Relative Dispersion of Particles in a Stratified Rotating Atmosphere

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ABSTRACT

An analysis of the kinetics and dynamics of the relative dispersion of particles in a stratified rotating fluid is made. The expressions for the relative displacement tensor, and the power- and cross-spectra of the relative velocity are derived. Their characteristics for large and small diffusion times are examined. The governing equations for the motion of marked fluid particles are separated into two sets of equations, one governing the motion of the center of mass and the other governing the motion of individual particles relative to the center of mass. Discussions of the concentration distribution in clusters of marked fluid particles are made. A turbulent diffusion model is constructed for the estimate of the effects of thermal stratification and rotation on the dispersion of particles in the atmosphere.

1. Introduction

Study of the characteristics of relative particle displacement is fundamental in the understanding of particle dispersion in instantaneous sources. Relative diffusion has been investigated by many investigators (Roberts, 1923; Richardson, 1926; Sutton, 1932; Batchelor, 1952; Frenkiel and Katz, 1956; Kellogg, 1956; Gifford, 1957; Matchta *et al.*, 1957; Lin, 1960; Smith and Hay, 1961; Corrsin, 1962; Kao, 1962a,b; Mesinger, 1965; Kao and Wendell, 1967; Kao and Al-Gain, 1968). These studies were primarily concerned with the kinematics aspect of the dispersion. Indeed, diffusion in a field of homogeneous and stationary turbulence can be regarded as almost wholly a kinematic problem, depending only indirectly on the dynamics of the turbulent motion.

In the atmosphere, however, the motion of fluid particles is not only affected by the pressure and Coriolis forces but complicated by the lack of homogeneity and stationarity in the field of turbulence. Several analyses have been made of the characteristics of particle oscillations and trajectories in a rotating viscous fluid (Kao, 1962a,b, 1965). These studies have provided some information regarding the effects of pressure and Coriolis forces on the motion of fluid particles in the atmosphere. However, in these studies models are constructed under the assumptions that the fluid is neutrally stratified and that the geostrophic velocity is uniform. In the atmosphere, inversion layers frequently occur in the lower level, and velocity shear generally exists near the jet streams and frontal zones. To analyze the dispersion of fluid particles in the atmosphere it is necessary to take into account the effects of thermal stratification and shear flow in the atmosphere. The purpose of this paper is to make such a study.

2. Kinematics analysis

In problems of relative diffusion, we are dealing with the motion of a cluster of marked fluid particles. For simplicity in the analysis we assume that the marked fluid particles are hydrodynamically indistinguishable from a normal fluid particle. To analyze the motion of such a cluster of marked fluid particles, we shall make use of a well-known theorem in mechanics that the motion of a system of particles may be resolved into the motion of the center of mass of the system of particles and the motion of individual particles relative to the center of mass.

Let the coordinates of the trajectory of the center of mass of a cluster of N marked fluid particles be

$$x_{m,j}(t)=\frac{1}{N}\sum_{i=1}^{N} x_j(a_i,b_i,c_i,t), \quad j=1, 2, 3, \tag{1}$$

averaged over the marked particles. In the above equation a_i, b_i and c_i are the initial coordinates of particle i.

The coordinates of particle i relative to the center of mass may therefore be expressed as

$$x_{r,j}(a_i,b_i,c_i,t)=x_j(a_i,b_i,c_i,t)-x_{m,j}(t). \tag{2}$$

It is obvious that the relative motion of the fluid particles determines the change in the shape of the cluster of marked particles and the exchange of a transferable property with the surroundings. The motion of the center of mass of the cluster of marked fluid particle is independent of the shape of the cluster of fluid particles. The advantage in the treatment of relative dispersion of particles is that it automatically takes into account the effects of velocity shear and rotation of the fluid. The rate of particle separation generally increases with increasing distance of particle separation. However, recent analyses (Kao and Al-Gain, 1968) indicate that the rate of separation varies little, about 5 m sec^{-1}, for large-scale diffusion of cluster size of magnitude 1000 km, which indicates a quasi-stationary process may be assumed for the large-scale relative diffusion in the atmosphere.

