Some Properties of Hadley Regimes on Rotating and Non-Rotating Planets

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ABSTRACT

The problem of the steady symmetric motion of a Boussinesq fluid is considered for a system with small aspect ratio. It is assumed that the motion is driven by applying a periodic heat flux to the horizontal boundaries. Solutions are first found for a non-rotating system in which nonlinear effects are small, but not zero. The solutions show that if the fluid is heated from above, the meridional circulation tends to be concentrated near the upper boundary at the point where the cooling is a maximum; when the fluid is heated from below the meridional circulation tends to be concentrated near the lower boundary at the point where the heating is a maximum.

Then, it is shown for a non-rotating system that when nonlinear effects are dominant, vertical boundary layers must form. These vertical boundary layers form at points where the horizontal velocity is zero, and are characterized by small horizontal velocities and temperature gradients, but large vertical velocities and horizontal diffusion. By means of scaling analysis, the scales and magnitudes of the variables are determined for both the internal boundary layers and the boundary layers along the horizontal boundaries, when nonlinear effects are dominant.

Next, the effect of rotation is considered, and it shown that exactly the same sorts of vertical boundary layers will form in a rotating system. Scaling analysis is again used to show that in this case the horizontal boundary layers near the internal boundary layers are of the same kind as in the non-rotating case, but far enough away from the internal boundary layers they merge into a nonlinear Ekman layer.

Finally, some possible geophysical applications are considered. The model of the atmospheric circulations on Venus proposed by Goody and Robinson is found to agree qualitatively with the results presented here, but the quantitative results for the internal boundary layer, or mixing region, are found to differ considerably. Also, estimates are made for the internal boundary layer which would accompany a Hadley cell similar to that found in the earth's tropical region. It is found that the rising motions will occur over a region about 200 km in width. This result suggests that the nonlinear process which produces these internal boundary layers may be one of the important processes in determining the structure of the Intertropical Convergence Zone. Finally, the identification of the narrow sinking regions as another example of the kind of internal boundary layer studied here is considered, but in this case the magnitudes and scales are not plausible.

1. Introduction

Determining the steady, axially-symmetric state of motion of a differentially heated atmosphere or ocean is essentially a problem in sideways convection. The resulting state of motion in geophysical contexts is generally referred to as the Hadley regime of motion. Such motions have been studied extensively for non-rotating systems in the fluid dynamics literature (e.g., see Batchelor, 1954; Weinbaum, 1964; Elder, 1965; Rossby, 1965; Gill, 1966). In fact, it has been suggested that the results of the non-rotating case may have relevance for certain geophysical problems; in particular, by Stommel (1962) who was concerned with the oceanic sinking regions, and by Goody and Robinson (1966) who were concerned with the circulations in the atmosphere of Venus. The motions that occur in rotating systems have been even more extensively studied, particularly in connection with the annulus experiments [e.g., see Fowlis and Hide (1965) and Williams (1967), both of whom have extensive references to the literature], because of their importance in the theory of the general circulations of the terrestrial atmosphere. Another example of geophysical relevance is the model of the oceanic thermal circulation produced by Robinson and Welander (1963), which is essentially a solution for sideways convection in a rotating system.

However, all of the published studies of sideways convection in both rotating and non-rotating systems differ in two important respects from many geophysical situations. In the first place they produce the differential heating by applying at the boundaries a temperature that varies horizontally, instead of a flux that varies horizontally. Secondly, they do not consider the case when the depth of the system is small compared to its horizontal extent.[1] In order to study the effects of such conditions, we will model a typical geophysical situation by a Boussinesq fluid in a rectangular coordinate system, unbounded horizontally, bounded below

[1] A referee has pointed out that similar effects have been considered by Rogers (1954) for a less general problem.

by a rigid surface, and bounded above by a flat, stress-free surface. We will only consider the steady symmetric (no zonal variations) motion.

We will first study the motion in a non-rotating system. The equations of motion are, in general, highly nonlinear, with the relative size of the nonlinear terms measured by a Rayleigh number Ra. In order to gain some insight about the kind of motions that occur, we will investigate the effects of shallowness and flux boundary conditions by solving the equations with a perturbation expansion in powers of Ra. In general, such expansions only give quantitatively accurate results for $\mathrm{Ra}<10^3$, whereas $\mathrm{Ra}\gg 10^3$ for geophysical situations. However, by going to high enough order in the perturbation series, the first effects of nonlinearities can be examined, and previous studies show that the tendencies revealed by these first nonlinear effects agree qualitatively with the results for $\mathrm{Ra}\gg 10^3$. In particular, the concentration of the flow near the boundaries and the development of asymmetries in the flow when $\mathrm{Ra}\gg 10^3$ are correctly revealed by the Rayleigh number expansions (see Batchelor, 1954; Weinbaum, 1964; Covelli, 1966).

Next we will consider how the flow is modified for large Rayleigh numbers and draw some qualitative conclusions. These conclusions enable us to determine many features of the flow by use of scaling arguments. Then we will consider how these features are modified in a rotating system. Finally, we will consider several geophysical phenomena for which the properties deduced for the nonlinear Hadley regime motions may be relevant.

2. The mathematical model

We consider first the case of no rotation. The Boussinesq equations for steady, two dimensional flow, are then:

$$\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.1}$$

$$v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-\frac{1}{\rho_0}\frac{\partial P}{\partial y}+\nu\left(\frac{\partial^2 v}{\partial z^2}+\frac{\partial^2 v}{\partial y^2}\right), \tag{2.2}$$

$$v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho_0}\frac{\partial P}{\partial z}+\nu\left(\frac{\partial^2 w}{\partial z^2}+\frac{\partial^2 w}{\partial y^2}\right)+\alpha g T, \tag{2.3}$$

$$v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}=\kappa\left(\frac{\partial^2 T}{\partial z^2}+\frac{\partial^2 T}{\partial y^2}\right), \tag{2.4}$$

where v and w are the meridional and vertical velocity components, y and z the meridional and vertical space variables, ρ_0 density, P hydrodynamic pressure, and T is the deviation from the mean value of either the temperature (for a liquid) or the potential temperature (for a gas). In addition, α is the expansion coefficient, g the acceleration of gravity, ν kinematic viscosity, κ kinematic conductivity and for the time being we will assume $\kappa=\nu$ (although the analysis would be no more difficult in principle if $\kappa\neq\nu$).

Eq. (2.1) allows us to introduce a stream function,

$$v=-\phi_z, \tag{2.5}$$

$$w=\phi_y. \tag{2.6}$$

In terms of ϕ Eq. (2.4) may be written

$$\kappa\left(\frac{\partial^2 T}{\partial z^2}+\frac{\partial^2 T}{\partial y^2}\right)+\phi_z T_y-\phi_y T_z=0. \tag{2.7}$$

If we use Eq. (2.2) to eliminate P from Eq. (2.3), we obtain

$$\nu\left(\frac{\partial^2}{\partial z^2}+\frac{\partial^2}{\partial y^2}\right)^2\phi+\left(\phi_z\frac{\partial}{\partial y}-\phi_y\frac{\partial}{\partial z}\right)\left(\frac{\partial^2\phi}{\partial z^2}+\frac{\partial^2\phi}{\partial y^2}\right)+\alpha g T_y=0. \tag{2.8}$$

To model a geophysical situation, we will solve Eqs. (2.7) and (2.8) in a region bounded by two horizontal planes at $z=0$ and $z=d$. We will assume that the lower boundary is rigid so that

$$\phi=\phi_z=0 \quad \text{at } z=0, \tag{2.9}$$

but that the upper boundary is flat and stress-free, so that

$$\phi=\phi_{zz}=0 \quad \text{at } z=d. \tag{2.10}$$

At these surfaces we will specify the vertical temperature gradient,

$$T_z=F_0(y) \quad \text{at } z=0, \tag{2.11}$$

$$T_z=F_1(y) \quad \text{at } z=d, \tag{2.12}$$

where F_0 and F_1 would, of course, be determined by specifying the flux at the boundaries. We will assume that the flux boundary conditions are periodic with a scale $y\sim L$.

It is convenient to rewrite Eqs. (2.7) and (2.8) in dimensionless form. We will measure y in units of L, z in units of d, T in units of $\Delta T=Gd$, where G is the magnitude of the boundary conditions applied to T_z, and ϕ in units of $(\alpha g d^3\Delta T)^{\frac{1}{2}}$. Thus, Eqs. (2.7) and (2.8) become

$$\nabla^2 T+\epsilon\mathrm{Ra}^{\frac{1}{2}}[\phi_z T_y-\phi_y T_z]=0, \tag{2.13}$$

$$\nabla^4\phi+\epsilon\mathrm{Ra}^{\frac{1}{2}}[\phi_z\nabla^2\phi_y-\phi_y\nabla^2\phi_z]+\epsilon\mathrm{Ra}^{\frac{1}{2}}T_y=0, \tag{2.14}$$

where

$$\nabla^2=\frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial y^2}, \tag{2.15}$$

$$\epsilon=\frac{d}{L}, \tag{2.16}$$

$$\mathrm{Ra}=\frac{\alpha g d^4 G}{\kappa^2}. \tag{2.17}$$

It is understood that ϕ, T, y, and z are now dimensionless variables.

