Phase Space Solution of Buoyant Jets

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ABSTRACT

The solution of buoyant jets in a calm stratified atmosphere is considered. The Morton model yields a set of differential equations that can be solved in a closed form in a phase space having for coordinates the integral with height of the mass flux, the mass flux, and the derivative with height of the mass flux. By using an approximate expression that relates the mass flux integral with height to a given height, we are able to transfer knowledge of the phase space solution to physical space.

1. Introduction

The present paper considers the steady flow field in a turbulent jet moving in stagnant surroundings. A revival of interest in turbulent jets in nonrotating fluids has taken place in the last few years with a number of papers using such a jet model in order to describe the updraft found in cloud formation and thunderstorm convection. These mechanisms and a good survey of the literature are presented in Squires and Turner (1962), Henry and Srivastava (1964), while various aspects of the dynamics of these flows are treated in detail in the papers of Morton *et al.* (1956) and Morton (1959a,b, 1961, 1967). In the problem at hand the axial flow is driven by buoyancy and momentum, and can be characterized by a single nondimensional number, a Froude number. Outside the jet, stagnant fluid is entrained and mixed with the rising lighter fluid. As a result, in stable surroundings, the available buoyancy decreases with height, eventually vanishes and finally reverses direction opposing the further rise of the core fluid. Entrainment takes place at the core edge and the reader is referred to Morton (1967) for various entrainment models. These models, although approximate, have the advantage of giving simplified theoretical solutions that compare very favorably with experimental observations. To quote Morton, they enable us "to relate the entrainment inflow directly to the inner flow variables so that an approximate solution may be obtained for the inner region in isolation"; here the inner region stands for the core region. An entrainment parameter is used as a measure of the ratio of the radial velocity at the core edge at a given height to the axial velocity at that height. As such, entrainment is defined as a local property, and the magnitude of this parameter is small for slender plumes or jets since, from the continuity equation, the ratio of radial to axial velocity is of the order of the ratio of radial to vertical scales. The specific value of the entrainment parameter must be determined experimentally and depends upon the chosen shape of the velocity profiles.

The assumption of a point source leads to similarity solutions in which the core appears as an inverted cone, of small angular aperture, with its apex at the source (Morton *et al.*, 1956). This assumption was used extensively by Morton and was extended (1959b, 1961) to sources of finite size by introducing a virtual source below the $z=0$ plane. Such a source would then account for the far-field dynamics of the plume. By using this method, Morton (1959b) was able to treat analytically and in great detail the dynamics of plumes in unstratified surroundings and to compute the maximum height reached by these plumes when the surroundings are stratified. In the 1961 paper, Morton relates momentum flow to mass flow by elimination of the coordinate along the plume axis. This partial solution provides useful information about the nature of the flow found in jets, plumes and wakes, but requires numerical integration if one wants to transfer these results into physical space where the vertical coordinate appears explicitly. Also, the vertical source concept was used to characterize the different flows and it should be emphasized that the surroundings were assumed unstratified. Then, the r-averaged energy equation which expresses the constancy of the buoyancy flux reduces to an algebraic equation. The system is then characterized by only two equations, *viz.*, the conservation of mass and the conservation of vertical momentum. This constitutes the essence of the momentum-mass diagram. For stratified surroundings, the energy equation retains its differential character and one must extend the momentum-mass plane to a three-dimensional space. On the other hand, Murghai and Emmons (1960) considered a similar problem in an attempt to understand natural convection above fires. They considered finite sources and a piece-wise constant atmospheric lapse rate. The problem was solved numerically

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for a wide range of values of the governing dimensionless parameter.

The main departure of this paper from that of Morton's (1959b, 1961) is that the present work consists of a phase space formulation of the problem with stratification, where the representative point moves in a coordinate system having for its axis the mass flux integral with height, the mass flux, and its derivative with height. These considerations enable us, first, to obtain a closed solution to the dynamics in that space and, second, to derive an approximate expression which enables us to connect such a solution to physical space. Furthermore, we do not invoke, as Morton does, the presence of a vitual source to explain the far-field dynamics of these plumes and we do obtain the flow behavior in the immediate vicinity of a source of finite area. Depending upon the value of the Froude number, it is found that in some cases the core radius decreases first with height before eventually increasing. This "pinch effect" cannot be modeled by means of a virtual source located *below* the $z=0$ plane. As in Murghai and Emmons (1960) we will assume a piece-wise constant atmospheric lapse rate. Then, the plume will move in a stratum of atmospheric layers where $(d/dz)\times\log\theta_e'=\beta_n$ (β_n is a constant pertaining to the nth of these layers). We do not have to invoke the presence of n virtual sources to solve for the dynamics in these layers.

The assumptions made are similar to those made by Morton *et al.* (1956) whereby 1) the rate of entrainment is proportional to the vertical velocity at the height, and 2) the profile of the vertical velocity and of the buoyancy force are similar in the radial direction at all heights.

It was shown in Murhgai and Emmons that the variations of the potential density with height are unimportant if we are not dealing with extremely large fires that extend very deeply into the atmosphere. In what follows we will then relax the third assumption used by Morton *et al.*; namely, that the relative local density changes are to be small compared to unity.