We attempt in this section to describe the statistical behavior of an ensemble of marked fluid particles in terms of $\overline{x_{ri}(t)x_{rj}(t)}$, the ensemble average of the correlation of the ith and jth components of the particle coordinate relative to the center of mass. To this end we denote the ith component of particle velocity relative to the center of mass by $v_{ri}(t)=dx_{ri}/dt$, and define the correlation function of the particle velocity relative to the center of mass as

$$R_{v_{ri}v_{rj}}(\tau)=\frac{\overline{v_{ri}(t)v_{rj}(t+\tau)}}{(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}}.$$

For simplicity we assume that the dispersion process is statistically stationary and random; thus,

$$R_{v_{ri}v_{rj}}(\tau)=R_{v_{rj}v_{ri}}(-\tau).$$

It can be shown by extending the analysis of absolute displacement of particles (Taylor, 1921) to relative particle displacement that

$$\overline{x_{ri}(t)x_{rj}(t)}=(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}\times\int_0^t\int_0^\eta[R_{v_{ri}v_{rj}}(\tau)+R_{v_{rj}v_{ri}}(\tau)]d\tau d\eta+\overline{x_{ri}(0)x_{rj}(0)},\quad(3)$$

which can be expressed as

$$\overline{x_{ri}(t)x_{rj}(t)}=(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}\times\int_0^t(t-\tau)[R_{v_{ri}v_{rj}}(\tau)+R_{v_{rj}v_{ri}}(\tau)]d\tau+\overline{x_{ri}(0)x_{rj}(0)}.\quad(4)$$

It may be noted that for small diffusion times, $R_{v_{ri}v_{rj}}(\tau)\approx R_{v_{ri}v_{rj}}(0)$; thus,

$$\overline{x_{ri}(t)x_{rj}(t)}=(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}R_{v_{ri}v_{rj}}(0)t^2+\overline{x_{ri}(0)x_{rj}(0)},\quad(5)$$

while for large diffusion times,

$$\overline{x_{ri}(t)x_{rj}(t)}=2(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}T_{rij}t+\overline{x_{ri}(0)x_{rj}(0)},\quad(6)$$

where

$$T_{rij}=\frac{1}{2}\int_0^\infty[R_{v_{ri}v_{rj}}(\tau)+R_{v_{rj}v_{ri}}(\tau)]d\tau$$

is a Lagrangian relative time scale.

It can be shown that the cross spectrum of the relative velocity takes the form

$$E_{v_{ri}v_{rj}}(n)=2(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}\times\int_0^\infty[R_{v_{ri}v_{rj}}(\tau)+R_{v_{rj}v_{ri}}(\tau)]\cos 2\pi n\tau d\tau,\quad(7)$$

where n is the frequency. By substituting the inverse transform of Eq. (7),

$$R_{v_{ri}v_{rj}}(\tau)+R_{v_{rj}v_{ri}}(\tau)=\frac{2}{(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}}\int_0^\infty E_{v_{ri}v_{rj}}(n)\cos 2\pi n\tau\, dn,\quad(8)$$

into (4), and integrating with respect to τ, we obtain

$$\overline{x_{ri}(t)x_{rj}(t)}=t^2\int_0^\infty E_{v_{ri}v_{rj}}(n)\left(\frac{\sin\pi nt}{\pi nt}\right)^2dn+\overline{x_{ri}(0)x_{rj}(0)}.\quad(9)$$

For small diffusion times, (9) may be approximated by

$$\overline{x_{ri}(t)x_{rj}(t)}\approx t^2\int_0^\infty E_{v_{ri}v_{rj}}(n)dn+\overline{x_{ri}(0)x_{rj}(0)}.\quad(10)$$

Thus, at the initial stage of diffusion the separation of particles is affected by turbulent motion of all scales.

For large diffusion times, (9) may be approximated by

$$\overline{x_{ri}(t)x_{rj}(t)}\approx\tfrac{1}{2}E_{v_{ri}v_{rj}}(0)t+\overline{x_{ri}(0)x_{rj}(0)}.\quad(11)$$

Thus, for large diffusion times the separation of particles is primarily affected by the turbulent motion of large scale.

We define a time dependent turbulence diffusivity

$$D_{ij}(t)=\frac{1}{2}\frac{d}{dt}\overline{x_{ri}(t)x_{rj}(t)}.\quad(12)$$

For small diffusion times, substitution of (10) and (5) into (12) gives

$$D_{ij}(t)=t\int_0^\infty E_{v_{ri}v_{rj}}(n)dn=(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}t,\quad(13)$$

while for large diffusion times [substitution of (6) and (11) into (12)], we have

$$D_{ij}(t)=\tfrac{1}{4}E_{v_{ri}v_{rj}}(0)=(\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}}T_{rij}.$$

Therefore, only for large diffusion times does turbulence diffusivity become a constant.

