to the conversion of kinetic and potential energies, and to the meridional transports of angular momentum and sensible heat in the atmosphere.

1. Introduction

The foundation of the statistical theory of turbulence lies in G. I. Taylor's pioneering work in 1935 and 1938 (Taylor, 1935, 1938) in which he introduced the correlation between velocities at two points as one of the quantities needed to describe the turbulence, and the Fourier transform of the correlation to obtain an energy spectrum function depicting the distribution of kinetic energy over various Fourier wave-number components of the turbulence. In recent years, a great deal of theoretical and experimental work on turbulence has been done. Comprehensive lists of literature on turbulence may be found in monographs by Batchelor and Townsend (Batchelor, 1953; Townsend, 1956).

Because of the complex nature of turbulent motion, mathematical treatment of turbulence has mostly been confined to the microscale under the assumptions that the field of turbulent motion is statistically homogeneous and isotropic. Indeed, analysis of microscale turbulent motion becomes greatly simplified, since the effects of Coriolis and pressure forces are comparatively small and may be neglected. In the large-scale motion in the atmosphere, however, the motion is not only affected by the presence of the pressure and Coriolis forces but complicated by the lack of homogeneity and isotropy in the field of turbulence. The best way of studying atmospheric turbulence is to analyze the motion in three-dimensional wave-number space. Such an analysis, though it proves to be most general, is extremely complex. However, the mean motion in the atmosphere and the movement of the large-scale atmospheric systems are primarily parallel to the latitude circles. These suggest that the large-scale atmospheric motion and transport processes may be examined in the longitude-time space. The purpose of this paper is to make such an analysis.

2. Power and cross spectra in wave-number, frequency space

One of the important studies of the large-scale atmospheric turbulence and transport processes is the analysis of the power and cross spectra of the turbulent motion and transports in the atmosphere. The former, which deals primarily with the kinetic, potential and internal energies, is basic to the understanding of the mechanism of turbulence. The latter, which concerns primarily the transport and conversion of energies, is fundamental in the maintenance of the general circulation in the atmsophere.

Studies of the longitude spectra (Benton and Kahn, 1958; Eliasen, 1958; Kao, 1954; Saltzman, 1957, 1958; Van Mieghem *et al.*, 1960) and the Lagrangian and Eulerian time spectra (Kao, 1962, 1965; Kao and Bullock, 1964) have recently been made. A comprehensive list of references of the analyses of the former may be found in a recent monograph by Van Mieghem (1961). These studies have provided a great deal of information regarding the contributions of the large-scale motion due either to the longitude- or time-eddies. However, to gain an insight into the contribution due to the motion of eddies of various sizes and frequencies, it is necessary to analyze the power and cross spectra of the large-scale atmospheric motion and transports in the frequency, wave-number space. In this section we shall develop a tool for such an analysis.

Let $q(\lambda,t)$ be a real, single-valued function, which is piecewise differentiable in a normalized domain, $0\leq\lambda$, $t\leq 2\pi$, where, λ and t stand, respectively, for the longitude and time. $q(\lambda,t)$ has a Fourier transform which may be written

$$Q(k,n)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} q(\lambda,t)e^{-i(k\lambda+nt)}d\lambda dt, \tag{1}$$

where Q is the complex coefficient and k and n are the wave number and frequency, respectively. The inverse transform of (1) gives $q(\lambda,t)$ expressed in terms of its complex coefficient as follows:

$$q(\lambda,t)=\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty}Q(k,n)e^{i(k\lambda+nt)}dn. \tag{2}$$

Here the summation of the complex coefficient Q with respect to the integer wave numbers is the consequence of the cyclic distribution of $q(\lambda,t)$ along latitude circles.

For convenience of computations in this study, we express

$$Q(k,n)=Q_r(k,n)+iQ_i(k,n), \tag{3}$$

where

$$Q_r(k,n)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}q(\lambda,t)\cos(k\lambda+nt)d\lambda dt$$

and

$$Q_i(k,n)=-\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}q(\lambda,t)\sin(k\lambda+nt)d\lambda dt$$

are the real and imaginary parts, respectively, of $Q(k,n)$.