We will consider, in turn, the cases of neutral and stable surroundings and we will formulate the problem in phase space. Finally, an example, very similar to the one found in Murghai and Emmons, will be used to illustrate the proposed method of solution. With a problem as simple as the one considered, integration in physical space is the straightforward, logical approach to solving it. However, phase space considerations are sometimes valuable when more intricate problems are considered. Examples of such problems are found in Barcilon (1967a,b) where buoyancy and/or rotation are present in jets rising in free vortex surroundings similar to those found in dust devils.

2. Formulation of the problem

If r denotes radial distance measured from the vertical axis z along which the jet is moving, we will anticipate that $\partial/\partial r \gg \partial/\partial z$, so that the boundary layer approximations apply in the core of the jet. Furthermore, we will assume that the vertical velocity, the buoyancy force, and the turbulent shear stresses are vanishingly small at the core edge $r_e(z)$. We will also take the pressure departure from hydrostatic to be constant at any level. Then, if $u'(r,z)$ and $w'(r,z)$ denote the radial and vertical dimensional velocities, respectively, the governing boundary layer equations are as follows:

Continuity

$$\frac{\partial}{\partial r}(\rho' r u')+\frac{\partial}{\partial z}(\rho' r w')=0 \tag{1}$$

Vertical momentum

$$\frac{\partial}{\partial r}(r\rho' u'w')+\frac{\partial}{\partial z}(r\rho' w'^2)=(\rho_e'-\rho')gr+\frac{\partial}{\partial r}(r\tau_{zr}') \tag{2}$$

Entropy

$$\frac{C_pT_e'}{\theta_e'}\left\{\frac{\partial}{\partial r}(r\rho' u'\theta')+\frac{\partial}{\partial z}(r\rho' w'\theta')\right\}=\frac{\partial}{\partial r}(rq_r') \tag{3}$$

In the above

C_p = specific heat at constant pressure (a constant),
τ_{zr}' = dimensional turbulent stress in the z direction on the r surface,
q_r' = dimensional heat flux by conduction and turbulence in the r direction,
T_e' = absolute dimensional temperature at the core edge, and
θ_e' = potential temperature at the core edge.

Finally, the state equation for a perfect gas is taken as

$$\frac{\rho_e'-\rho'}{\rho_e'}=\frac{\theta'-\theta_e'}{\theta_e'}. \tag{4}$$

We define a set of radially averaged (barred) quantities:

r-averaged mass

$$\rho_e'\bar{w}'(z)\bar{b}'^2(z)=\int_0^{r_e}\rho' w' d(r^2), \tag{5}$$

where $\bar{b}'(z)$ denotes a measure of the core radius at height z.

r-averaged vertical momentum

$$\rho_e'\bar{w}'^2(z)\bar{b}'^2(z)=\int_0^{r_e}\rho' w'^2 d(r^2) \tag{6}$$

r-averaged buoyancy

$$\bar{b}'^2\overline{\Delta\gamma'}(z)\equiv\int_0^{r_e}\left(\frac{\theta'-\theta_e'(z)}{\theta_e'(z)}\right)d(r^2) \tag{7}$$

We also define the entrainment parameter α as

$$\alpha=\frac{-2r_eu_e'}{\bar{w}'(z)\bar{b}'(z)},$$

where u_e' is the radial velocity at the core edge. The ratio $r_e/\bar{b}'(z)$ which depends on the shape of the velocity profile enters in the defining equation for α.

After integrating (1) from $r=0$ to $r=r_e$, we arrive at

$$\frac{1}{\rho_e'}\frac{d}{dz}(\rho_e'\bar{w}'\bar{b}'^2)=\frac{d}{dz}(\bar{w}'\bar{b}'^2)+w'\bar{b}'^2\frac{d}{dz}(\ln\rho_e')=\alpha\bar{w}'\bar{b}'. \tag{8.1}$$

By similar manipulations (2) can be written as

$$\frac{1}{\rho_e'}\frac{d}{dz}(\rho_e'\bar{w}'^2\bar{b}'^2)=\frac{d}{dz}(\bar{w}'^2\bar{b}'^2)+\bar{w}'^2\bar{b}'^2\frac{d}{dz}(\ln\rho_e')$$
$$=g\bar{b}'^2\overline{\Delta\gamma'}, \tag{8.2}$$

and by using the similarity assumption between vertical velocity and buoyancy force, we can write the r-averaged entropy equation as

$$\frac{d}{dz}(\rho_e'\theta_e'\bar{w}'\bar{b}'^2\overline{\Delta\gamma'})=-\rho_e'\bar{w}'\bar{b}'^2\frac{d}{dz}\theta_e',$$

or

$$\frac{d}{dz}(\bar{w}'\bar{b}'^2\overline{\Delta\gamma'})-\frac{g}{\gamma RT_e'}(\bar{w}'\bar{b}'^2\overline{\Delta\gamma'})=-\bar{w}'\bar{b}'^2\frac{d}{dz}(\ln\theta_e'). \tag{8.3}$$