It can be shown that the variances of the relative displacement and velocity between particles are equal to twice those relative to the center of mass. Therefore, the above results apply also to relative dispersion between particles.

3. Concentration distribution

To obtain the distribution of marked fluid particles relative to their center of mass, let (x_r', y_r', z_r') be a Cartesian coordinate system of which the coordinate axes coincide with the principal axes of diffusion. It can be shown that the following Gaussian expression for the distribution of the concentration of marked fluid particles,

$$C(x_r',y_r',z_r',t)=\frac{S}{(2\pi)^{\frac{3}{2}}[\overline{x_r'^2(t)}\ \overline{y_r'^2(t)}\ \overline{z_r'^2(t)}]^{\frac{1}{2}}}\times\exp\left\{-\frac{1}{2}\left[\frac{x_r'^2}{\overline{x_r'^2(t)}}+\frac{y_r'^2}{\overline{y_r'^2(t)}}+\frac{z_r'^2}{\overline{z_r'^2(t)}}\right]\right\}, \quad (14)$$

satisfies the diffusion equation

$$\frac{\partial C}{\partial t}=D_{x'x'}(t)\frac{\partial^2 C}{\partial x_r'^2}+D_{y'y'}(t)\frac{\partial^2 C}{\partial y_r'^2}+D_{z'z'}(t)\frac{\partial^2 C}{\partial z_r'^2}, \quad (15)$$

where S is the source strength, and

$$D_{x_i'x_i'}(t)=\frac{1}{2}\frac{d}{dt}\overline{x_{ri}'^2(t)} \quad (16)$$

is the turbulence diffusivity.

It may be noted that in (14), for small diffusion times,

$$\overline{x_{ri}'^2(t)}\approx\overline{v_{ri}'^2}t^2+\overline{x_{ri}'^2(0)},$$

and, for large diffusion times,

$$\overline{x_{ri}'^2(t)}\approx 2K_{x'x'}t+\overline{x_{ri}'^2(0)}.$$

In the atmosphere the z_r' axis is generally perpendicular to the isentropic surface which is practically horizontal outside the frontal zones. In practice, z_r' may be replaced by z_r.

To express the concentration distribution (14) as function of x_r, y_r, z_r, and t, let α be the angle between the x_r and x_r' axes, measured in the counterclockwise direction. We have

$$\left.\begin{aligned} x_r' &= x_r\cos\alpha+y_r\sin\alpha\\ y_r' &= -x_r\sin\alpha+y_r\cos\alpha\end{aligned}\right\}. \quad (17)$$

Since, by definition,

$$\overline{x_r'(t)y_r'(t)}=0, \quad (18)$$

we obtain

$$[\overline{x_r^2(t)}-\overline{y_r^2(t)}]\sin\alpha\cos\alpha+\overline{x_r(t)y_r(t)}(\sin^2\alpha-\cos^2\alpha)=0.$$

Therefore,

$$\alpha(t)=\frac{1}{2}\tan^{-1}\left(\frac{2\overline{x_r(t)y_r(t)}}{\overline{x_r^2(t)}-\overline{y_r^2(t)}}\right), \quad (19)$$

and

$$\left.\begin{aligned}\overline{x_r'^2(t)}&=\overline{x_r^2(t)}+\overline{x_r(t)y_r(t)}\tan\alpha(t)\\ \overline{y_r'^2(t)}&=\overline{y_r^2(t)}-\overline{x_r(t)y_r(t)}\tan\alpha(t)\end{aligned}\right\}. \quad (20)$$

Using (17), (19) and (20), we can express the concentration distribution (14) as a function of x_r, y_r, z_r and t.

Since $D_{ij}(t)$ is a tensor of second order, its relation with the turbulence diffusivity $D_{i''j''}(t)$ in a Cartesian coordinate system (x'',y'',z'') is therefore

$$D_{j''k''}=D_{ij}c_{ij''}c_{jk''},$$

where $c_{ij''}$ and $c_{jk''}$ are, respectively, the direction cosines between the i and j'' axes and between the j and k'' axes. Let (x'',y'',z'') be so oriented that $D_{x''y''}=0$, and let α_D be the angle between the x and x'' axes. It can be shown that

$$\alpha_D(t)=\frac{1}{2}\tan^{-1}\left(\frac{D_{xy}(t)}{D_{xx}(t)-D_{yy}(t)}\right), \quad (21)$$

and

$$D_{x''x''}=D_{xx}+D_{xy}\tan\alpha_D,$$
$$D_{y''y''}=D_{yy}-D_{xy}\tan\alpha_D.$$

Therefore,

$$\alpha_D=\frac{1}{2}\tan^{-1}\left\{\tan 2\alpha+\frac{(\overline{x_r^2}-\overline{y_r^2})}{\frac{d}{dt}(\overline{x_r^2}-\overline{y_r^2})}\frac{d}{dt}\tan 2\alpha\right\}.$$

It can be seen that $\alpha_0=\alpha$ when $d\alpha/dt=0$.

