On the Distinction Between "Total" Heat Flux and Eddy Heat Flux[1]

J. A. Businger and J. W. Deardorff[2]

University of Washington, Seattle

21 August 1967

Often when preparing lecture notes a teacher finds himself in a dilemma on how to present a basic aspect of the subject because seemingly contradictory points of view may be taken. These dilemmas generally are resolved to the mutual benefit of students and teacher. Occasionally, however, the resolution of such a dilemma seems worthy of a larger audience, and we feel this note is an example of such a case.

When we add to the first law of thermodynamics,

$$\rho c_p \frac{dT}{dt} = \frac{dp}{dt} + \rho \frac{dh}{dt}, \quad (1)$$

(where dh/dt is the rate of heating per unit mass and c_p is the specific heat at constant pressure) the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho u_i) = 0, \quad (2)$$

after multiplying it with c_pT, we obtain the flux form of the first law, i.e.,

$$c_p\left[\frac{\partial}{\partial t}(\rho T) + \frac{\partial}{\partial x_i}(\rho u_i T)\right] = \frac{dp}{dt} + \rho \frac{dh}{dt}. \quad (3)$$

Here ρ is density, T absolute temperature, u_i the velocity component in the x_i direction, and p pressure.

Combining (3) with the equation of state for a perfect gas,

$$p = \rho R T, \quad (4)$$

where R is the specific gas constant for air, and taking the average, we have

$$\frac{c_p}{R}\frac{\partial \bar{p}}{\partial t} = -\frac{\partial}{\partial x_i}\overline{(\rho u_i c_p T)} + \overline{\frac{dp}{dt}} + \overline{\rho \frac{dh}{dt}}. \quad (5)$$

This result is nothing new. However, the dilemma is in the interpretation of this equation. We note that if the last two terms on the right-hand side of (5) are negligible the convergence of the flux of c_pT (the specific enthalpy), which is normally considered the heat flux, tends to increase locally the mean pressure and not the mean temperature. We must conclude, therefore, that the heat flux $\overline{\rho u_i c_p T}$ is not at all relevant as a cause of local temperature change. We call this conclusion a dilemma because we are accustomed to expect the convergence of the heat flux to produce a warming just as the convergence of the momentum flux tends to increase locally the mean momentum. We recognize, therefore, that the dilemma is a consequence of the existence of the equation of state.

To see what *is* relevant, we may expand the heat-flux term in (5) and utilize the averaged equation of continuity to obtain

$$\begin{aligned}-\frac{\partial}{\partial x_i}\overline{(\rho u_i c_p T)} &= -\frac{\partial}{\partial x_i}\left[\overline{\rho u_i} c_p \bar{T} + \overline{(\rho u_i)' c_p T'}\right] \\ &= -\overline{\rho u_i}\frac{\partial}{\partial x_i}(c_p \bar{T}) + c_p \bar{T}\frac{\partial \bar{\rho}}{\partial t} \\ &\quad -\frac{\partial}{\partial x_i}\left[\overline{(\rho u_i)' c_p T'}\right]. \quad (6)\end{aligned}$$

Substitution of (6) into (5), with the aid of (4), then gives

$$\begin{aligned}\bar{\rho} c_p \frac{\partial \bar{T}}{\partial t} + c_p \frac{\partial}{\partial t}\overline{(\rho' T')} \\ = -\frac{\partial}{\partial x_i}\left[\overline{(\rho u_i)' c_p T'}\right] - \overline{\rho u_i}\frac{\partial}{\partial x_i}(c_p \bar{T}) + \overline{\frac{dp}{dt}} + \overline{\rho \frac{dh}{dt}}. \quad (7)\end{aligned}$$

[1] Contribution No. 165, Department of Atmospheric Sciences, University of Washington, Seattle.

[2] On leave of absence from the National Center for Atmospheric Research, Boulder, Colo.

The second term on the left and the second and third terms on the right will be shown to be negligibly small in the atmospheric surface layer when horizontal homogeneity prevails. Hence, Eq. (7) demonstrates that it is the vertical convergence of the *eddy* heat flux $\overline{(\rho w)'c_pT'}$, and not of the "total" heat flux $\overline{\rho w c_p T}$ which causes local warming under these conditions in the absence of other heating mechanisms.