But,

$$\frac{d}{dz}(\ln\rho_e')=\frac{1}{\gamma}\frac{d}{dz}(\ln p_e')=\frac{g}{\gamma RT_e'},$$

where R is the gas constant, and if h denotes the effective height of the plume we will assume that $gh/\gamma RT_e'\ll 1$. Then Eqs. (8) reduce to

$$\frac{d}{dz}(\bar{w}'\bar{b}'^2)=\alpha\bar{w}'\bar{b}', \tag{9.1}$$

$$\frac{d}{dz}(\bar{w}'^2\bar{b}'^2)=g\bar{b}'^2\overline{\Delta\gamma'}, \tag{9.2}$$

$$\frac{d}{dz}(\bar{w}'\bar{b}'^2\overline{\Delta\gamma'})=-\beta'\bar{w}'\bar{b}'^2, \tag{9.3}$$

where

$$\beta'=\frac{d}{dz}(\ln\theta_e'). \tag{10}$$

In what follows, β' will be taken as a constant.

3. Phase space solution

Let us scale all the dependent quantities in terms of their respective values at the base of the jet, i.e., at $z=0$. Thus,

$$\left.\begin{aligned}w&=\bar{w}'(z)/\bar{w}'(0)\\ b&=\bar{b}'(z)/\bar{b}'(0)\\ \Delta\gamma&=\overline{\Delta\gamma'}(z)/\overline{\Delta\gamma'}(0)\end{aligned}\right\}. \tag{11}$$

We define a Froude number F_0 at the base of the core as

$$F_0^{-2}=g\bar{b}'(0)\overline{\Delta\gamma'}(0)/2\alpha\bar{w}'^2(0), \tag{12}$$

and a stratification parameter M as

$$M=F_0^2\bar{b}'(0)\beta'/\alpha\overline{\Delta\gamma'}(0). \tag{13}$$

Finally, let

$$\left.\begin{aligned}x&=\frac{\alpha}{F_0}\left(\frac{z}{\bar{b}'(0)}\right)\\ \sigma&=wb\\ F_0q&=wb^2\\ t&=\int_0^x qdx'\end{aligned}\right\}. \tag{14}$$

Upon substitution of (11)–(14) in (9) we obtain the following:

Continuity

$$\frac{dq}{dx}=\sigma \tag{15.1}$$

Vertical momentum

$$\frac{1}{F_0}\frac{d}{dx}(\sigma^2)=2\left(\frac{b^2}{F_0^2}\right)\Delta\gamma=2\frac{q^2}{\sigma^2}\Delta\gamma \tag{15.2}$$

Entropy

$$\frac{d}{dx}[F_0q\Delta\gamma+Mt]=0, \tag{15.3}$$

with the auxiliary condition that

$$\frac{dt}{dx}=q, \tag{15.4}$$

and the boundary conditions at the base, i.e., $x=0$,

$$t=0:\quad \sigma=\Delta\gamma=F_0q=1. \tag{15.5}$$

The entropy equation integrates at once, and with the use of the boundary conditions at $x=0$, we obtain

$$\Delta\gamma=\frac{1-Mt}{F_0q}. \tag{16}$$

By using (16) and rearranging (15), there results a set of three first-order differential equations that do not contain x explicitly. Such a system is called autonomous and reads

$$\frac{dt}{dx}=q, \tag{17.1}$$

$$\frac{dq}{dx}=\sigma, \tag{17.2}$$

$$\frac{d\sigma}{dx}=\frac{q(1-Mt)}{\sigma^3}, \tag{17.3}$$

and at $x=0$

$$t(0)=0, \quad \sigma(0)=F_0 q(0)=1. \tag{17.4}$$

Let us first consider the phase space solution when $M=0$.

4. Neutral surroundings

The fluid outside the core of the jet is at a constant potential temperature, i.e, $M=\beta=0$. The order of the system diminishes by one since the system no longer depends on V, and the entropy equation integrates at once to give

$$F_0 q\Delta\gamma=1, \tag{18}$$

implying that the flux of buoyancy remains constant with height. For the buoyancy to vanish, the jet must entrain an infinite amount of mass, and this can only take place over an infinite distance. We anticipate the jet to extend to infinity in the vertical direction.

For $M=0$, the autonomous system reduces to

$$\left.\begin{aligned}\frac{dq}{dx}&=\sigma\\ \frac{d\sigma}{dx}&=\frac{q}{\sigma^3}\end{aligned}\right\}, \tag{19.1}$$

with

$$\sigma(0)=1, \quad q(0)=F_0^{-1}. \tag{19.2}$$

The numerical integration of this system in the x space is straightforward; let us instead consider some of its properties in the phase space. The elimination of x in (19.1) and the use of (19.2) lead to an integration for the stretched, nondimensional flux q, i.e.,

$$q^2=F_0^{-2}+\tfrac{2}{5}(\sigma^5-1), \tag{20.1}$$

or

$$\sigma=\left[\left(1-\frac{5}{2F_0^2}\right)+\frac{5}{2}q^2\right]^{1/5}. \tag{20.2}$$

For large values of q, σ behaves like $q^{2/5}$. The height x relates to σ through the equation

$$x=\int_1^\sigma \frac{\sigma^3 d\sigma}{q(\sigma)}=\int_1^\sigma \frac{\sigma^3 d\sigma}{\{F_0^{-2}+\frac{2}{5}(\sigma^5-1)\}^{\frac{1}{2}}}. \tag{21}$$