3. Behavior for small Rayleigh number

If Ra=0, Eq. (2.13) reduces to

$$\frac{\partial^2 T}{\partial z^2}+\epsilon^2\frac{\partial^2 T}{\partial y^2}=0. \tag{3.1}$$

Suppose, for example, that $F_0=(G/d)k_0\cos y$ and $F_1=(G/d)k_1\cos y$. Then the exact solution of Eq. (3.1), found by separation of variables, is

$$T(y,z)=\cos y\left(\frac{k_0}{\epsilon}\sinh\epsilon z+\frac{(k_1-k_0\cosh\epsilon)}{\epsilon\sinh\epsilon}\cosh\epsilon z\right). \tag{3.2}$$

In the limit as $\epsilon\rightarrow 0$, this solution becomes

$$T=\frac{1}{\epsilon^2}(\cos y)(k_1-k_0)+\cos y\left[\left(\frac{k_1-k_0}{2}\right)z^2+k_0 z-\frac{k_1}{6}-\frac{k_0}{3}\right]+O(\epsilon^2). \tag{3.3}$$

Consequently, the limit $\epsilon\rightarrow 0$ is a singular one, and this must be taken into account if we wish to take advantage of the smallness of ϵ in solving our problem.

The mathematical reason for the singular behavior is clear. In the limit $\epsilon\rightarrow 0$, Eq. (3.1) reduces to an ordinary differential equation which is only first-order in T_z. Consequently, if we wish to apply two boundary conditions to T_z, we cannot neglect the second term in Eq. (3.1). The singular behavior is not present if one or more boundary condition is applied to T. Physically, the flux divergence between the two boundaries must be disposed of in order to maintain a steady state. When Ra=0, this can only be accomplished by horizontal diffusion, and thus the horizontal diffusion term must be as large as the vertical term, even though $\epsilon^2\ll 1$.

If $T\sim O(\epsilon^{-2})$, then from Eq. (2.14) we see that $\phi\sim O(\mathrm{Ra}^{\frac{1}{2}}\epsilon^{-1})$. Furthermore, if we re-scaled ϕ and T by these magnitudes, the parameters which occur in the equations are Ra and ϵ^2. Consequently, the proper expansion for the solution when both Ra and ϵ are small is

$$T=\frac{1}{\epsilon^2}[T_{00}+\mathrm{Ra}T_{01}+\mathrm{Ra}^2T_{02}+\cdots]+[T_{10}+\mathrm{Ra}T_{11}+\cdots]+\epsilon^2[T_{20}+\cdots]+\cdots, \tag{3.4}$$

$$\phi=\frac{\mathrm{Ra}^{\frac{1}{2}}}{\epsilon}[\phi_{00}+\mathrm{Ra}\phi_{01}+\mathrm{Ra}^2\phi_{02}+\cdots]+\epsilon\mathrm{Ra}^{\frac{1}{2}}[\phi_{10}+\mathrm{Ra}\phi_{11}+\cdots]+\epsilon^3\mathrm{Ra}^{\frac{1}{2}}[\phi_{20}+\cdots]+\cdots. \tag{3.5}$$

Substituting these expansions into Eqs. (2.13) and (2.14) and equating coefficients of equal powers of the parameters, we obtain

$$\frac{\partial^2 T_{00}}{\partial z^2}=0, \tag{3.6}$$

$$\frac{\partial^2 T_{01}}{\partial z^2}+\frac{\partial\phi_{00}}{\partial z}\frac{\partial T_{00}}{\partial y}-\frac{\partial\phi_{00}}{\partial y}\frac{\partial T_{00}}{\partial z}=0, \tag{3.7}$$

$$\frac{\partial^2 T_{10}}{\partial z^2}+\frac{\partial^2 T_{00}}{\partial y^2}=0, \tag{3.8}$$

$$\frac{\partial^2 T_{11}}{\partial z^2}+\frac{\partial^2 T_{01}}{\partial y^2}+\frac{\partial\phi_{10}}{\partial z}\frac{\partial T_{00}}{\partial y}+\frac{\partial\phi_{00}}{\partial z}\frac{\partial T_{10}}{\partial y}-\frac{\partial\phi_{10}}{\partial y}\frac{\partial T_{00}}{\partial z}-\frac{\partial\phi_{00}}{\partial y}\frac{\partial T_{10}}{\partial z}=0, \tag{3.9}$$

$$\frac{\partial^4\phi_{00}}{\partial z^4}+\frac{\partial T_{00}}{\partial y}=0, \tag{3.10}$$

$$\frac{\partial^4\phi_{01}}{\partial z^4}+\frac{\partial T_{01}}{\partial y}+\frac{\partial\phi_{00}}{\partial z}\frac{\partial^3\phi_{00}}{\partial y\partial z^2}-\frac{\partial\phi_{00}}{\partial y}\frac{\partial^3\phi_{00}}{\partial z^3}=0, \tag{3.11}$$

$$\frac{\partial^4\phi_{10}}{\partial z^4}+2\frac{\partial^4\phi_{00}}{\partial y^2\partial z^2}+\frac{\partial T_{10}}{\partial y}=0, \text{ etc.} \tag{3.12}$$

Substituting Eqs. (3.4) and (3.5) into the dimensionless forms of Eqs. (2.9)–(2.12), we find for the boundary conditions

$$\phi_{mn}=\frac{\partial\phi_{mn}}{\partial z}=0 \text{ at } z=0 \text{ for all } m,n, \tag{3.13}$$

$$\phi_{mn}=\frac{\partial^2\phi_{mn}}{\partial z^2}=0 \text{ at } z=1 \text{ for all } m,n, \tag{3.14}$$

$$\frac{\partial T_{10}}{\partial z}=f_0(y) \text{ at } z=0, \tag{3.15}$$

$$\frac{\partial T_{10}}{\partial z}=f_1(y) \text{ at } z=1, \tag{3.16}$$

$$\frac{\partial T_{mn}}{\partial z}=0 \text{ at } z=0 \text{ and } z=1 \text{ for all } m,n \text{ except } m=1,\ n=0. \tag{3.17}$$

Here, f_0 and f_1 are the dimensionless forms of the flux boundary conditions and are therefore of order unity.

The form of the expansions (3.4) and (3.5) show that when $\epsilon\ll 1$, the relative size of the nonlinear terms depends on the magnitude of Ra. On the other hand, if instead of flux boundary conditions we apply at least one boundary condition to the temperature, then there is no singular behavior as $\epsilon \to 0$ and we could directly expand the solution to Eqs. (2.13) and (2.14) in powers of $\epsilon \mathrm{Ra}^{\frac{1}{2}}$. In fact, only the alternate terms in such an expansion are non-zero, and therefore the relative size of the nonlinear terms when we do not have two flux conditions depends on the magnitude of $\epsilon^2\mathrm{Ra}$. Referring to our definition of Ra [Eq. (2.17)] and recalling that we have scaled the temperature by $\Delta T = Gd$, we see that one must apply a boundary condition on T of magnitude $\Delta T \sim O(Gd\epsilon^{-2})$ in order to produce nonlinear affects equal in magnitude to those produced when one applies a boundary condition on T_z of magnitude G.

The solution of Eqs. (3.6)–(3.12) is straightforward since they are all of the form $\partial^2 T_{mn}/\partial z^2$ or $\partial^4 \phi_{mn}/\partial z^4$ equal to known functions. In each order there are some undetermined functions of y which are determined by applying the boundary conditions to equations of next higher order. Thus, using boundary conditions (3.13)–(3.17), we readily find

$$T_{00} = B(y), \tag{3.18}$$

$$T_{01} = \left(\frac{dB}{dy}\right)^2 \left[\frac{z^5}{120} - \frac{5z^4}{192} + \frac{z^3}{48}\right] + A(y), \tag{3.19}$$

$$\phi_{00} = \frac{dB}{dy}\frac{z^2}{48}(z-1)(3-2z), \tag{3.20}$$

$$\begin{aligned}\phi_{01} = \frac{1}{(48)^2}\frac{dB}{dy}\frac{d^2B}{dy^2}&\left[-\frac{2}{45}z^9 + \frac{1}{4}z^8 - \frac{37}{70}z^7 + \frac{1}{2}z^6\right.\\ &\left.-\frac{3}{10}z^5 + \frac{1357}{2520}z^3 - \frac{349}{840}z^2\right]\\ &+\frac{dA}{dy}\left[-\frac{1}{24}z^4 + \frac{5}{48}z^3 - \frac{1}{16}z^2\right],\end{aligned} \tag{3.21}$$

where

$$\frac{d^2B}{dy^2} = f_0(y) - f_1(y), \tag{3.22}$$

$$\frac{d^2A}{dy^2} = -\frac{1}{1920}\frac{d^2}{dy^2}\left(\frac{dB}{dy}\right)^2 - \frac{1}{320}\frac{d}{dy}\left(f_0\frac{dB}{dy}\right). \tag{3.23}$$

Since $\epsilon \ll 1$ in geophysical situations, we will not consider the solutions for the first-order terms in the ϵ^2 expansion. The first-order solutions in the Rayleigh number expansion T_{01} and ϕ_{01} contain the first nonlinear effects. Once the boundary conditions f_0 and f_1 are specified, we can integrate Eqs. (3.22) and (3.23), using the boundary condition that T is periodic in y to complete the solution. The constant terms in $A(y)$ and $B(y)$ are determined by the requirement that the mean value of T is zero (T is the deviation from the mean temperature), but we will arbitrarily set these constants equal to zero since they do not affect any of the properties we will discuss.