4. Dynamics analysis

In the analysis of the dispersion of marked fluid particles, the Lagrangian system is more natural for describing the motion. The equations of motion for a fluid particle in the atmosphere may be written as

$$\frac{d^2x}{dt^2}-f_z\frac{dy}{dt}+f_y\frac{dz}{dt}=-f_zv_g+F_x, \quad (22)$$

$$\frac{d^2y}{dt^2}+f_z\frac{dx}{dt}=f_zu_g+F_y, \quad (23)$$

$$\frac{d^2z}{dt^2}-f_y\frac{dx}{dt}=-\frac{1}{\rho}\frac{\partial p}{\partial z}-g+F_z, \quad (24)$$

where f_y and f_z are, respectively, twice the y and z component of the angular velocity of the earth, u_g and v_g are, respectively, the x and y component of the geostrophic velocity, and F_x, F_y and F_z are the x, y and z component of the frictional force per unit mass.

By substituting (2) into (22), (23) and (24), we obtain

$$\frac{d^2x_m}{dt^2}+\frac{d^2x_r}{dt^2}-f_z\left(\frac{dy_m}{dt}+\frac{dy_r}{dt}\right)+f_y\left(\frac{dz_m}{dt}+\frac{dz_r}{dt}\right)=-f_zv_g+F_x, \quad (25a)$$

$$\frac{d^2y_m}{dt^2}+\frac{d^2y_r}{dt^2}+f_z\left(\frac{dx_m}{dt}+\frac{dx_r}{dt}\right)=f_zu_g+F_y, \quad (25b)$$

$$\frac{d^2z_m}{dt^2}+\frac{d^2z_r}{dt^2}-f_y\left(\frac{dx_m}{dt}+\frac{dx_r}{dt}\right)=-\Gamma z_r+F_z, \quad (25c)$$

where $\Gamma=g\theta^{-1}\partial\theta/\partial z$. In the above equation the Boussinesq approximation has been employed.

Taking means over the marked fluid particles, we obtain the equations of motion for the center of mass of the fluid particles, i.e.,

$$\frac{d^2x_m}{dt^2}-f_z\left(\frac{dy_m}{dt}-\bar{v}_g\right)+f_y\frac{dz_m}{dt}=\bar{F}_x, \quad (26a)$$

$$\frac{d^2y_m}{dt^2}+f_z\left(\frac{dx_m}{dt}-\bar{u}_g\right)=\bar{F}_y, \quad (26b)$$

$$\frac{d^2z_m}{dt^2}+f_y\frac{dx_m}{dt}=\bar{F}_z, \quad (26c)$$

where the overbars denote the mean over the marked fluid particles. It may be noted that the acceleration of the center of mass may be neglected if the mean frictional force is negligibly small and the difference between velocity of the center of mass and the mean geostrophic velocity is small. In this case, the trajectory of the center of mass may be predicted by the geostrophic wind.

Subtracting Eqs. (26) from Eqs. (25), we obtain the equations of motion for fluid particles relative to the center of mass; thus,

$$\frac{d^2x_r}{dt^2}-f_z\left(\frac{dy_r}{dt}-v_{gr}\right)+f_y\frac{dz_r}{dt}=F_{xr}, \quad (27a)$$

$$\frac{d^2y_r}{dt^2}+f_z\left(\frac{dx_r}{dt}-u_{gr}\right)=F_{yr}, \quad (27b)$$

$$\frac{d^2z_r}{dt^2}-f_y\frac{dx_r}{dt}+\Gamma z_r=F_{zr}, \quad (27c)$$

where

$$\left.\begin{aligned}u_{gr}&=u_g-\bar{u}_g\\ v_{gr}&=v_g-\bar{v}_g\\ F_{x_ir}&=F_{x_i}-\bar{F}_{x_i}\end{aligned}\right\}, \quad (28)$$

for $i=1, 2, 3$, where u_{gr} and v_{gr} are, respectively, the x and y components of the geostrophic velocity relative to the mean geostrophic velocity. They, therefore, take into account the effect of the velocity shear in the mean flow.