Since $dx=\sigma^3 d\sigma/q$, and because q and σ are positive, we can assert that increasing x corresponds to increasing σ, i.e., we can look at the plot of several physical quantities vs σ and deduce their corresponding behavior in the x space.

a. Core radius. The radius b of the jet relates to σ and $q(\sigma)$ as

$$b=\frac{F_0 q(\sigma)}{\sigma}=\left[\sigma^{-2}+\frac{2F_0^2}{5}(\sigma^3-\sigma^{-2})\right]^{\frac{1}{2}}. \tag{22}$$

We see that as $\sigma\rightarrow\infty$, $b\propto F_0\sigma^{\frac{3}{2}}$ and $q\propto\sigma^{5/2}$. Therefore, b varies linearly with x at large σ. Also,

$$\left(\frac{db}{d\sigma}\right)_{\sigma=1}=F_0\left(\frac{db}{dx}\right)_{x=0}=(F_0^2-1),$$

i.e., for $F_0>1$, b is a monotonic increasing function of σ (therefore of x), while for $F_0<1$, b decreases at first and then increases.

b. Vertical velocity. The vertical velocity w is such that

$$w=\frac{\sigma^2}{F_0 q}=\left(\frac{\sigma^4}{1+\frac{2}{5}F_0^2(\sigma^5-1)}\right)^{\frac{1}{2}}. \tag{23}$$

At the base we find that

$$\left(\frac{dw}{d\sigma}\right)_{\sigma=1}=F_0\left(\frac{dw}{dx}\right)_{x=0}\propto(2F_0^{-2}-1).$$

This imples that if $F_0>\sqrt{2}$, w is a monotonic, decreasing function with height; while if $F_0<\sqrt{2}$, w increases first with height before decreasing. Let us note that as $\sigma\rightarrow\infty$, i.e., as $x\rightarrow\infty$, $w\sim F_0^{-1}\sigma^{-\frac{1}{2}}$.

5. Stably stratified surroundings

The addition of a stable stratification, even small, will improve the mechanism by which the buoyancy force is reduced because of mixing. Since the destruction of buoyancy is more efficient when $M>0$, we anticipate that the jet will stop at some *finite* height z_f. Actually, the buoyancy will vanish at a height $z_1<z_f$, and because the jet still possesses vertical momentum at that level, it will proceed upward, the buoyancy force now opposing its motion.

a. Projection of phase space trajectory onto the (t,σ) plane. By combining (17.1) and (17.3), after an integration, we arrive at

$$\sigma^4=1+4t-2Mt^2, \tag{24}$$

i.e.,

$$t=\frac{1}{M}\left(1\pm\sqrt{1-\frac{M}{2}(\sigma^4-1)}\right). \tag{25}$$

These equations represent the projection of the phase space trajectory on the (t,σ) plane. At the base of the core $\sigma=1$ and $t=0$, i.e., the representative point moves on the negative determination; that is,

$$t^{(-)}=\frac{1}{M}\left(1-\sqrt{1-\frac{M}{2}(\sigma^4-1)}\right). \tag{26}$$

For $M>0$, σ will increase from 1 to $(1+2/M)^{1/4}$, at which point $t^{(-)}=1/M$ and $(dt^{(-)}/d\sigma)=\infty$. The representative point then continues on the positive determination, i.e.,

$$t^{(+)}=\frac{1}{M}\left(1+\sqrt{1-\frac{M}{2}(\sigma^4-1)}\right). \tag{27}$$

On this branch, σ decreases steadily from $(1+2/M)^{1/4}$ to $\sigma=0$ at which point

$$t^{(+)}=\left[1+\sqrt{1+M/2}\right]M^{-1} \quad \text{and} \quad \left(\frac{dt^{(+)}}{d\sigma}\right)_{\sigma=0}=0.$$

For $M<0$, i.e., for unstable stratifications, the representative point moves on the negative determination from $\sigma=1$ to $\sigma=\infty$. Fig. 1 represents the plot of Mt vs σ for positive and negative values of M. Fig. 2 sketches the phase space with the projection curves when $M>0$.

6. Mass flux

a. Stable surroundings: $M>0$. The nondimensional mass flux wb^2 can be expressed in terms of t as

$$(wb^2)^2=\mathrm{F_0}^2q^2=1+2\mathrm{F_0}^2\int_0^t(1+4t-2Mt^2)^{1/4}dt, \tag{28}$$

which, as shown in the Appendix, may be expressed as

$$(wb^2)^2=1+\frac{\mathrm{F_0}^2}{2}\left(\frac{2}{M}\right)^{5/4}\left(1+\frac{M}{2}\right)^{3/4}\left[\int_0^s x^{-1/2}dx(1-x)^{1/4}\right.$$
$$\left.-\frac{(1-Mt)}{|1-Mt|}\int_0^{s(Mt-1)^2}x^{-1/2}dx(1-x)^{1/4}\right]. \tag{29}$$