It can easily be shown that

$$\left.\begin{aligned}Q^*(\pm k,\mp n)&=Q(\mp k,\pm n)\\ Q_r(\pm k,\mp n)&=Q_r(\mp k,\pm n),\\ Q_i(\mp k,\pm n)&=-Q_i(\pm k,\mp n)\\ Q(\pm k,\mp n)Q^*(\pm k,\mp n)&=Q_r^2(\pm k,\mp n)\\ &\quad+Q_i^2(\pm k,\mp n)=|Q(\pm k,\mp n)|^2\end{aligned}\right\}, \tag{4}$$

where Q^* denotes the conjugate of Q.

Consider the same conditions for another scalar function $s(\lambda,t)$ with a Fourier transform $S(k,n)$. It can be shown that for functions $s(\lambda,t)$ and $q(\lambda,t)$, we have

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}s(\lambda,t)q(\lambda,t)e^{-i(k\lambda+nt)}d\lambda dt=\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{+\infty}S(j,m)Q(k-j,\,n-m)dm, \tag{5}$$

where

$$S(j,m)=\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}s(\lambda,t)e^{-i(j\lambda+mt)}d\lambda dt.$$

Letting $k,\,n\rightarrow 0$, we have the generalized Parseval's formula,

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}s(\lambda,t)q(\lambda,t)d\lambda dt&=\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{+\infty}S(j,m)Q(-j,\,-m)dm,\\ &=\sum_{k=-\infty}^{+\infty}\int_{-\infty}^{+\infty}S(k,n)Q^*(k,n)dn.\end{aligned} \tag{6}$$

It can further be shown that

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}s(\lambda,t)q(\lambda,t)d\lambda dt&=\frac{1}{4}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty}[S(k,n)Q(-k,\,-n)+S(-k,\,-n)Q(k,n)\\ &\quad+S(k,\,-n)Q(-k,\,n)+S(-k,\,n)Q(k,\,-n)]dn.\end{aligned} \tag{7}$$

In view of the integrand of the right-hand side of the above equation being an even function, (7) may be written as

$$\begin{aligned}\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}s(\lambda,t)q(\lambda,t)d\lambda dt&=\int_0^{\infty}[S_r(0,n)Q_r(0,n)+S_i(0,n)Q_i(0,n)\\ &\quad+S_r(0,\,-n)Q_r(0,\,-n)+S_i(0,\,-n)Q_i(0,\,-n)]dn\\ &\quad+2\sum_{k=1}^{\infty}\int_0^{\infty}[S_r(k,n)Q_r(k,n)+S_i(k,n)Q_i(k,n)\\ &\quad+S_r(k,\,-n)Q_r(k,\,-n)+S_i(k,\,-n)Q_i(k,\,-n)]dn.\end{aligned} \tag{8}$$

Here relation (4) has been employed.

Denote the cross spectrum of $s(\lambda,t)$ and $q(\lambda,t)$ due to eddies of wave-number k and frequency n moving toward the direction of increasing and decreasing longitude, respectively, by

$$\left.\begin{aligned}E_{sq}(0\ \mp n)&=S_r(0,\,\mp n)Q_r(0,\,\mp n)\\ &\quad+S_i(0,\,\mp n)Q_i(0,\,\mp n)\\ E_{sq}(k,\,\mp n)&=2[S_r(k,\,\mp n)Q_r(k,\,\mp n)\\ &\quad+S_i(k,\,\mp n)Q_i(k,\,\mp n)]\end{aligned}\right\}, \tag{9}$$

for $k\neq 0$

The contribution of $s(\lambda,t)$ and $q(\lambda,t)$ integrated over a latitude circle and over a normalized time interval 2π may then be expressed in terms of the sum of $E_{sq}(k,n)$ and $E_{sq}(k,\,-n)$ integrated over the frequency and wave-number domain. Thus,

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}s(\lambda,t)q(\lambda,t)d\lambda dt=\sum_{k=0}^{\infty}\int_0^{\infty}[E_{sq}(k,n)+E_{sq}(k,\,-n)]dn. \tag{10}$$