Multiplying Eqs. (27) by x_r, y_r and z_r, respectively, and then taking means, we obtain the set of equations for the mean square of the particle displacement relative to the center of mass; thus,

$$\left(\frac{d^2}{dt^2}\right)\frac{\overline{x_r^2}}{2}=\overline{u_r^2}+f_z\overline{x_r(v_r-v_{gr})}-f_y\overline{x_rw_r}+\overline{x_rF_{xr}}, \quad (29a)$$

$$\left(\frac{d^2}{dt^2}\right)\frac{\overline{y_r^2}}{2}=\overline{v_r^2}-f_z\overline{y_r(u_r-u_{gr})}+\overline{y_rF_{yr}}, \quad (29b)$$

$$\left(\frac{d^2}{dt^2}+2\Gamma\right)\frac{\overline{z_r^2}}{2}=\overline{w_r^2}+f_y\overline{z_ru_r}+\overline{z_rF_{zr}}. \quad (29c)$$

It may be noted that to solve for the trajectory of the center of mass of the marked fluid particles from Eqs. (26), for the particle displacement relative to the center of mass from Eqs. (27), and for the mean square of the particle displacement from Eqs. (29), an assumption regarding the x, y, and z components of the frictional force needs to be made. A model for the frictional force will be constructed, and the equations for the trajectory of the center of mass and those for the mean square particle displacement will be solved in the next section.

5. A model for turbulent diffusion

Similar to Einstein's assumption of Brownian motion, we assume that the frictional force acting on a fluid particle consists of two parts: one of regular nature, the other of random turbulent nature. With regard to the former, we assume that the deviation of any particle of fluid from the surrounding current is resisted by a force proportional to the velocity of the particle relative to that of the surrounding current. In large-scale atmospheric motion, the geostrophic velocity, which results from the balance between the pressure and Coriolis forces, may be considered as a first approximation to the velocity of the mean flow. The frictional force per unit mass may thus be assumed to take the form

$$\left.\begin{aligned}F_x&=-\epsilon_x\left(\frac{dx}{dt}-u_g\right)+\varphi_x\\ F_y&=-\epsilon_y\left(\frac{dy}{dt}-v_g\right)+\varphi_y\\ F_z&=-\epsilon_z\left(\frac{dz}{dt}-w_m\right)+\varphi_z\end{aligned}\right\}, \quad (30)$$

where ϵ_i can be shown to be inversely proportional to the i component of the integral time scale of the turbulence, φ_i the i component of the random part of the frictional force, and w_m the mean vertical velocity of the surrounding air, which may be assumed to be zero as a first approximation.

By substituting (30) into Eqs. (26), (27) and (29), we obtain

$$\frac{d^2x_m}{dt^2}-f_z\left(\frac{dy_m}{dt}-\bar{v}_g\right)+f_y\frac{dz_m}{dt}+\epsilon_x\left(\frac{dx_m}{dt}-\bar{u}_g\right)=0, \tag{31a}$$

$$\frac{d^2y_m}{dt^2}+f_z\left(\frac{dx_m}{dt}-\bar{u}_g\right)+\epsilon_y\left(\frac{dy_m}{dt}-\bar{v}_g\right)=0, \tag{31b}$$

$$\frac{d^2z_m}{dt^2}-f_y\frac{dx_m}{dt}+\epsilon_z\frac{dz_m}{dt}=0, \tag{31c}$$

$$\frac{d^2x_r}{dt^2}-f_z\left(\frac{dx_r}{dt}-v_{gr}\right)+f_y\frac{dz_r}{dt}+\epsilon_x\left(\frac{dx_r}{dt}-u_{gr}\right)-\varphi_x=0, \tag{32a}$$

$$\frac{d^2y_r}{dt^2}+f_z\left(\frac{dx_r}{dt}-u_{gr}\right)+\epsilon_y\left(\frac{dy_r}{dt}-v_{gr}\right)-\varphi_y=0, \tag{32b}$$

$$\frac{d^2z_r}{dt^2}-f_y\frac{dx_r}{dt}+\epsilon_z\frac{dz_r}{dt}+\Gamma z_r-\varphi_z=0, \tag{32c}$$

$$\left(\frac{d^2}{dt^2}+\epsilon_x\frac{d}{dt}\right)\frac{\overline{x_r^2}}{2}=\overline{u_r^2}+\epsilon_x\overline{x_ru_{gr}}+f_z\overline{x_r(v_r-v_{gr})}-f_y\overline{x_rw_r}, \tag{33a}$$