To show that $\partial/\partial t\,(\overline{\rho'T'})$ in (7) is small under most conditions, we make use of the good approximation that

$$\frac{\rho'}{\bar{\rho}}\simeq-\frac{T'}{\bar{T}}, \tag{8}$$

which is due to the small magnitude of relative fluctuations in pressure. Hence, we have

$$\frac{\partial}{\partial t}\overline{\rho'T'}\simeq-\frac{\partial}{\partial t}\left(\frac{\bar{\rho}}{\bar{T}}\overline{T'^2}\right), \tag{9}$$

which is smaller than the first term in (7) by the approximate factor $\overline{T'^2}/\bar{T}^2$, i.e., four or five orders of magnitude smaller.

When we assume horizontal uniformity the term $\overline{\rho u_i}\partial\bar{T}/\partial x_i$ may be written as

$$\overline{\rho w}\frac{\partial\bar{T}}{\partial z}=-\frac{\partial\bar{T}}{\partial z}\int_0^z\frac{\partial\bar{\rho}}{\partial t}dz \tag{10}$$

from (2). For sufficiently small z (the lowest 20 m of the atmosphere), (10) becomes approximately

$$\overline{\rho w}\frac{\partial\bar{T}}{\partial z}=-\frac{z}{\bar{T}}\frac{\partial\bar{T}}{\partial z}\left(\bar{\rho}\frac{\partial\bar{T}}{\partial t}+\frac{1}{R}\frac{\partial\bar{p}}{\partial t}\right). \tag{11}$$

Both terms on the right are typically of the same order of magnitude so that $\overline{\rho w}(\partial/\partial z)(c_p\bar{T})$ in (7) is smaller than $\bar{\rho}c_p\partial\bar{T}/\partial t$ by the approximate factor $-(z/\bar{T})\partial\bar{T}/\partial z$. This factor is small when $|\partial\bar{T}/\partial z|$ is large adjacent to a horizontal boundary because z is then small; at considerably larger z it is still small because $|\partial\bar{T}/\partial z|$ generally decreases as z^{-1} or more rapidly. We estimate this factor to be of magnitude 0.01 or less.

The magnitude of $\overline{dp/dt}$ may be evaluated most easily from

$$\overline{\frac{dp}{dt}}=\frac{\partial\bar{p}}{\partial t}+\overline{u_i\frac{\partial p}{\partial x_i}}, \tag{12}$$

where $\partial\bar{p}/\partial t$ has a magnitude of 0.3 erg cm^{-3} sec^{-1} for a change in pressure of 1 mb per hour. The horizontal component of the last term of (12) is typically 0.1 erg cm^{-3} sec^{-1}, i.e., a cross-isobaric speed of 10 m sec^{-1} and a pressure gradient of 1 mb (100 km)$^{-1}$. The vertical component, which is essentially $\bar{w}\partial\bar{p}/\partial z$, also has a magnitude of 0.1 erg cm^{-3} sec^{-1}, assuming a rather large eddy heat flux of 25 mW cm^{-2}. A conservative estimate of $\overline{dp/dt}$ is therefore 0.4 erg cm^{-3} sec^{-1}. This we should compare with $\bar{\rho}c_p\partial\bar{T}/\partial t$ which has typically a magnitude of 4 ergs cm^{-3} sec^{-1}, for $\partial\bar{T}/\partial t\simeq$ 1C hr^{-1}, and is therefore an order of magnitude larger than $\overline{dp/dt}$. These order of magnitude estimates confirm the importance of the eddy heat flux in (7).

We note that with both horizontal homogeneity and steady state our dilemma disappears, and that then

$$\overline{\rho w c_p T}=\overline{(\rho w)'c_pT'}\simeq\bar{\rho}c_p\overline{w'T'}.$$

As far as we know, standard textbooks consider only these conditions.

Acknowledgments. The authors are indebted to Mr. A. Bernstein for helpful criticisms. The work reported in this paper has been supported in part by the National Science Foundation under grant GP-4689.