4. Solutions for specific examples

First of all we will consider the model discussed by Goody and Robinson (1966) which has an insulating bottom, i.e.,

$$f_0 = 0, \tag{4.1}$$

and periodic differential heating from above, which we shall represent by

$$f_1 = \cos y. \tag{4.2}$$

These choices correspond to an atmosphere in which all solar radiation is absorbed at the top, with $y=0$ being the subsolar point and $y=\pi$ the antisolar point.

Substituting the above choices for the boundary conditions into Eqs. (3.22) and (3.23) and integrating, we obtain

$$B = \cos y, \tag{4.3}$$

$$A = -\frac{1}{1920}\sin^2 y. \tag{4.4}$$

Using Eqs. (4.3), (4.4), and (3.18)–(3.21), we can calculate the solution with the first nonlinear effects included from the expressions, i.e.,

$$T = \frac{1}{\epsilon^2}(T_{00} + \mathrm{Ra}T_{01}) + O\left(1, \frac{\mathrm{Ra}^2}{\epsilon^2}\right), \tag{4.5}$$

$$\phi = \frac{\mathrm{Ra}^{\frac{1}{2}}}{\epsilon}(\phi_{00} + \mathrm{Ra}\phi_{01}) + O\left(\epsilon \mathrm{Ra}^{\frac{1}{2}}, \frac{\mathrm{Ra}^{\frac{5}{2}}}{\epsilon}\right). \tag{4.6}$$

Fig. 1 shows the isotherms and Fig. 2 the streamlines calculated from Eqs. (4.5) and (4.6) when the nonlinear effects are zero (Ra=0).[2] These fields are completely symmetrical about $y=\pi/2$. The velocities are strongest near the top since it is stress-free while on the bottom the velocity goes to zero.

In general, $\phi_{01}/\phi_{00} \sim T_{01}/T_{00} \sim 10^{-3}$. In Figs. 3 and 4 we plot, respectively, the isotherms and streamlines when Ra=500 as calculated from Eqs. (4.5) and (4.6). The main effect of the nonlinearities is to tend to concentrate the flow and the horizontal temperature gradients toward the right and toward the top, especially the former. Also, the solution is everywhere stably stratified ($T_z > 0$).

As a second example we shall consider a case with heating from below; in particular, we choose

$$f_0 = -\cos y, \tag{4.7}$$

$$f_1 = 0. \tag{4.8}$$

These choices correspond to a flux from the ground into the fluid varying from a maximum at $y=0$ to a minimum at $y=\pi$, with an insulating top. Substituting

[2] In Figs. 1 through 8 the values given for T and ϕ have had the respective factors $1/\epsilon^2$ and $\mathrm{Ra}^{\frac{1}{2}}/\epsilon$ factored out.

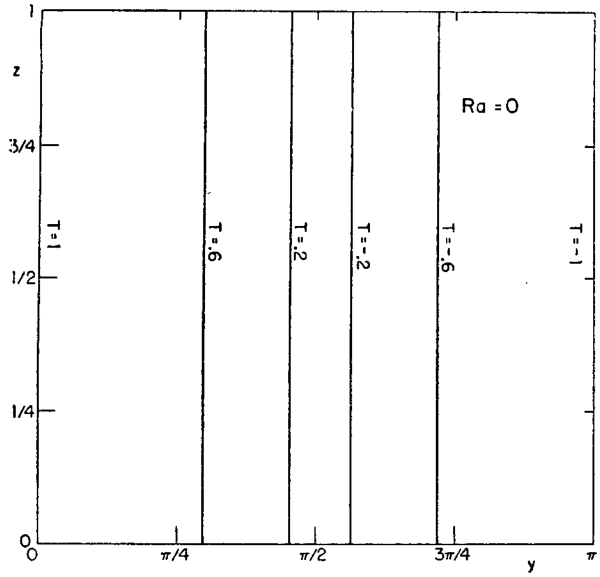

FIG. 1. Isotherms when Ra=0.

Eqs. (4.7) and (4.8) into Eqs. (3.22) and (3.23) and integrating, we obtain

$$B=\cos y, \tag{4.9}$$

$$A=-\frac{1}{1920}\sin^2 y+\frac{\cos 2y}{1280}. \tag{4.10}$$

Since B is the same for this case as for the previous one of heating from above, the solution when Ra=0, which depends only on B, will be identical in the two cases (see Figs. 1 and 2).

Figs. 5 and 6 show the isotherms and streamlines, respectively, for Ra=500 for the case of heating from below. In contrast to our first case, the flow and horizontal temperature gradients now tend to be concentrated by the nonlinear terms into a region near the lower boundary towards the left. Again the flow is everywhere stably stratified.

As a final example we consider a model with the same amount of heating from both above and below, i.e.,

$$f_0=-\cos y, \tag{4.11}$$

$$f_1=\cos y. \tag{4.12}$$

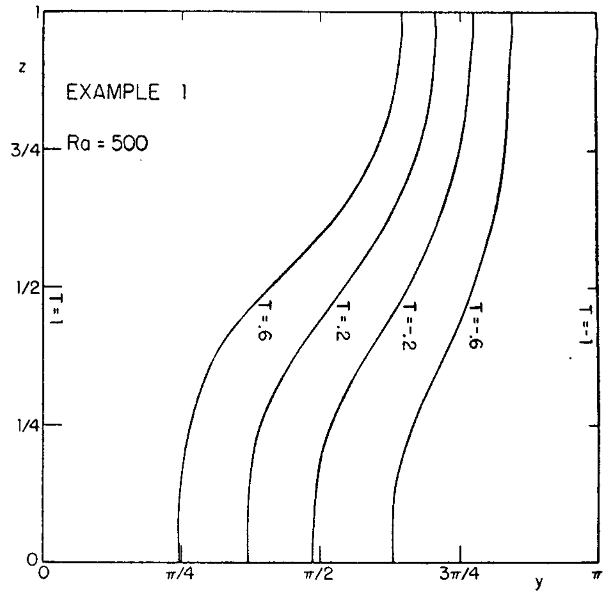

FIG. 3. Isotherms for the case of heating from above when Ra=500.

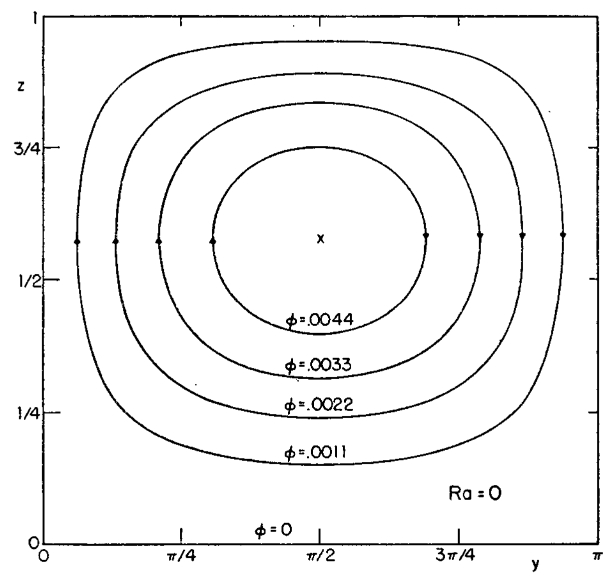

FIG. 2. Streamlines when Ra=0.

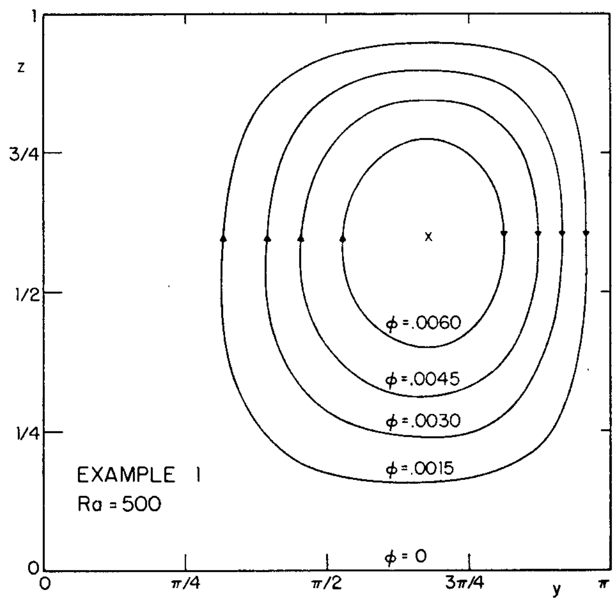

FIG. 4. Streamlines for the case of heating from above when Ra=500.