$$\left(\frac{d^2}{dt^2}+\epsilon_y\frac{d}{dt}\right)\frac{\overline{y_r^2}}{2}=\overline{v_r^2}+\epsilon_y\overline{y_rv_{gr}}-f_z\overline{y_r(u_r-u_{gr})}, \tag{33b}$$

$$\left(\frac{d^2}{dt^2}+\epsilon_z\frac{d}{dt}\right)\frac{\overline{z_r^2}}{2}=\overline{w_r^2}+f_y\overline{z_ru_r}. \tag{33c}$$

Because of the nature of the random turbulent part of the frictional force, the terms $\overline{\varphi_x}=\overline{\varphi_y}=\overline{\varphi_z}=\overline{x_r\varphi_x}=\overline{y_r\varphi_y}=\overline{z_r\varphi_z}=0$ in the above equations.

In the atmosphere, the vertical velocity is at least three orders of magnitude smaller than the horizontal velocity. Thus, we may neglect the Coriolis term due to the vertical velocity in (31a). Let $R_m(t)=x_m(t)+iy_m(t)$ be a complex variable representing the horizontal radius vector of the center of mass of the particles. If (31b) is multiplied by the imaginary number i and added term by term to (31a), we obtain

$$\frac{d^2R_m(t)}{dt^2}+(\epsilon+if_z)\frac{dR_m(t)}{dt}=(\epsilon+if_z)[\bar{u}_g(x_m,y_m,t)+i\bar{v}_g(x_m,y_m,t)]. \tag{34}$$

In the above, $\epsilon=\epsilon_x=\epsilon_y$ has been assumed.

If $\bar{u}_g$ and $\bar{v}_g$ have continuous partial derivatives of all orders and satisfy the Cauchy-Riemann differential equation for all t, we may define

$$G(R_m,t)=\bar{u}_g(x_m,y_m,t)+i\bar{v}_g(x_m,y_m,t).$$

Therefore, (34) may be written as

$$\frac{d^2R_m(t)}{dt^2}+(\epsilon+if_z)\frac{dR_m(t)}{dt}=(\epsilon+if_z)G(R_m,t). \tag{35}$$

It is seen from (35) that the complex function $G(R_m,t)$ is equal to $R_m'(t)$ whenever $R_m''(t)$ is zero, where single and double primes denote the first and second time derivatives.

If G is a function of time only, (35) has the solution

$$R_m(t)=R_m(0)+(\epsilon+if_z)^{-1}R_m'(0)\times\{1-\exp[-(\epsilon+if_z)t]\}+\int_0^t G(\tau)d\tau-\int_0^t \exp[(\epsilon+if_z)(\tau-t)]G(\tau)d\tau. \tag{36}$$

To analyze the characteristics of the above equation, let $\lambda=-(\epsilon+if_z)^{-1}$. If $n+1$ derivatives of $G(\tau)$ are continuous in the interval $0\leq\tau\leq t$, (36) may be written as

$$R_m(t)=R_m(0)-\lambda R_m'(0)\{1-\exp[-(\epsilon+if_z)t]\}+\int_0^t G(\tau)d\tau+\lambda\{G(t)-G(0)\exp[-(\epsilon+if_z)t]\}$$
$$+\lambda^2\{G'(t)-G'(0)\exp[-(\epsilon+if_z)t]\}+\cdots+\lambda^{n+1}\{G^{(n)}(t)-G^{(n)}(0)\exp[-(\epsilon+if_z)t]\}$$
$$+\lambda^{n+2}\int_0^t \exp[(\epsilon+if_z)(\tau-t)]G^{(n+1)}(\tau)d\tau.$$

Should convergence be satisfied, one may pass to the limit $n \to \infty$, and the above equation becomes

$$R_m(t)=R_m(0)+\int_0^t G(\tau)d\tau+\lambda[-R_m'(0)+G(t)+\lambda G'(t)+\lambda^2 G''(t)+\lambda^3 G'''(t)+\cdots]$$
$$+\lambda\exp[-(\epsilon+if_z)t][-R_m'(0)+G(0)+\lambda G'(0)+\lambda^2 G''(0)+\cdots].$$

It may be noted that for large values of t the terms containing $\exp[-(\epsilon+if_z)t]$ drop out. The effect of the frictional force, therefore, is to damp out the inertial oscillation of the motion of the center of mass. Analysis of the nonlinear solution of the system of Eqs. (31a) and (31b) for the motion of the center of mass in a long atmospheric wave (Kao, 1962a) shows that the frequency of oscillation predicted by the theory agrees well with that of the observed.