The above equation may also be expressed as

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}s(\lambda,t)q(\lambda,t)d\lambda dt=\sum_{k=0}^{+\infty}\{E_{sq}(k,\,+)+E_{sq}(k,\,-)\}, \tag{11}$$

or

$$\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} s(\lambda,t)q(\lambda,t)d\lambda dt = \int_0^{+\infty}\{E_{sq}(+n)+E_{sq}(-n)\}dn, \quad (12)$$

where

$$E_{sq}(k,\mp)=\int_0^{\infty} E_{sq}(k,\mp n)dn$$

is the cross spectrum of $s(\lambda,t)$ and $q(\lambda,t)$ due to eddies of wave number k and all frequencies, moving, respectively, in the direction of increasing and decreasing longitude, and

$$E_{sq}(\mp n)=\sum_{k=0}^{+\infty} E_{sq}(k,\mp n)$$

is the cross spectrum due to eddies of frequency n and all wave numbers, moving, respectively, in the direction of increasing and decreasing longitude.

To compare the longitude-time power and cross spectra for different latitudes and time intervals, it is convenient to introduce the normalized cross spectra,

$$F_{sq}(k,\pm n) = \frac{2[S_r(k,\pm n)Q_r(k,\pm n)+S_i(k,\pm n)Q_i(k,\pm n)]}{\frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} s(\lambda,t)q(\lambda,t)d\lambda dt}, \quad k\neq 0,$$

(for $k=0$, the factor 2 in the numerator should be replaced by 1), and

$$\left.\begin{aligned} F_{sq}(k,\pm) &= \int_0^{+\infty} F_{sq}(k,\pm n)dn \\ F_{sq}(\pm n) &= \sum_{k=0}^{+\infty} F_{sq}(k,\pm n) \end{aligned}\right\}, \quad (13)$$

such that

$$\left.\begin{aligned} \sum_{k=0}^{+\infty}[F_{sq}(k,+)+F_{sq}(k,-)] &= 1 \\ \int_0^{+\infty}[F_{sq}(+n)+F_{sq}(-n)]dn &= 1 \end{aligned}\right\}. \quad (14)$$

For the analysis of the power spectrum of the scalar quantity $q(\lambda,t)$, the quantities $s(\lambda,t)$ and $S(k,\pm n)$ in the equations of this section should be replaced, respectively, by $q(\lambda,t)$ and $Q(k,\pm n)$.

Computations of the longitude-time power and cross spectra of the large-scale atmospheric motion and the meridional transports of angular momentum, kinetic and potential energies, etc., for the year 1964 have been made (Kao *et al.*, 1966). Analysis of these results, which provide information regarding the mechanism of the large-scale turbulence and transport in the atmosphere, will be presented in a later paper.

3. Equations for the large-scale atmospheric motion and transports in the longitude, latitude, pressure coordinate system

In the longitude λ, latitude φ, pressure p coordinate system, the equations of motion, the hydrostatics equation, the continuity equation, and the energy equation may be written as:

$$\left(\frac{\partial}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}+\omega\frac{\partial}{\partial p}\right)u - \left(f+u\frac{\tan\varphi}{a}\right)v = -\frac{g}{a\cos\varphi}\frac{\partial z}{\partial\lambda}+F_1, \quad (15)$$

$$\left(\frac{\partial}{\partial t}+\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}+\omega\frac{\partial}{\partial p}\right)v + \left(f+u\frac{\tan\varphi}{a}\right)u = -\frac{g}{a}\frac{\partial z}{\partial\varphi}+F_2, \quad (16)$$

$$\frac{\partial z}{\partial p}+\frac{R}{g}\frac{T}{p}=0, \quad (17)$$

$$\frac{\partial\omega}{\partial p}+\frac{1}{a\cos\varphi}\left(\frac{\partial u}{\partial\lambda}+\frac{\partial v\cos\varphi}{\partial\varphi}\right)=0, \quad (18)$$

$$c_p\left(\frac{\partial}{\partial\lambda}+\frac{u}{a\cos\varphi}\frac{\partial}{\partial\lambda}+\frac{v}{a}\frac{\partial}{\partial\varphi}+\omega\frac{\partial}{\partial p}\right)T=R\frac{\omega T}{p}-h, \quad (19)$$

where a is the radial distance from the center of the earth, f the Coriolis parameter, g the acceleration of gravity, z the height of isobaric surfaces, ω the individual rate of change of pressure, T the temperature, R the gas constant, h the rate of heat addition per unit mass, u and v the longitudinal and meridional component of the velocity, respectively, and F_1 and F_2 the corresponding longitudinal and meridional component of the frictional force. For large-scale atmospheric motion, F_1 and F_2 represent the sum of molecular frictional force and the Reynolds stress force due to eddies of high frequencies.