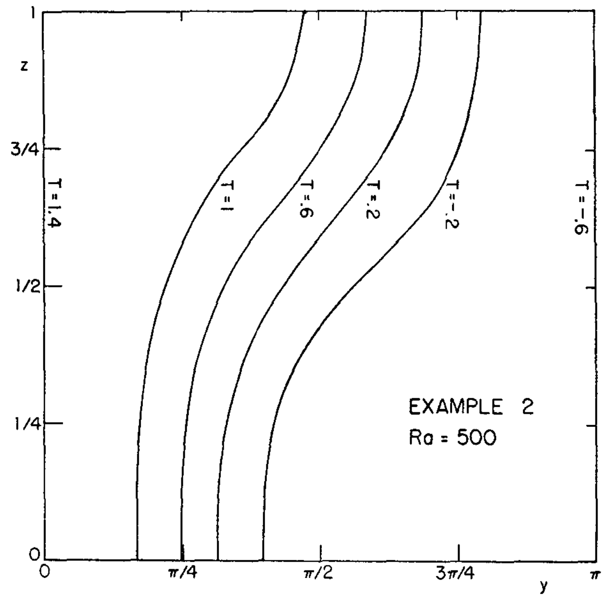

FIG. 5. Isotherms for the case of heating from below when Ra=500.

In this case, integrating Eqs. (3.22) and (3.23), we obtain

$$B=2\cos y, \tag{4.13}$$

$$A=-\frac{1}{480}\sin^2 y+\frac{1}{640}\cos 2y. \tag{4.14}$$

Once more the solution calculated by substituting Eqs. (4.13) and (4.14) into Eqs. (3.18)–(3.21) and then substituting the results into Eqs. (4.5) and (4.6) gives the same solution, when Ra=0, as our previous two models (aside from a multiplicative factor of 2).

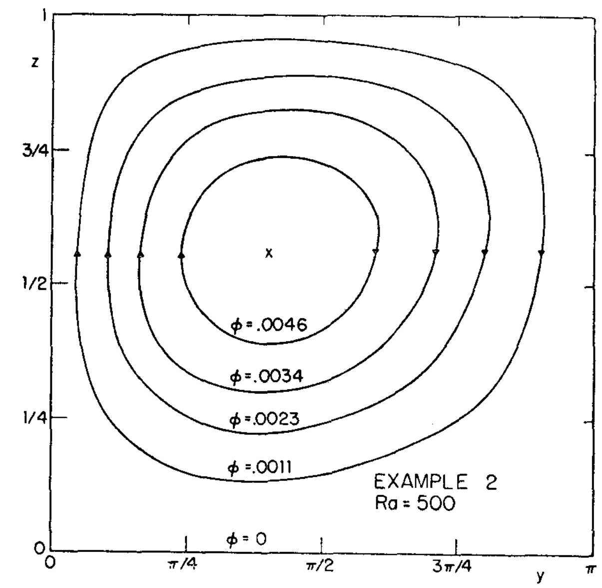

FIG. 6. Streamlines for the case of heating from below when Ra=500.

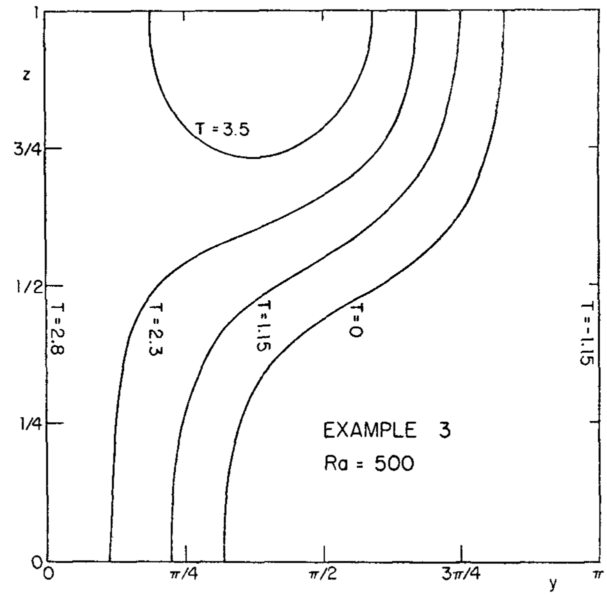

FIG. 7. Isotherms for the case of heating from both above and below when Ra=500.

The solution for this last model with equal heating from above and below when Ra=500 is shown in Fig. 7 (isotherms) and Fig. 8 (streamlines). In this case the nonlinear effects are intermediate between the effects in our first two models. The flow and horizontal temperature gradients tend to be concentrated toward both the upper right-hand and lower left-hand corners, but particularly toward the former. (Neither corner would have been favored if we had applied the same velocity boundary conditions above and below.) Once more the flow is stably stratified.

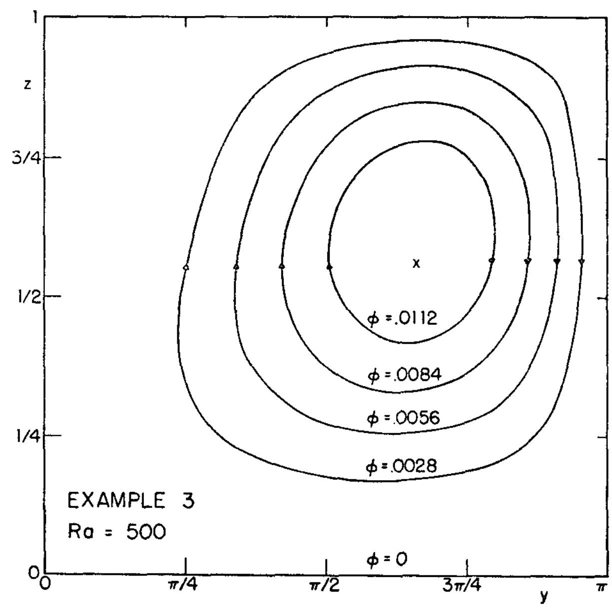

FIG. 8. Streamlines for the case of heating from both above and below when Ra=500.

The above simple solutions show that, in general, the meridional dependence of the flow will be asymmetric when nonlinear effects are important [cf. Stommel (1962) and Rossby (1965)]. In addition, they clearly demonstrate that what kind of asymmetry occurs depends crucially on the boundary conditions.

5. Behavior for large Rayleigh number

In geophysical problems not only is $\mathrm{Ra}\gg 10^3$, but in general the horizontal and vertical eddy coefficients are not equal. Consequently, it is preferable to use in place of Eqs. (2.13) and (2.14) the equations one would derive from Eqs. (2.1)–(2.4) if one replaces κ by κ_v in the vertical diffusion terms, and by κ_H in the horizontal diffusion terms. In the definition of Ra [Eq. (2.17)] κ is now replaced by κ_v. The resulting equations, derived in exactly the same way as Eqs. (2.13) and (2.14) are

$$\frac{\partial^2 T}{\partial z^2}+\frac{\epsilon^2}{\mu}\frac{\partial^2 T}{\partial y^2}+\epsilon \mathrm{Ra}^{\frac{1}{2}}(\phi_z T_y-\phi_y T_z)=0, \qquad (5.1)$$

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\epsilon^2}{\mu}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial y^2}\right)\phi + \epsilon \mathrm{Ra}^{\frac{1}{2}}\left[\phi_z\left(\frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial y^2}\right)\phi_y - \phi_y\left(\frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial y^2}\right)\phi_z\right]+\epsilon \mathrm{Ra}^{\frac{1}{2}} T_y=0, \qquad (5.2)$$

where

$$\mu=\frac{\kappa_v}{\kappa_H}. \qquad (5.3)$$

The most interesting question when $\mathrm{Ra}\gtrsim 10^3$ is whether or not the solutions still have a singular behavior as $\epsilon\to 0$. If we apply different flux boundary conditions at $z=0$ and $z=1$ and neglect the T_{yy} term in Eq. (5.1), the advection terms are now available to remove the flux divergence and maintain a steady state. Thus, in general, we have no reason to expect a singular behavior.

Suppose, however, that the flux boundary conditions are symmetric about some value of y, say $y=y_0$ (as they would normally be in a planetary atmosphere). Eqs. (5.1) and (5.2) preserve symmetry, the solution for T would consequently be symmetric about $y=y_0$, and the solution for ϕ would be antisymmetric. Thus, at $y=y_0$, $\phi_z=T_y=0$, and in the limit $\epsilon\to 0$, $\epsilon\mathrm{Ra}^{\frac{1}{2}}\gtrsim 1$, Eq. (5.1) reduces to

$$\frac{\partial^2 T}{\partial z^2}-\epsilon \mathrm{Ra}^{\frac{1}{2}}\phi_y T_z=0, \text{ at } y=y_0. \qquad (5.4)$$

This equation is of first order in T_z, and one cannot therefore satisfy two flux conditions. Physically, at points of symmetry there is no horizontal advection, and if we also neglect horizontal diffusion, we are only left with vertical advection to balance a vertical flux divergence. In general, however, this is not possible, since vertical advection can redistribute the flux in the vertical, but cannot redistribute it so as to match two arbitrary flux boundary conditions.