In view of the fact that

$$\overline{x_{ri}\frac{dx_{rj}}{dt}}=(\overline{v_{ri}^2 v_{rj}^2})^{\frac{1}{2}}\int_0^t R_{v_{ri}v_{rj}}(\tau)d\tau,\quad i,\ j=1,\ 2,\ 3,$$

Eqs. (31) can be written as

$$\left(\frac{d^2}{dt^2}+\epsilon_x\frac{d}{dt}\right)\frac{\overline{x_r^2}}{2}=\overline{u_r^2}+\epsilon_x(\overline{u_r^2}\,\overline{u_{gr}^2})^{\frac{1}{2}}\int_0^t R_{u_r u_{gr}}(\tau)d\tau+f_z[\overline{u_r^2}\,\overline{(v_r-v_{gr})^2}]^{\frac{1}{2}}\int_0^t R_{u_r(v_r-v_{gr})}(\tau)d\tau$$
$$-f_y(\overline{u_r^2}\,\overline{w_r^2})^{\frac{1}{2}}\int_0^t R_{u_r w_r}(\tau)d\tau,\tag{37a}$$

$$\left(\frac{d^2}{dt^2}+\epsilon_y\frac{d}{dt}\right)\frac{\overline{y_r^2}}{2}=\overline{v_r^2}+\epsilon_y(\overline{v_r^2}\,\overline{v_{gr}^2})^{\frac{1}{2}}\int_0^t R_{v_r v_{rg}}(\tau)d\tau-f_z[\overline{v_r^2}\,\overline{(u_r-u_{gr})^2}]^{\frac{1}{2}}\int_0^t R_{v_r(u_r-u_{gr})}(\tau)d\tau,\tag{37b}$$

$$\left(\frac{d^2}{dt^2}+\epsilon_z\frac{d}{dt}+2\Gamma\right)\frac{\overline{z_r^2}}{2}=\overline{w_r^2}+f_y(\overline{w_r^2}\,\overline{u_r^2})^{\frac{1}{2}}\int_0^t R_{w_r u_r}(\tau)d\tau.\tag{37c}$$

The solutions of (37) are as follows:

$$\overline{x_r^2(t)}=2\epsilon_x^{-1}\overline{u_r^2}\{t-\epsilon_x^{-1}(1-e^{-\epsilon_x t})\}+2(\overline{u_r^2}\,\overline{u_{gr}^2})^{\frac{1}{2}}\int_0^t\int_0^\eta\{1-\exp[\epsilon_x(\eta-t)]\}R_{u_r u_{gr}}(\tau)d\tau\,d\eta$$
$$+2\epsilon_x^{-1}f_z[\overline{u_r^2}\,\overline{(v_r-v_{gr})^2}]^{\frac{1}{2}}\int_0^t\int_0^\eta\{1-\exp[\epsilon_x(\eta-t)]\}R_{u_r(v_r-v_{gr})}(\tau)d\tau d\eta$$
$$-2\epsilon_x^{-1}f_y(\overline{u_r^2}\,\overline{w_r^2})^{\frac{1}{2}}\int_0^t\int_0^\eta\{1-\exp[\epsilon_x(\eta-t)]\}R_{u_r w_r}(\tau)d\tau\,d\eta,\tag{38a}$$

$$\overline{y_r^2(t)}=2\epsilon_y^{-1}\overline{v_r^2}\{t-\epsilon_y^{-1}(1-e^{-\epsilon_y t})\}+2(\overline{v_r^2}\,\overline{v_{gr}^2})^{\frac{1}{2}}\int_0^t\int_0^\eta\{1-\exp[\epsilon_y(\eta-t)]\}R_{v_r v_{gr}}(\tau)d\tau d\eta$$
$$-2\epsilon_y^{-1}f_z[\overline{v_r^2}\,\overline{(u_r-u_{gr})^2}]^{\frac{1}{2}}\int_0^t\int_0^\eta\{1-\exp[\epsilon_y(\eta-t)]\}R_{v_r(u_r-u_{gr})}(\tau)d\tau d\eta,\tag{38b}$$

$$\overline{z_r^2(t)}=\overline{w_r^2}\Gamma^{-1}\{1-e^{-at}(ab^{-1}\sin bt+\cos bt)\}+4b^{-1}f_y(\overline{w_r^2}\,\overline{u_r^2})^{\frac{1}{2}}e^{-at}\int_0^t\int_0^\eta e^{a\eta}\sin b(t-\eta)R_{u_r w_r}(\tau)d\tau d\eta,\tag{38c}$$

where

$$a=\tfrac{1}{2}\epsilon_z,\quad b=\tfrac{1}{2}(8\Gamma-\epsilon_z)^{\frac{1}{2}}.\tag{39}$$