Let us, for convenience, introduce the notation

$$K(\xi:a,b)=\int_0^\xi x^{a-1}(1-x)^{b-1}dx;\quad 0<\xi<1. \tag{30}$$

The above integral is the indefinite Beta function[1] and can be expressed as

$$K(\xi;a,b)=\frac{\xi^2(1-\xi)^b}{a}$$
$$\times\left\{1+\sum_{n=0}^{\infty}\frac{B(a+1,\,n+1)}{B(a+b,\,n+1)}\xi^{n+1}\right\},\quad a,b>0, \tag{31}$$

where

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}.$$

b. Unstable surroundings: $M<0$. Let us first introduce the quantity μ defined as $\mu=-M$. With similar manipulations, outlined in the Appendix, we arrive at the following expressions for wb^2:

1. When $\mu/2<1$.

$$(wb^2)^2=1+\frac{\mathrm{F_0}^2}{2}\left(\frac{2}{\mu}\right)^{5/4}\left(1-\frac{\mu}{2}\right)^{3/4}\left[\int_\epsilon^{1-\mu/2}x^{-7/4}(1-x)^{1/4}dx\right.$$
$$\left.-\int_\epsilon^{(1-\mu/2)/(\mu t+1)^2}x^{-7/4}dx(1-x)^{1/4}\right]. \tag{32}$$

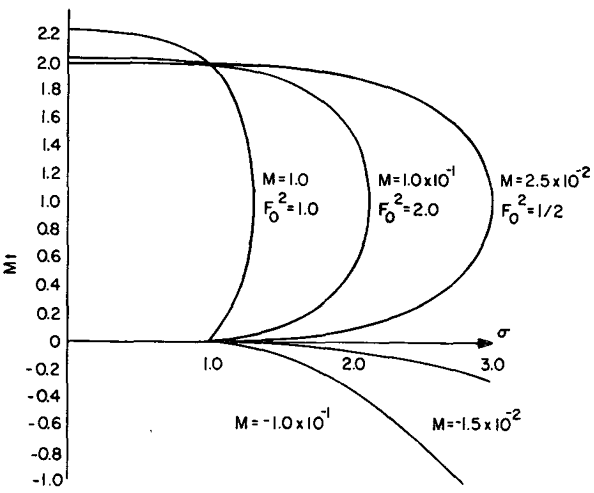

FIG. 1. Plot of various phase space trajectories onto the (Mt,σ) plane for different values of the stratification parameter M.

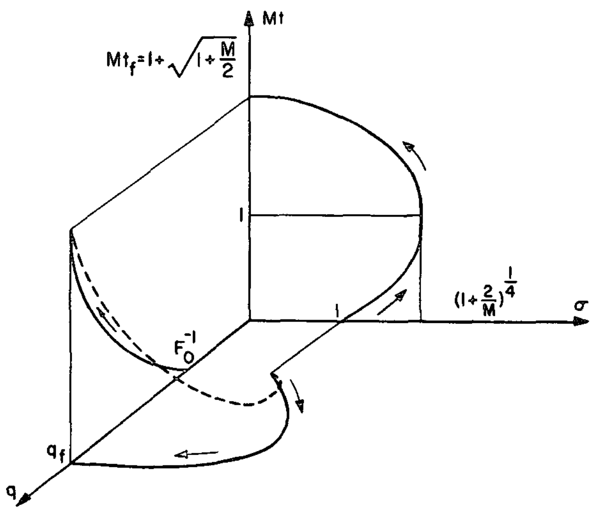

FIG. 2. Schematic representation of the phase space trajectory with its various projections in the different planes (for $M>0$).

[1] National Bureau of Standards, 1964: Handbook of Mathematical Functions, p. 944.

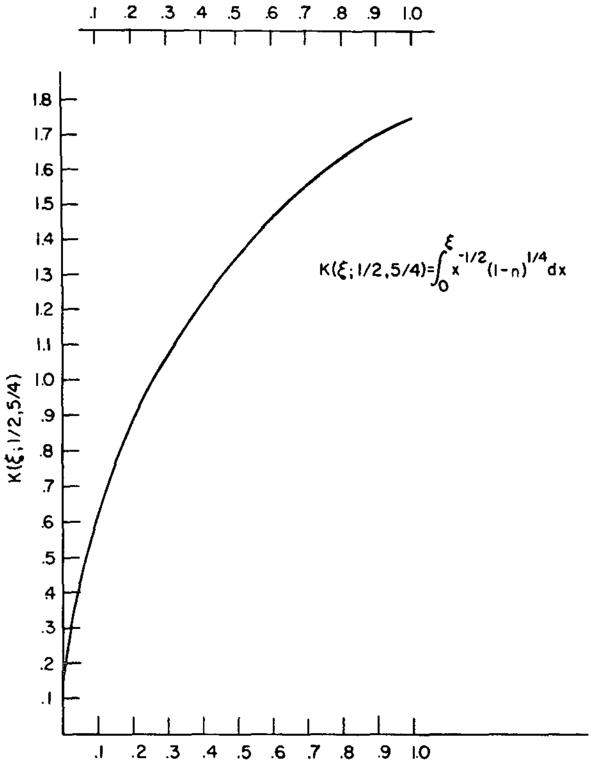

FIG. 3a. Plot of $K(\xi;\frac{1}{2},5/4)$ vs ξ.