In the study of the maintenance of the general circulation in the atmosphere, we are particularly interested in the local rate of change of the kinetic and internal energies, and the rates of the meridional flux of sensible heat and angular momentum. With the use of Eqs. (15)–(19), they can be shown to be

$$\frac{\partial}{\partial t}\left[\frac{u^2+v^2}{2}\right]=\frac{-1}{a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}[u(u^2+v^2)]+\frac{\partial}{\partial\varphi}[u(u^2+v^2)\cos\varphi]\right\}$$

$$-\frac{\partial}{\partial p}[\omega(u^2+v^2)]-\frac{g}{a}\left[\frac{u}{\cos\varphi}\frac{\partial z}{\partial\lambda}+v\frac{\partial z}{\partial\varphi}\right]-(uF_1+vF_2), \quad (20)$$

$$\frac{\partial}{\partial t}(c_VT)=\frac{-c_V}{a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}(uT)+\frac{\partial}{\partial\varphi}(vT\cos\varphi)\right\}+\frac{c_V}{c_p}\alpha\omega+\frac{c_V}{c_p}h, \quad (21)$$

$$\frac{\partial}{\partial t}(c_pvT)=\frac{-c_p}{a\cos\varphi}\left\{\frac{\partial}{\partial\lambda}(uvT)+\frac{\partial}{\partial\varphi}(v^2T\cos\varphi)\right\}$$

$$-c_p\frac{\partial}{\partial p}(wvT)+\frac{R}{p}\omega vT-\left(f+u\frac{\tan\varphi}{a}\right)c_puT-\frac{g}{a}c_pT\frac{\partial z}{\partial\varphi}+vh+c_pTF_2, \quad (22)$$

$$\frac{\partial}{\partial t}(vua\cos\varphi)=-\left\{\frac{\partial}{\partial\lambda}(u^2v)+\frac{\partial}{\partial\varphi}(v^2u\cos\varphi)\right\}$$

$$-a\cos\varphi\frac{\partial}{\partial p}(\omega vu)-a\cos\varphi\left\{\left(f+u\frac{\tan\varphi}{a}\right)(u^2-v^2)+\frac{g}{a}\left(\frac{v}{\cos\varphi}\frac{\partial z}{\partial\lambda}+u\frac{\partial z}{\partial\varphi}\right)-(vF_1+uF_2)\right\}. \quad (23)$$

These equations of transports will be compared with those transformed to the frequency, wave-number space in the next section.

4. Governing equations for the large-scale atmospheric motion and transports in the wave-number, frequency space

To transform the governing equations for the large-scale atmospheric motion and transports to the wave-number, frequency space, we introduce the following notations for the Fourier coefficients of the quantities used in this study:

$q(\lambda,t,p,\varphi)$	u	v	ω	z	T	h	F_1	F_2
$Q(k,n,p,\varphi)$	U	V	W	Z	θ	H	G_1	G_2

Applying the Fourier transform formula (1) to the zonal component of the equation of motion (15), we obtain the complex coefficient of the zonal component of the velocity, i.e.,

$$U(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}jU(j,\pm m)U(k-j,\pm n\mp m)+\frac{1}{a}U_\varphi(j,\pm m)V(k-j,\pm n\mp m)\right.$$

$$\left.+U_p(j,\pm m)W(k-j,\pm n\mp m)-\frac{\tan\varphi}{a}U(j,\pm m)V(k-j,\pm n\mp m)\right\}dm$$

$$\mp\frac{1}{n}\left\{\frac{g}{a\cos\varphi}kZ(k,\pm n)+ifV(k,\pm n)+iG_1(k,\pm n)\right\}, \quad (24)$$