We conclude that the solution for large Ra is also singular as $\epsilon\to 0$ in the vicinity of points where the horizontal advection goes to zero, such as at points of symmetry. To properly satisfy the flux boundary conditions at these points, we must rescale the equations so as to retain the T_{yy} term in Eq. (5.1). In the geophysically interesting case when $\epsilon\mathrm{Ra}^{\frac{1}{2}}\gg 1$, the horizontal advection terms cannot balance the vertical flux divergence except in a very small region about $y=y_0$. Consequently, y is the proper variable to rescale in order to retain the T_{yy} term near $y=y_0$. Such a scaling is also suggested by the solutions found in Section 4 which show that the nonlinear effects tend to concentrate the rapid variations near the points of symmetry $y=0$ and $y=\pi$. Thus, internal boundary layers will form at points of symmetry in the boundary conditions.

When $\epsilon\mathrm{Ra}^{\frac{1}{2}}\gg 1$ the solution of Eq. (5.1) will also be of boundary layer character in the z direction, since we must retain the T_{zz} terms in order to satisfy the boundary conditions; these terms will be in balance with the nonlinear terms only if the z scale is small. Such a boundary layer is necessary for all values of y. Therefore, the internal boundary layers will have a small z scale associated with them and will be confined near the horizontal boundaries. (If $\epsilon\mathrm{Ra}^{\frac{1}{2}}\sim 1$, there is no small vertical scale associated with these internal boundary layers, and they will extend vertically throughout the interior of the fluid.) Internal boundary layers like these have been termed mixing regions by Goody and Robinson (1966). We shall also use this terminology.

In general, however, only one mixing region is necessary at symmetry points $y=y_0$. This conclusion follows from the fact that when we do neglect T_{yy} in the temperature equation we can still obtain an interior solution which satisfies one flux boundary condition, and we therefore only need rescale y near the boundary where the other flux condition is not satisfied. If we integrate Eq. (5.4), since $w=\phi_y$, we obtain

$$\frac{\partial T}{\partial z}=\left.\frac{\partial T}{\partial z}\right|_{z=0}\exp\left\{\epsilon\mathrm{Ra}^{\frac{1}{2}}\int_0^z w\,dz\right\}, \text{ at } y=y_0, \qquad (5.5)$$

or

$$\frac{\partial T}{\partial z}=\left.\frac{\partial T}{\partial z}\right|_{z=1}\exp\left\{-\epsilon\mathrm{Ra}^{\frac{1}{2}}\int_z^1 w\,dz\right\}, \text{ at } y=y_0, \qquad (5.6)$$

depending on whether we satisfy the boundary condition at $z=0$ or $z=1$. If $y=y_0$ is a point where the fluid is being heated, then rising motions will occur near $y=y_0$, $w>0$. Thus, if we chose to apply the boundary condition at $z=0$, the flux would grow exponentially into the interior, whereas if we chose to apply the condition at $z=1$, it would decay exponentially into the interior. The former case is physically implausible compared to the latter, since it implies fluxes in the interior of the fluid exponentially larger than those applied at the boundaries, and would require a mixing region near the top which makes an exponentially large correction in the flux. Consequently, at symmetry points where rising motions occur, one should use the interior solution to satisfy the boundary condition at the top, and then use a mixing region to produce the correct flux near the bottom. Conversely, at symmetry points where cooling is applied, sinking motions occur, and one should use the interior solution to satisfy the lower boundary condition, and a mixing region will be necessary near the upper boundary in order to produce the correct flux. These conclusions are also suggested by the solutions for small Ra which showed that the nonlinear effects tend to concentrate the large gradients near the lower boundary at symmetry points where heating is occurring, and near the upper boundary at symmetry points where cooling is occurring.

Because there are so many constraints on the mathematical balances which must occur in the equations for the mixing regions, it is possible to determine uniquely the scales of y, z, T and ϕ appropriate to these regions. In particular, from the above considerations, we know that when $\epsilon\mathrm{Ra}^{\frac{1}{2}}\gg 1$, we must retain both the T_{zz} and T_{yy} terms in Eq. (5.1) in order to satisfy the flux conditions. In addition, we must retain the ϕ_{zzzz} term in Eq. (5.2) in order to satisfy the velocity boundary conditions, and finally, T_z must be of order unity since it is specified to be so at the boundary of the region. These four conditions can only be met if the dimensionless variables have the following scales in the mixing regions:

$$z\sim O\left(\frac{1}{(\mu\mathrm{Ra})^{\frac{1}{4}}}\right), \tag{5.7}$$

$$y\sim O\left(\frac{\epsilon}{(\mu^3\mathrm{Ra})^{\frac{1}{4}}}\right), \tag{5.8}$$

$$T\sim O\left(\frac{1}{(\mu\mathrm{Ra})^{\frac{1}{4}}}\right), \tag{5.9}$$

$$\phi\sim O\left(\frac{1}{(\mu\mathrm{Ra})^{\frac{1}{2}}}\right). \tag{5.10}$$

These scalings yield the following magnitudes for the corresponding dimensional quantities in the mixing regions:

$$z_{mr}=\left(\frac{\kappa_v\kappa_H}{\alpha g G}\right)^{\frac{1}{4}}, \tag{5.11}$$

$$y_{mr}=\left(\frac{{\kappa_H}^3}{\kappa_v\alpha g G}\right)^{\frac{1}{4}}, \tag{5.12}$$

$$T_{mr}=\left(\frac{\kappa_v\kappa_H G^3}{\alpha g}\right)^{\frac{1}{4}}, \tag{5.13}$$

$$w_{mr}=\left(\frac{{\kappa_v}^3\alpha g G}{\kappa_H}\right)^{\frac{1}{4}}, \tag{5.14}$$

$$v_{mr}=(\kappa_H\kappa_v\alpha g G)^{\frac{1}{4}}. \tag{5.15}$$

Since the mixing regions occur at points of symmetry, where $v=T_y=0$, they are not regions of large horizontal temperature gradients (only T_{yy} is large) or of large horizontal velocities. However, the small y scale does mean that they are regions of relatively large vertical velocities ($w=\phi_y$) and of large horizontal diffusion. The approximate equations for these regions, when we assume that $\epsilon^2\ll\epsilon^2/\mu\ll 1$, $\epsilon\mathrm{Ra}^{\frac{1}{2}}\gg 1$, and that the above scales hold, are

$$\frac{\partial^2 T}{\partial z^2}+\frac{\epsilon^2}{\mu}\frac{\partial^2 T}{\partial y^2}+\epsilon\mathrm{Ra}^{\frac{1}{2}}(\phi_z T_y-\phi_y T_z)=0, \tag{5.16}$$

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\epsilon^2}{\mu}\frac{\partial^2}{\partial y^2}\right)\frac{\partial^2\phi}{\partial z^2}+\epsilon\mathrm{Ra}^{\frac{1}{2}} \times[\phi_z\phi_{yzz}-\phi_y\phi_{zzz}+T_y]=0. \tag{5.17}$$

It is also possible to find the scaling for the boundary layers along the horizontal boundaries adjacent to the mixing regions. In this region the y scale is larger and horizontal diffusion of momentum and temperature is no longer necessary to satisfy the vertical boundary conditions. Consequently, the horizontal diffusion terms can be neglected. Then the conditions that T_{zz} and ϕ_{zzzz} be retained in the equations when $\epsilon\mathrm{Ra}^{\frac{1}{2}}\gg 1$, and that $T_z\sim 1$, are sufficient to find the scales of z, T and ϕ in terms of the y scale. The results for the boundary layer scales are:

$$z\sim O\left(\frac{y^{\frac{1}{3}}}{\epsilon^{\frac{1}{3}}\mathrm{Ra}^{\frac{1}{6}}}\right), \tag{5.18}$$

$$T\sim O\left(\frac{y^{\frac{1}{3}}}{\epsilon^{\frac{1}{3}}\mathrm{Ra}^{\frac{1}{6}}}\right), \tag{5.19}$$

$$\phi\sim O\left(\frac{y^{\frac{2}{3}}}{\epsilon^{\frac{2}{3}}\mathrm{Ra}^{\frac{1}{3}}}\right). \tag{5.20}$$

In dimensional terms these scales imply the corresponding magnitudes:

$$z_{bl}=\left(\frac{{\kappa_v}^2L^2}{\alpha g G}\right)^{\frac{1}{3}}, \tag{5.21}$$

$$T_{bl}=\left(\frac{{\kappa_v}^2L^2G^5}{\alpha g}\right)^{\frac{1}{6}}, \tag{5.22}$$

$$w_{bl}=\left(\frac{\kappa_v{}^4\alpha gG}{L^2}\right)^{\frac{1}{6}}, \tag{5.23}$$

$$v_{bl}=(\alpha gG\kappa_v L)^{\frac{1}{3}}, \tag{5.24}$$

where L is the y scale. The approximate equations for these regions, when we assume that $\epsilon^2 \ll \epsilon^2/\mu \ll 1$, $\epsilon \mathrm{Ra}^{\frac{1}{2}} \gg 1$, and that the above scales hold, are

$$\frac{\partial^2 T}{\partial z^2}+\epsilon \mathrm{Ra}^{\frac{1}{2}}(\phi_z T_y-\phi_y T_z)=0, \tag{5.25}$$

$$\frac{\partial^4 \phi}{\partial z^4}+\epsilon \mathrm{Ra}^{\frac{1}{2}}[\phi_z\phi_{yzz}-\phi_y\phi_{zzz}+T_y]=0. \tag{5.26}$$

It would be natural to choose for L the horizontal scale of the boundary conditions. However, one could specify boundary conditions that had no such scale, e.g., conditions that contained a single point of symmetry. Furthermore, if there were such a scale in the boundary conditions and we identified it with L, then the boundary layer would not, in general, join on to the mixing region, in the sense that they would have different depths. However, there is one other horizontal scale in the problem which we will identify with L because it is not subject to these difficulties; namely, the distance from the point of symmetry in the boundary conditions. Thus, the magnitudes of the boundary layer quantities given by Eqs. (5.21)–(5.24) vary with distance from the mixing region. In particular, the boundary layer depth grows like $L^{\frac{1}{3}}$, and at $L=y_{mr}$, its depth is equal to the depth of the mixing region, so the two regions join smoothly. Also, the vertical velocities decay very slowly ($\sim L^{-\frac{1}{3}}$) away from the mixing region. Since the vertical velocity averaged over y must be zero, this implies that strong rising motions (or sinking motions as the case may be) will not, in general, be confined to the mixing regions. The horizontal velocities will be relatively large in the boundary layer because of the small vertical scale ($v=-\phi_z$).