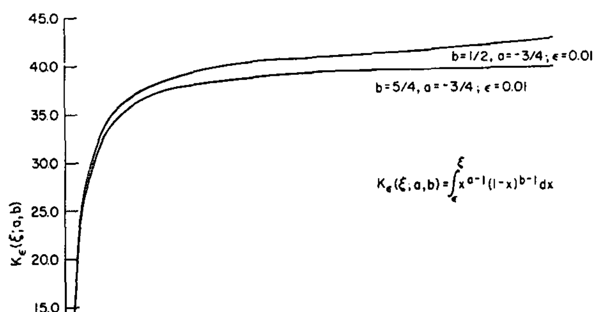

FIG. 3b. Plot of $K_\epsilon(\xi;a,b)$ vs ξ.

2. When $\mu/2>1$.

$$(wb^2)^2=1+\frac{F_0^2}{2}\left(\frac{2}{\mu}\right)^{5/4}\left(1+\frac{\mu}{2}\right)^{3/4}\left[\int_\epsilon^{1-2/\mu} x^{-7/4}(1-x)^{-1/2}dx - \int_\epsilon^{(1-2/\mu)/(1+4t+2\mu t^2)} x^{-7/4}dx(1-x)^{-1/2}\right]. \quad (33)$$

The lower limit ϵ has been introduced to avoid the singularity at $x=0$. The exact value of ϵ is irrelevant since differences are taken. In the curves of Fig. 3b, $\epsilon=0.01$.

3. When $\mu/2=1$.

$$(wb^2)^2=1-\tfrac{2}{3}F_0^2[(1+2t)^{\frac{3}{2}}-1].$$

From (29), (32) and (33) the evaluation of wb^2 requires the knowledge of integrals of the type $K(\xi;\frac{1}{2},5/4)$, $K_\epsilon(\xi;-\frac{3}{4},5/4)$ and $K_\epsilon(\xi;-\frac{3}{4},\frac{1}{2})$. These three integrals are plotted as functions of ξ in Fig. 3.

If we could establish a correspondence between x, the nondimensional height, and t, we then would be in a position to evaluate all the relevant physical quantities, since

$$\text{core radius } b=\frac{wb^2}{\sigma(t)},\quad \sigma^4=1+4t-2Mt^2,$$

$$\text{vertical velocity } w=\frac{\sigma^2(t)}{wb^2},$$

$$\text{buoyancy } \Delta\gamma=\frac{1-Mt}{(wb^2)},$$

i.e., if for a given x we could find the corresponding t, by using the analytical expressions for wb^2, we would then be able to determine b, w, $\Delta\gamma$.

7. Relation between x and t

In most problems of interest, the logarithmic derivative with height of the potential temperature can be taken to be piece-wise constant, i.e., for each slice in which M can be treated as constant we must formulate the problem using the conditions at the bottom of the slice, these conditions being obtained by the solution of the previous slices. The problem then proceeds in small steps upwards. We then anticipate that we ought to look for an approximate expression for $x(t)$, which is most accurate for small values of t. (Here t can be thought of as a measure of height in phase space.)

The exact expression for $x(t)$ is

$$x(t)=\int_0^t \frac{F_0 d\xi}{\left[1+2F_0^2\int_0^\xi \sigma(\eta)d\eta\right]^{\frac{1}{2}}}, \quad (34)$$

where

$$\sigma(\eta)=(1+4\eta-2M\eta^2)^{\frac{1}{4}}.$$

We approximate the curve $\sigma(t)$ by a straight line that pivots about $t=0$, $\sigma=1$ (see Fig. 1), i.e.,

$$\sigma_{\text{approx}}=S(\eta)=\left[\frac{\sigma(t)-1}{t}\right]\eta+1=a(t)\eta+1, \quad (35)$$

where t must be thought of as the fixed point at which

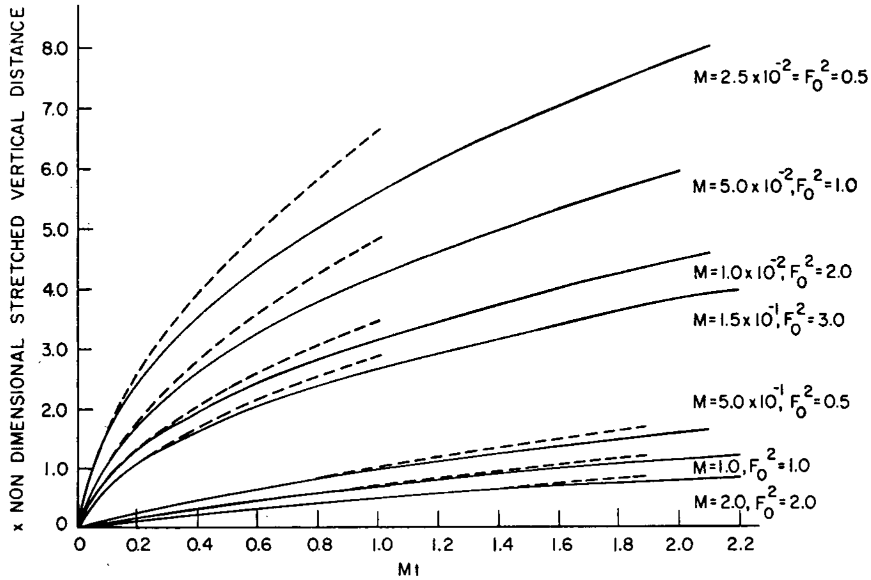

FIG. 4. Plot of computed and approximate curves of x vs Mt for different M and F_0.