where the subscripts represent the variables with respect to which the partial differentiations have been taken. On the right-hand side of Eq. (24), the first term represents the contribution to the complex coefficient of the zonal component of the velocity due to the nonlinear interaction of the waves of the zonal velocity, the second term represents the contribution due to the interaction between the waves of the zonal and meridional components of the velocity, and the third term represents the contribution due to the interaction of the zonal and vertical components of the velocity. The fourth and sixth terms represent, respectively, the effect of the earth's curvature and rotation, whereas the fifth term represents the contribution of the pressure waves. The last term gives the effect of the frictional force in the atmosphere; for large-scale atmospheric motion, it represents the complex coefficient of the Fourier transform of the molecular frictional force and the Reynolds stress force due to eddies of high frequency, which is nonlinear in nature.

Eq. (24) may be used to analyze the linear and nonlinear effects of waves of various wavelength and frequency on the zonal component of the velocity. It may be noted that the first and fifth terms of the right-hand side of (24) are weighted by kn^{-1}, whereas the rest of the terms of the right-hand side have a factor n^{-1}. This indicates that waves of small frequency would generally contribute more to the zonal component of the velocity. As a by-product, Eq. (24) may be used to evaluate the Fourier coefficient of the frictional forces.

Applying the Fourier transform formula to Eqs. (16), (17), (18) and (19), we obtain, respectively, the complex coefficient of the meridional component of the velocity, the height of the pressure surface, the individual rate of change of the pressure, and the temperature as follows:

$$V(k,\pm n)=\pm\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}jV(j,\pm m)U(k-j,\pm n\mp m)+\frac{1}{a}V_\varphi(j,\pm m)V(k-j,\pm n\mp m)\right.$$
$$\left.+V_p(j,\pm m)W(k-j,\pm n\mp m)+\frac{\tan\varphi}{a}U(j,\pm m)U(k-j,\pm n\mp m)\right\}dm$$
$$\pm\frac{i}{n}\left\{\frac{g}{a}Z_\varphi(k,\pm n)+fU(k,\pm n)-G_2(k,\pm n)\right\},\quad(25)$$

$$Z_p(k,\pm n)=-\frac{R}{c_pp}\theta(k,\pm n),\quad(26)$$

$$W_p(k,\pm n)=-\frac{1}{a\cos\varphi}\{ikU(k,\pm n)+V_\varphi(k,\pm n)-V(k,\pm n)\sin\varphi\},\quad(27)$$

$$\theta(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}j\theta(j,\pm m)U(k-j,\pm n\mp m)+\frac{1}{a}\theta_\varphi(j,\pm m)V(k-j,\pm n\mp m)\right.$$
$$\left.+\theta_p(j,\pm m)W(k-j,\pm n\mp m)-\frac{R}{c_pp}\theta(j,\pm m)W(k-j,\pm n\mp m)\right\}dm\mp\frac{i}{n}\frac{1}{c_p}H(k,\pm n).\quad(28)$$

The above equations will be used to analyze the kinematic, dynamic and thermodynamic contributions to velocity and temperature fields in the frequency, wave-number space.

One of the objectives of this study is to analyze the contribution of the large-scale atmospheric motion to the kinetic energy, the rate of the meridional transports of sensible heat and angular momentum, and the available potential energy in the atmosphere. To do so, spectra of $\frac{1}{2}(u^2+v^2)$, vT, $vua\cos\varphi$ and ωT need to be computed. They can, respectively, be shown to be

$$E_{\frac{1}{2}(u^2+v^2)}(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}jU(k-j,\pm n\mp m)[U(j,\pm m)U(-k,\mp n)+V(j,\pm m)V(-k,\mp n)]\right.$$
$$+\frac{1}{a}V(k-j,\pm n\mp m)[U_\varphi(j,\pm m)U(-k,\mp n)+V_\varphi(j,\pm m)V(-k,\mp n)]$$
$$+W(k-j,\pm n\mp m)[U_p(j,\pm m)U(-k\mp n)+V_p(j,\pm m)V(-k,\mp n)]$$
$$\left.+\frac{\tan\varphi}{a}U(j,\pm m)[V(-k,\mp n)U(k-j,\pm n\mp m)-U(-k,\mp n)V(k-j,\pm n\mp m)]\right\}dm$$
$$\pm\frac{1}{n}\left\{\frac{g}{a}[iZ_\varphi(k,\pm n)V(-k,\mp n)-kZ(k,\pm n)U(-k,\mp n)]\right.$$
$$+if[U(k,\pm)V(-k,\mp n)-U(-k,\mp n)V(k,\pm n)]$$
$$\left.-i[U(-k,\mp n)G_1(k,\pm n)+V(-k,\mp n)G_2(k,\pm n)]\right\},\quad(29)$$