The fact that the scalings depend on the y scale suggests that Eqs. (5.25) and (5.26) contain a similarity solution of the form indicated by Eqs. (5.18)–(5.20). A little manipulation shows that this is indeed true, and that one solution of Eqs. (5.25) and (5.26) is of the form

$$T=\epsilon^{-1/3}\mathrm{Ra}^{-1/6}y^{1/3}f(x), \tag{5.27}$$

$$\phi=\epsilon^{-2/3}\mathrm{Ra}^{-1/3}y^{2/3}g(x), \tag{5.28}$$

where

$$x=\epsilon \mathrm{Ra}^{\frac{1}{2}}\frac{z^3}{y}.$$

When this dimensionless solution is put in dimensional form, L completely drops out of the solution.

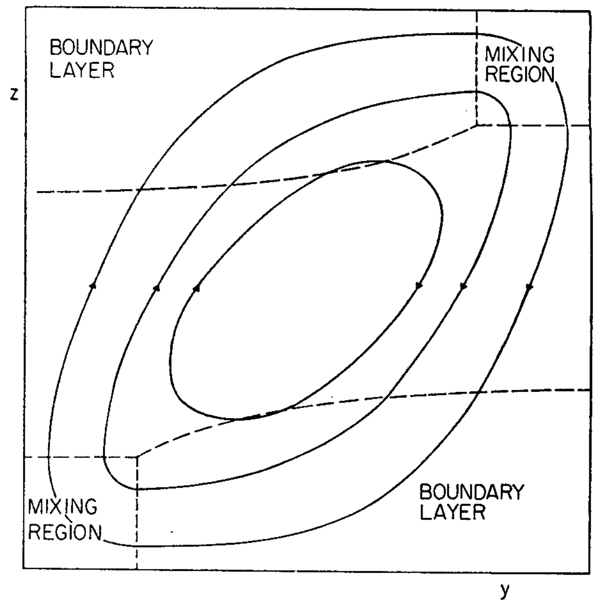

FIG. 9. Schematic diagram of the streamlines without rotation when $\epsilon \mathrm{Ra}^{\frac{1}{2}} \gg 1$.

The functions $f(x)$ and $g(x)$ satisfy the equations

$$9\frac{d^2f}{dx^2}+\frac{6}{x}\frac{df}{dx}+\frac{1}{x^{\frac{2}{3}}}\left(f\frac{dg}{dx}-2g\frac{df}{dx}\right)=0, \tag{5.29}$$

$$81x^2\frac{d^4g}{dx^4}+324x\frac{d^3g}{dx^3}+168\frac{d^2g}{dx^2}-\frac{1}{x^{\frac{2}{3}}}\left(4g\frac{dg}{dx}+36xg\frac{d^2g}{dx^2}+18x^2g\frac{d^3g}{dx^3}+\frac{1}{3}f-x\frac{df}{dx}\right)=0. \tag{5.30}$$

These equations, although ordinary, are still highly nonlinear and therefore very difficult to solve. In any case the natural boundary condition for a solution of the form (5.27) is a constant flux on the horizontal boundary $x=0$, which is not the kind of condition we are concerned with here. It is, however, interesting to note that this solution [Eq. (5.27)] corresponds to a temperature which increases along the horizontal boundary even though the flux is constant.

Unfortunately, there are not sufficient *a priori* constraints available for the interior motions to determine a unique scaling for them. Only after the boundary layer equations have been actually solved so that one knows what scales of the variables emerge into the interior can such a scaling be found. However, using the information available from the scalings for the mixing regions and boundary layers, it is possible to draw a schematic diagram of what the motions will be like. Fig. 9 shows such a diagram for the motion when the boundary conditions contain symmetry points at $y=0$ and $y=\pi$, with heating occurring at the former point and cooling at the latter.

6. Effects of rotation

When we add rotation to the regime we have been considering, the system will contain a zonal component of velocity u. If the rotation vector is vertical, then Eq. (2.2) is modified to

$$v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}=-fu-\frac{1}{\rho_0}\frac{\partial P}{\partial y}+\kappa_v\frac{\partial^2 v}{\partial z^2}+\kappa_H\frac{\partial^2 v}{\partial y^2}, \quad (6.1)$$

where f is the Coriolis parameter. To have a determinate set of equations, we must now add the zonal momentum equation,

$$v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=fv+\kappa_v\frac{\partial^2 u}{\partial z^2}+\kappa_H\frac{\partial^2 u}{\partial y^2}. \quad (6.2)$$

The continuity, vertical momentum, and energy equations are unaffected by rotation, since we are assuming a symmetrical state. The boundary conditions on u are the same as those on v.

If we now proceed in the same manner as in Section 2 in order to reduce equations (2.1), (2.3) (2.4), (6.1) and (6.2) to dimensionless form, we obtain in place of Eqs. (5.1) and (5.2) the equations

$$\frac{\partial^2 T}{\partial z^2}+\frac{\epsilon^2}{\mu}\frac{\partial^2 T}{\partial y^2}+\epsilon\mathrm{Ra}^{\frac{1}{2}}(\phi_z T_y-\phi_y T_z)=0, \quad (6.3)$$

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\epsilon^2}{\mu}\frac{\partial^2}{\partial y^2}\right)\left(\frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial y^2}\right)\phi +\epsilon\mathrm{Ra}^{\frac{1}{2}}\left[\phi_z\left(\frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial y^2}\right)\phi_y -\phi_y\left(\frac{\partial^2}{\partial z^2}+\epsilon^2\frac{\partial^2}{\partial y^2}\right)\phi_z\right] +\epsilon\mathrm{Ra}^{\frac{1}{2}}\left[T_y+\frac{1}{R_0}u_z\right]=0, \quad (6.4)$$

$$\frac{\partial^2 u}{\partial z^2}+\frac{\epsilon^2}{\mu}\frac{\partial^2 u}{\partial y^2}+\epsilon\mathrm{Ra}^{\frac{1}{2}}[\phi_z u_y-\phi_y u_z]-\frac{\epsilon\mathrm{Ra}^{\frac{1}{2}}}{\mathrm{R_0}}\phi_z=0, \quad (6.5)$$

where u has been made dimensionless in terms of the same units as v, namely $(\alpha g d\Delta T)^{\frac{1}{2}}$, and the new parameter which appears in the equations is a thermal Rossby number,

$$\mathrm{Ro}=\frac{(\alpha g d^2 G)^{\frac{1}{2}}}{fL}. \quad (6.6)$$

The conclusion drawn in Section 5 that an internal boundary layer was necessary at symmetry points in order to satisfy flux boundary conditions was based solely on an analysis of the energy equation. Since Eq. (6.3) is unchanged when rotation is added to the problem, the same conclusion holds. In addition, the y scale of this region is very small compared to the scaling length. Consequently, the Rossby number for the mixing region will in general be very large, and to a first approximation the u_z term may be neglected in (6.4). Thus, the equations for the mixing region will be identical for both the rotating and non-rotating Hadley regime, and the scalings we derived in the preceding section are equally valid for the mixing region in a rotating system.