we want to evaluate $x(t)$. We then substitute (35) in (34), and after some straightforward algebra, to obtain

$$x(t)=\frac{1}{\sqrt{a(t)}}\ln\sigma(t)\left(\frac{1+\sqrt{1+(a(t)F_0^{-2}-1)/\sigma^2(t)}}{1+\sqrt{a(t)F_0^{-2}}}\right), \tag{36}$$

where

$$a(t)=\frac{\sigma(t)-1}{t}.$$

Fig. 4 shows a plot of x vs Mt for different values of M and of F_0; the solid lines represent the computed values obtained by an x-space integration of (17). The dashed lines in Fig. 4 are the values of x obtained by using (36). The approximation appears to be very accurate for small values of Mt and x and the accuracy improves as M becomes larger.

8. Example

To illustrate the method, consider a plume where $\bar{b}'(0)=10$ m, $\bar{w}'(0)=3.13$ m sec^{-1} and $\overline{\Delta\gamma}'(0)=0.6$, rising in an atmosphere with the lapse rate shown in Fig. 5. Take $\alpha=0.1$.

We have the following relations:

$$\sigma^4=1+4t-2Mt^2, \tag{37}$$

$$a=\frac{\sigma(t)-1}{t}, \tag{38}$$

$$x=\frac{1}{\sqrt{a}}\ln\left[\frac{\sigma+\sqrt{\sigma^2+aF_0^{-2}-1}}{1+\sqrt{aF_0^{-2}}}\right], \tag{39}$$

$$\Delta z=\frac{x\bar{b}'(0)}{\alpha F_0^{-1}}, \tag{40}$$

with

$$F_0^{-2}=\left(\frac{g}{2\alpha}\right)\frac{\bar{b}'(0)\overline{\Delta\gamma}'(0)}{\bar{w}'^2(0)}=49\frac{\bar{b}'(0)\overline{\Delta\gamma}'(0)}{\bar{w}'^2(0)}, \tag{41}$$

$$M=\left(\frac{2}{g}\right)\frac{\bar{w}'^2(0)}{\overline{\Delta\gamma}'^2(0)}\beta'=\frac{1}{4.9}\left(\frac{\bar{w}'(0)}{\overline{\Delta\gamma}'(0)}\right)^2\beta', \tag{42}$$

where β' is the lapse rate.

The mass flux is determined by Eqs. (29) for $M>0$, by (32) for $-M/2<1$ and by (33) for $-M/2>1$. For the case where $M=0$ the mass flux is given by

$$(wb^2)^2=1+\frac{2}{5F_0^{-2}}(\sigma^5-1). \tag{43}$$

Finally,

$$x(t)=\int_0^t\frac{F_0d\xi}{\left[1+2F_0^2\int_0^\xi\sigma(\eta)dy\right]^{\frac{1}{2}}}<\int_0^t F_0d\xi=F_0t, \tag{44}$$

i.e.,

$$t>\frac{x}{F_0}=\alpha F_0^{-2}\frac{\Delta z}{\bar{b}'(0)}, \tag{45}$$

or

$$t>t_{\min},$$

where

$$t_{\min}=\alpha F_0^{-2}\frac{\Delta z}{\bar{b}'(0)}. \tag{46}$$

The numerical calculations proceed in the following manner. Conditions at the base of a given layer,

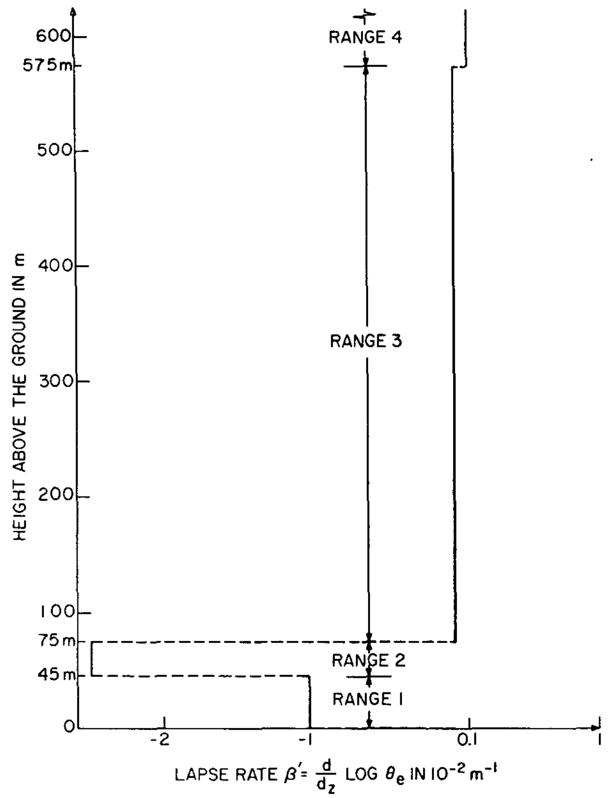

FIG. 5. Model lapse rate to illustrate the proposed method.