$$E_{vT}(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}U(k-j,\pm n\mp m)\left[jV(j,\pm m)\theta(-k,\mp n)\right.\right.$$
$$\left.+jV(-k,\mp n)\theta(j,\pm m)+\frac{\tan\varphi}{a}\theta(-k,\mp n)U(j,\pm m)\right]$$
$$+\frac{1}{a}V(k-j,\pm n\mp m)[V_\varphi(j,\pm m)\theta(-k,\mp n)+V(-k,\mp n)\theta_\varphi(j,\pm m)]$$
$$+W(k-j,\pm n\mp m)\left[V_p(j,\pm m)\theta(-k,\mp n)+V(-k,\mp n)\theta_p(j,\pm m)\right.$$
$$\left.\left.-\frac{R}{c_p p}V(-k,\mp n)\theta(j,\pm m)\right]\right\}dm\pm\frac{i}{n}\left\{\theta(-k,\mp n)\left[\frac{g}{a}Z_\varphi(k,\pm n)+fU(k,\pm n)-G_2(k,\pm n)\right]\right.$$
$$\left.-\frac{1}{c_p}V(-k,\mp n)H(k,\pm n)\right\},\quad(30)$$

$$E_{vu}(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}jU(k-j,\pm n\mp m)[V(j,\pm m)U(-k,\mp n)+V(-k,\mp n)U(j,\pm m)]\right.$$
$$+\frac{1}{a}V(k-j,\pm n\mp m)[V_p(j,\pm m)U(-k,\mp n)+U_p(j,\pm m)V(-k,\mp n)]$$
$$+\frac{\tan\varphi}{a}U(j,\pm m)[U(-k,\mp n)U(k-j,\pm n\mp m)-V(-k,\mp n)V(k-j,\pm n\mp m)]$$
$$\pm\frac{i}{n}\left\{\frac{g}{a}\left[\frac{i}{\cos\varphi}kZ(k,\pm n)V(-k,\mp n)+Z_\varphi(k,\pm n)U(-k,\mp n)\right]\right.$$
$$+f[U(k,\pm n)U(-k,\mp n)-V(k,\pm n)V(-k,\mp n)]$$
$$\left.-G_1(k,\pm n)V(-k,\mp n)+G_2(k,\pm n)U(-k,\mp n)\right\},\quad(31)$$

$$E_{\omega T}(k,\pm n)=\pm\frac{i}{n}\sum_{j=-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{i}{a\cos\varphi}j[W(k,\pm n)\theta(-j,\mp m)U(-k+j,\mp n\pm m)\right.$$
$$+W(-k,\mp n)\theta(j,\pm m)U(k-j,\pm n\mp m)]$$
$$+\frac{1}{a}[W(-k,\mp n)\theta_\varphi(j,\pm m)V(k-j,\pm n\mp m)-W(k,\pm n)\theta_\varphi(-j,\mp m)V(-k+j,\mp n\pm m)]$$
$$+W(-k,\mp m)W(k-j,\pm n\mp m)\left[\theta_p(j,\pm m)-\frac{R}{c_p p}\theta(j,\pm m)\right]$$
$$\left.-W(k,\pm m)W(-k+j,\mp n\pm m)\left[\theta_p(-j,\mp m)-\frac{R}{c_p p}\theta(-j,\mp m)\right]\right\}dm$$
$$\mp\frac{i}{c_p n}\{W(-k,\mp n)H(k,\pm n)-W(k,\pm n)H(-k,\mp n)\}.\quad(32)$$

Studies of the linear and nonlinear effects of the velocity and temperature fields on the kinetic and internal energies, and the meridional transports of sensible heat and angular momentum in the wave-number, frequency space are being made. The results of these studies will be reported in a later paper.

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