Furthermore, the scaling derived for the boundary layer region adjacent to the mixing region in the preceding section will also be valid for a rotating system so long as its y scale is small enough. In particular, it will be small enough if the rotational term $\mathrm{Ro}^{-1}u_z$ does not exceed the T_y term that was assumed to be in balance with the ϕ_{zzzz} term in Eq. (6.4). For these small scales u is found by first solving Eqs. (6.3) and (6.4) for ϕ and T, and then using the results to solve Eq. (6.5) for u. In order to satisfy the boundary conditions on u when $\epsilon\mathrm{Ra}^{\frac{1}{2}}\gg 1$, the u_{zz} term must be in balance with the driving term ϕ_z in Eq. (6.5). This condition enables us to determine the scale of u in the boundary layer. Using the results of Section 5, we find that

$$u\sim O\left(\frac{y}{\mathrm{Ro}}\right), \quad (6.7)$$

which corresponds to the dimensional magnitude

$$u_{bl}=fL. \quad (6.8)$$

If we now equate the magnitudes of u_z and $\mathrm{Ro}T_y$ for the boundary layer region, we find the value of L for which rotation first becomes important, i.e.,

$$L_r=\left(\frac{\kappa_v\alpha g G}{f^3}\right)^{\frac{1}{4}}. \quad (6.9)$$

Our scaling for the non-rotating case will be equally valid for the rotating case so long as we are concerned with distances from the symmetry point which do not exceed this value of L, and so long as this value of L exceeds the y scale of the mixing region, so that the separation of y scales assumed in our analysis of the two regions is valid. Using Eqs. (5.12) and (6.9), this latter condition can be expressed as

$$f^2\kappa_H<\kappa_v\alpha g G. \quad (6.10)$$

If this condition is not met, then rotation must be taken into account even near the symmetry points.

When the distance from the symmetry point exceeds L_r, a new balance is formed in Eq. (6.4). The old balance between the ϕ_{zzzz} and T_y terms is now inconsistent since it leads to a u_z term larger than those assumed to be in balance. If we replace this condition by the condition that the ϕ_{zzzz} term be in balance with the u_z term, but otherwise use the same conditions used in determining the boundary layer scalings, we can solve for the ϕ, T, u and z scales in terms of the y scale. We then find that the depth of this new layer is just that of a conventional Ekman layer, and, in fact, in terms of our parameters (assuming $\epsilon^2 \ll \epsilon^2/\mu \ll 1 \ll \epsilon \mathrm{Ra}^{\frac{1}{5}}$) the various scales are

$$z \sim O\left(\frac{\mathrm{Ro}^2}{\epsilon^2 \mathrm{Ra}}\right)^{\frac{1}{4}}, \tag{6.11}$$

$$T \sim O\left(\frac{\mathrm{Ro}^2}{\epsilon^2 \mathrm{Ra}}\right)^{\frac{1}{4}}, \tag{6.12}$$

$$\phi \sim O\left(\frac{y^4}{\epsilon^2 \mathrm{Ro}^2 \mathrm{Ra}}\right)^{\frac{1}{4}}, \tag{6.13}$$

$$u \sim O\left(\frac{y}{\mathrm{Ro}}\right). \tag{6.14}$$

The magnitude of the corresponding dimensional quantities for this Ekman layer are

$$z_{el} = \left(\frac{\kappa_v}{f}\right)^{\frac{1}{2}} \tag{6.15}$$

$$T_{el} = G\left(\frac{K_v}{f}\right)^{\frac{1}{2}} \tag{6.16}$$

$$w_{el} = (\kappa_v f)^{\frac{1}{2}} \tag{6.17}$$

$$v_{el} = u_{el} = fL. \tag{6.18}$$

Since the last two depend on L, the requirement that the solutions in the Ekman layer join onto the adjacent boundary layer where rotation is unimportant, necessitates that we interpret L in the Ekman layer either as a variable scale, equal to the distance from the mixing region, or as the fixed scale $L = L_r$. Note that as L in Eq. (5.21) approaches the critical value L_r, $z_{bl} \rightarrow z_{el}$. Thus, the boundary layer effects a smooth transition from the mixing region to the Ekman layer.

The corresponding approximate equations in the Ekman layer are

$$\frac{\partial^2 T}{\partial z^2} + \epsilon \mathrm{Ra}^{\frac{1}{5}}(\phi_z T_y - \phi_y T_z) = 0, \tag{6.19}$$

$$\frac{\partial^4 \phi}{\partial z^4} + \epsilon \mathrm{Ra}^{\frac{1}{5}}\left[\phi_z \phi_{yzz} - \phi_y \phi_{zzz} + T_y + \frac{1}{\mathrm{Ro}} u_z\right] = 0, \tag{6.20}$$

$$\frac{\partial^2 u}{\partial z^2} + \epsilon \mathrm{Ra}^{\frac{1}{5}}\left[\phi_z u_y - \phi_y u_z - \frac{1}{\mathrm{Ro}} \phi_z\right] = 0. \tag{6.21}$$

Actually, in Eq. (6.20) the magnitude of the ratio of the T_y term to the $Ro^{-1}u_z$ term is $(L_r/L)^2$. Thus, if we interpret L as the distance from the mixing region, the T_y term will only be important in Eq. (6.20) in that

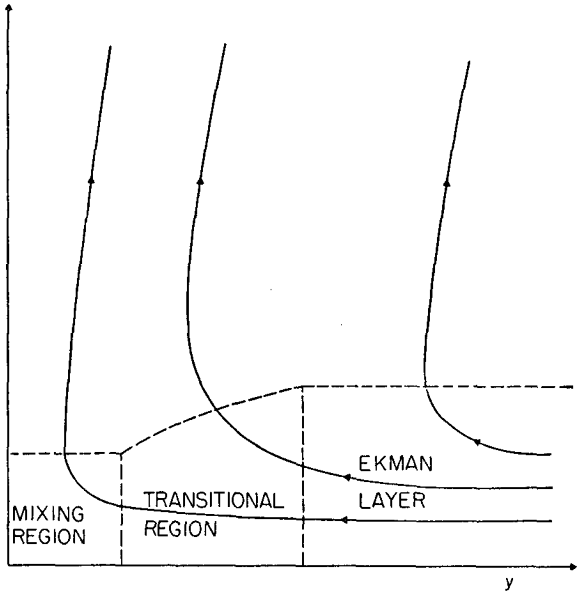

FIG. 10. Schematic diagram of the meridional streamlines with rotation near a point of heating at the lower boundary when $\epsilon \mathrm{Ra}^{\frac{1}{5}} \gg 1$.

part of the Ekman layer adjacent to the boundary layer where rotation is relatively unimportant (i.e., where $L \sim L_r$). Note that the Ekman layer equations for our problem are essentially nonlinear no matter how we interpret L.

Using the information gained from our scalings for the three regions in a rotating system, we can schematically picture the kind of flow that occurs when $\epsilon^2 \ll \epsilon^2/\mu \ll 1$ and $\epsilon \mathrm{Ra}^{\frac{1}{5}} \gg 1$. Such a picture is shown in Fig. 10, and for simplicity the streamlines are only indicated in the vicinity of a symmetry point where heating occurs. Once again no direct information can be obtained about the interior regions until the equations are actually solved.

7. Possible geophysical applications

a. The atmospheric circulations on Venus

Goody and Robinson (1966) proposed a heuristic model of the atmospheric circulations on Venus based on studies of sideways convection without rotation. In discussing their model they used exactly the same equations of motion as we have, except that in making them dimensionless they picked for the units of the vertical variable a value for d such that $\epsilon \mathrm{Ra}^{\frac{1}{5}} = 1$. This scale corresponds to the depth of the boundary layer [Eq. (5.21)]. They also attempted to scale the equations for the boundary layer and mixing regions, but arrived at different results from those we have given in Section 5.

The reason for the difference is that in the boundary layer, although they arrived at expressions identical to ours [Eqs. (5.21)–(5.24)], they interpreted L as the horizontal scale of the boundary conditions, rather than as the distance from the symmetry point. Consequently, in scaling the mixing region in order to have it join onto the boundary layer, they imposed the condition that $z_{bl}=z_{mr}$, with L interpreted in the above manner. This condition replaced the condition that the T_{zz} term be retained in Eq. (5.1), which we imposed so that we could satisfy the flux boundary condition at the edge of the mixing region. As a result Goody's and Robinson's scaling for the mixing region differs from ours, and their approximate equations for the mixing region do not contain any vertical diffusion terms, and cannot therefore satisfy any flux boundary condition. This is a nonphysical result, for if the fluid cannot see the boundary conditions, it cannot tell that a symmetry point is located there, and will not know that there is any need for a mixing region.

In Goody's and Robinson's model the flow is concentrated into one boundary layer, at the top of the atmosphere, in the cloud layer where most of the solar radiation is absorbed. This boundary layer merges into a mixing region located in the vicinity of the antisolar point. Our results indicate that, in general, one would also expect a second mixing region and boundary layer at the ground (see Fig. 9). However, Goody and Robinson considered the special case $T_z=0$ at $z=0$. Therefore, $G\equiv 0$ and in this case we see from Eqs. (5.11)–(5.15) that z_{mr}, $y_{mr}\rightarrow\infty$ and T_{mr}, v_{mr}, $w_{mr}\rightarrow 0$. Actually, z_{mr} will have an upper bound given by the depth of the system d, and corresponding to this value there will be a correspondingly larger value of y_{mr} and correspondingly smaller values of T_{mr}, v_{mr} and w_{mr}. Thus, although a much enlarged mixing region will still be necessary in order to satisfy the zero flux boundary condition at the ground, dynamically it will be inactive compared to the upper mixing region. Therefore, to a first approximation this lower region may be neglected in describing the flow appropriate to Goody's and Robinson's model.