For neutrally stratified atmosphere, i.e., $\Gamma=0$, the solution of (37c) becomes

$$\overline{z_r^2}(t)=2\epsilon_z^{-1}\overline{w_r^2}\{t-\epsilon_z^{-1}(1-e^{-\epsilon_z t})\}+2\epsilon_z^{-1}f_y(\overline{w_r^2}\,\overline{u_r^2})^{\frac{1}{2}}\int_0^t\int_0^\eta\{1-\exp[\epsilon_z(\eta-t)]\}R_{w_r u_r}(\tau)d\tau d\eta.\tag{40}$$

If there are good correlations between u_r and u_{gr}, and between v_r and v_{gr}, but no correlation between u_r and (v_r-v_{gr}), v_r and (u_r-u_{gr}), and w_r and u_r, it can be shown that for small dispersion times (37) may be approximated by

$$\left.\begin{aligned}\overline{x_r^2}(t)&\approx\overline{u_r^2}t+\tfrac{1}{3}\epsilon_x(\overline{u_r^2}\,\overline{u_{gr}^2})^{\frac{1}{2}}t^3\\ \overline{y_r^2}(t)&\approx\overline{v_r^2}t+\tfrac{1}{3}\epsilon_y(\overline{v_r^2}\,\overline{v_{gr}^2})^{\frac{1}{2}}t^3\\ \overline{z_r^2}(t)&\approx\overline{w_r^2}t\end{aligned}\right\}, \qquad (41)$$

and that for large dispersion times by

$$\left.\begin{aligned}\overline{x_r^2}(t)&\approx2\left[\frac{\overline{u_r^2}}{\epsilon_x}+(\overline{u_r^2}\,\overline{u_{gr}^2})^{\frac{1}{2}}I_x\right]t\\ \overline{y_r^2}(t)&\approx2\left[\frac{\overline{v_r^2}}{\epsilon_y}+(\overline{v_r^2}\,\overline{v_{gr}^2})^{\frac{1}{2}}I_y\right]t\\ \overline{z_r^2}(t)&\approx2\frac{\overline{w_r^2}}{\epsilon_z}\end{aligned}\right\}, \qquad (42)$$

where

$$I_x=\int_0^\infty R_{u_ru_{gr}}(\tau)d\tau$$

and

$$I_y=\int_0^\infty R_{v_rv_{gr}}(\tau)d\tau$$

are integral time scales.

For a stably stratified atmosphere, b is generally a positive quantity. In this case, for small dispersion times, (37c) may be approximated by

$$\overline{z_r^2}(t)\approx\frac{\overline{\epsilon_z^2w_r^2}}{8\Gamma}t^2,$$

and for large dispersion times by

$$\overline{z_r^2}(t)\approx\frac{\overline{w_r^2}}{\Gamma},$$

which provides an upper limit for vertical dispersion in an inversion layer.

For an unstably stratified atmosphere, b becomes an imaginary quantity and (37c) may be written as

$$\begin{aligned}\overline{z_r^2}(t)=\overline{w_r^2}\Gamma^{-1}\{1-e^{-at}(ac^{-1}\sinh ct+\cosh ct)\}\\ -4c^{-1}f_y(\overline{w_r^2}\,\overline{u_r^2})^{\frac{1}{2}}e^{-at}\int_0^t\int_0^\eta e^{a\eta}\\ \times\sinh[c(t-\eta)]R_{w_ru_r}(\tau)d\tau d\eta,\end{aligned} \qquad (43)$$

where

$$c=\tfrac{1}{2}(\epsilon_z^2-8\Gamma)^{\frac{1}{2}}.$$

In this case, for small dispersion times (43) may be approximated by

$$\overline{z_r^2}(t)\approx\overline{w_r^2}t^2,$$

and for large dispersion times by

$$\overline{z_r^2}(t)\approx-\frac{\overline{w_r^2}}{\Gamma}\left(1-\frac{a}{c}\right)e^{(c-a)t}.$$

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