Consequently, our results agree qualitatively with Goody's and Robinson's model. However, quantitatively, our results for the mixing region differ considerably from theirs. We will use their estimates of the values of the parameters for Venus to re-calculate the scales and magnitudes of the motion in the mixing region. The magnitude of the boundary value of T_z, which we have denoted by G, is related to the magnitude of the flux F by

$$F=\kappa_v\rho C_pG, \tag{7.1}$$

where C_p is the ratio of specific heats. For an atmosphere T is identified with the potential temperature, and this expression assumes that the main mechanism of vertical heat diffusion is turbulence. Goody's and Robinson's values for Venus are $d=40$ km, $L=20{,}000$ km, $F=1.6\times10^5$ ergs cm^{-2} sec^{-1}, $\rho C_p=2.7\times10^4$ ergs (°K)$^{-1}$ cm^{-3}, $g=870$ cm sec^{-2}, $\kappa_v=10^4$ cm^2 sec^{-1}, $\kappa_H=10^{10}$ cm^2 sec^{-1}, $\alpha=4.3\times10^{-3}$(° K)$^{-1}$, and $f=4\times10^{-7}$ sec^{-1}. From these values, using Eqs. (2.16), (2.17), (5.11)–(5.15), (6.9), and (7.1), we calculate

$$\epsilon=2\times10^{-3}$$
$$\mathrm{Ra}=6\times10^{15}$$
$$\epsilon\mathrm{Ra}^{\frac{1}{3}}=1.5\times10^{5}$$
$$z_{mr}=150\text{ m}$$
$$y_{mr}=150\text{ km}$$
$$T_{mr}=8\text{K}$$
$$w_{mr}=0.5\text{ cm sec}^{-1}$$
$$V_{mr}=5\text{ m sec}^{-1}$$
$$L_r=100{,}000\text{ km.}$$

Since the critical distance L_r for which rotational effects become important is much larger than the radius of Venus, which is about 7000 km, it is clear that the effects of rotation on the motions will be minor. The above values for the scales may be contrasted with the values found by Goody and Robinson, i.e., $z_{mr}=800$ m, $y_{mr}=3$ km, $T_{mr}=40$K, $w_{mr}=10$ m sec^{-1}, and $v=34$ m sec^{-1}. It is important to note that the values of κ_v and κ_H for Venus are not known and therefore estimates such as the above may be considerably in error. The value of y_{mr} is particularly uncertain since it is proportional to $\kappa_H^{\frac{3}{4}}$, and κ_H is the most uncertain quantity. Since the magnitude of the vertical velocity only falls off as $L^{-\frac{1}{3}}$ as one moves away from the mixing region, the magnitude of w at $L=20{,}000$ km (corresponding to the distance from subsolar to antisolar point on Venus), given by Eq. (5.23), is $w_{bl}=0.1$ cm sec^{-1}. Consequently, the requirement that there be no net mass flux suggests that the sinking motions are not confined to the mixing region, but extend a considerable distance into the boundary layer.

b. The tropical circulation

The tropical circulation on the earth in many respects resembles a Hadley type circulation of the kind we have been discussing. In particular, the tropical circulation frequently includes a narrow region where vertical velocities are relatively large and horizontal temperature gradients are relatively small, namely the Intertropical Convergence Zone [e.g., see Palmer (1951) and Bjerknes (1966)]. Such a region is analogous to the mixing and boundary layer regions which occur in our model near points of heating at the lower boundary. An additional similarity is that the characteristics of the mixing regions in our model are independent of latitude, and, in fact, the general structure of the Intertropical Convergence Zone does not appear to change with its position. This similarity could be coincidental in view of the fact that our model leaves out many processes that are probably important in determining the structure of the ITC. For example, we have not included condensation, and we have not allowed for any response of the lower boundary condition to the heating and wind stresses imposed on it by the atmosphere. Since the lower boundary on the earth is largely fluid, such effects can generate currents and a redistribution of the heat sources at the lower boundary. This might, for example, lead to a flux boundary condition whose horizontal variations were small compared to the flux itself. Under such circumstances our scaling analysis would not be valid, since we implicitly assumed throughout that the flux variations were comparable in magnitude to the flux itself.

In any case it is of interest to take typical values for the earth's equatorial regions to see how strong an asymmetry can be produced in a Hadley regime by the effects that we have considered by themselves. We will adopt the following values as typical: $d=10$ km, $F=1.4\times10^5$ ergs cm^{-2} sec^{-1} (10% of the solar constant), $\rho C_p=10^4$ ergs (°K)$^{-1}$ cm^{-3}, $g=10^3$ cm sec^{-2}, $\kappa_v=10^5$ cm^2 sec^{-1}, $\kappa_H=10^7$ cm^2 sec^{-1}, $\alpha=3.3\times10^{-3}$ (°K)$^{-1}$, and $f=3\times10^{-5}$ sec^{-1} (corresponding to 15° of latitude). Also, we shall interpret L as L_r in the Ekman layer. From these values, using Eqs. (2.16), (2.17), (5.11)–(5.15), (6.9), (6.15) through (6.18) and (7.1), we calculate the following magnitudes:

$$\epsilon=5\times10^{-3}$$
$$\mathrm{Ra}=4.6\times10^{10}$$
$$\epsilon\mathrm{Ra}^{\frac{1}{3}}=1.1\times10^3$$
$$z_{mr}=70\text{ m}$$
$$y_{mr}=700\text{ m}$$
$$T_{mr}=1.0\text{K}$$
$$w_{mr}=15\text{ cm sec}^{-1}$$
$$v_{mr}=1.5\text{ m sec}^{-1}$$
$$z_{vl}=600\text{ m}$$
$$L_r=400\text{ km}$$
$$T_{el}=8\text{K}$$
$$v_{el}=u_{el}=12\text{ m sec}^{-1}$$
$$w_{el}=1.7\text{ cm sec}^{-1}.$$

If we now assume that the Hadley cell has a total horizontal extent of 2000 km, then we can use the above values for the vertical velocities to estimate the width of the region where strong rising motions occur necessary to balance the slower sinking motions everywhere else. We find that rising motions will occur over a region about (1.7/15)2000 km≅200 km wide. The fact that this value corresponds well with the observed size of the ITC suggests that any complete theory of the ITC should include as one important process the interaction between flux boundary conditions, horizontal diffusion, and nonlinear advection of the kind present in our simplified model.

c. Oceanic sinking regions

Stommel (1962) suggested that the apparent smallness of the sinking regions in the earth's oceans might be due to the nonlinear effects in sideways convection. He made some calculations using a model that was essentially similar to ours, except that it neglected nonlinear terms in the momentum equations and applied thermal instead of flux boundary conditions. He found strong asymmetries, but did not attempt to estimate magnitudes for an oceanic circulation. Since the oceans are heated from above, the mixing region of our analysis in this instance would be located near the surface of the ocean, and would be located at the point where cooling is a maximum.

In the oceanic case there are again many important effects which have been left out of our simple model. In this connection we would particularly point out the ability of the flux boundary condition to adjust itself, as discussed above, and the fact that our assumption of symmetrical motion cannot be strictly valid in the oceans. However, we will again estimate how much of an asymmetry could arise solely from the kinds of effects included in our model, and for this purpose will adopt the following values as typical for oceanic sinking regions: $d=3$ km, $F=1.4\times10^5$ ergs cm^{-2} sec^{-1} (10% of the solar constant), $\rho C_p=4.2\times10^7$ ergs (°K)$^{-1}$ cm^{-3}, $g=10^3$ cm sec^{-2}, $\kappa_v=10^2$ cm^2 sec^{-1}, $\kappa_H=10^5$ cm^2 sec^{-1}, $\alpha=3.6\times10^{-3}$ (°K)$^{-1}$ and $f=1.3\times10^{-4}$ sec^{-1} (corresponding to 60° latitude). Again we will interpret L in the Ekman layer as L_r. In this case we calculate from our results the following magnitudes:

$$\epsilon=5\times10^4$$
$$\mathrm{Ra}=1\times10^{14}$$
$$\epsilon\mathrm{Ra}^{\frac{1}{3}}=5\times10^3$$
$$z_{mr}=50\text{ cm}$$
$$y_{mr}=170\text{ m}$$
$$T_{mr}=0.16\text{K}$$
$$w_{mr}=0.19\text{ cm sec}^{-1}$$
$$v_{mr}=6\text{ cm sec}^{-1}$$
$$z_{el}=10\text{ m}$$
$$L_r=700\text{ m}$$
$$T_{el}=0.001\text{K}$$
$$w_{el}=0.11\text{ cm sec}^{-1}$$
$$v_{el}=u_{el}=10\text{ cm sec}^{-1}.$$

The above results show that the magnitudes of the vertical velocities in the mixing region and the Ekman layer are not well separated. Consequently, we do not expect that the kind of interaction included in our model will lead to oceanic sinking regions appreciably narrower than the rising regions. The much stronger asymmetry that Stommel (1962) found can probably be attributed to the fact that he applied thermal boundary conditions.

The result was that his strongly asymmetric solutions were accompanied by a strongly asymmetric flux at the top boundary, with the fluid being cooled only above the sinking region, and heated everywhere else. Thus, it might be possible to explain narrow oceanic sinking if one could explain such a highly asymmetric flux boundary condition, but without such an asymmetric condition our results indicate that the nonlinear effects of sideways convection by themselves cannot explain narrow oceanic sinking regions.

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