namely, $\bar{b}'(0)$, $\bar{w}'(0)$, $\overline{\Delta\gamma}'(0)$, F_0, M, are known. A value of Δz is chosen at which quantities are to be determined. and the corresponding $t_{\min}$ is computed using (46). For moderate values of $t_{\min}$, $t=t_{\min}$ is used to calculate σ, a, x from (37), (38), (39), the corrected Δz then being recomputed using x in (40). Having obtained the height in physical space corresponding to the chosen height t in phase space, the flux in phase space is computed using (29), (32) or (33) as required. Knowing the mass flux, we then compute

$$b=\frac{wb^2}{\sigma},\quad w=\frac{\sigma^2}{wb^2},\quad \Delta\gamma=\frac{1-Mt}{wb^2},\tag{47}$$

and form

$$\left.\begin{aligned}\bar{b}'(\Delta z)&=b\bar{b}'(0)\\ \bar{w}'(\Delta z)&=w\bar{w}'(0)\\ \overline{\Delta\gamma}'(\Delta z)&=\Delta\gamma\overline{\Delta\gamma}'(0)\end{aligned}\right\}.\tag{48}$$

Let us illustrate the above outline.

a. Range 1: Unstable surroundings. At the start $F_0^{-2}=30$, $M=-5.55\times10^{-2}$ and for a 45-m layer $t_{\min}$ is 13.5. Because $t_{\min}$ is large we do not expect to get to the 45-m height in one step. Taking $t=2.0$, we compute σ $(=1.75)$ and a $(=0.375)$, and use (39) to find $x=0.344$, corresponding to a height $\Delta z=6.3$ m calculated using (40). Because M is negative and $|M|/2<1$, (32) is used for the mass flux calculations. (The differences in the square brackets of (29), (32) and (33) were calculated using tabulated values for these integrals.) Then, $wb^2=1.09$, and

$$\begin{aligned}b&=0.624,\ \text{giving}\ \bar{b}'(\Delta z_1)=0.624\times10\ \text{m}=6.24\ \text{m},\\ w&=2.82,\ \text{giving}\ \bar{w}'(\Delta z_1)=2.82\times3.12\ \text{m sec}^{-1}\\ &\qquad=8.82\ \text{m sec}^{-1},\\ \Delta\gamma&=1.02,\ \text{giving}\ \overline{\Delta\gamma}'(\Delta z_1)=1.02\times0.6=0.610,\end{aligned}$$

and

$$F_0^{-2}(\Delta z_1)=2.4,\quad M(\Delta z_1)=-0.425.$$

Again, taking $t=2$, we find that $\sigma=1.875$, $a=0.438$, $x=0.938$, $\Delta z_2=39$ m. We are now at the height $z=\Delta z_1+\Delta z_2=6.3$ m$+39$ m$=45.3$ m, which is close enough to the top of the first range. Because $|M|/2<1$, we use (32) for mass flux calculations, giving $wb^2=1.74$, and

$$\begin{aligned}b&=0.92,\ \text{yielding}\ \bar{b}'(\Delta z_2)=5.75\ \text{m},\\ w&=2.02,\ \text{yielding}\ \bar{w}'(\Delta z_2)=17.8\ \text{m sec}^{-1},\\ \Delta\gamma&=1.065,\ \text{yielding}\ \overline{\Delta\gamma}'(\Delta z_2)=0.65.\end{aligned}$$

b. Range 2: Unstable surroundings. The starting conditions are $F_0^{-2}=0.58$ and $M=-3.82$. For an interval of $\Delta z=30$ m $t_{\min}$ is 0.30. Taking $t=0.1$, we find that $\sigma=1.1$, $a=1.0$, $x=0.122$ and $\Delta z=9.25$ m. Therefore, at height $z=\sum_{i=1}^{3}\Delta z_i=54.25$ m, we find by using (33) that $wb^2=1.36$, and

$$\begin{aligned}b&=1.24,\ \text{yielding}\ \bar{b}'(\Delta z_3)=7.1\ \text{m},\\ w&=0.89,\ \text{yielding}\ \bar{w}'(\Delta z_3)=15.9\ \text{m sec}^{-1},\\ \Delta\gamma&=1.3,\ \text{yielding}\ \overline{\Delta\gamma}'(\Delta z_3)=0.842,\end{aligned}$$

and

$$F_0^{-2}(\Delta z_3)=1.15,\quad M(\Delta z_3)=-1.82.$$

We now form $t_{\min}$ and find a value of 0.345 based on a thickness of 20.75 m. Taking $t=0.3$, we have the result that $\sigma=1.26$, $a=0.867$, $x=0.248$, $\Delta z=16.4$ m. At the height $z=\sum_{i=1}^{4}\Delta z_i=70.6$ m, we find using (32) that $wb^2=1.27$, with

$$\begin{aligned}b&=1.005,\ \text{giving}\ \bar{b}'(\Delta z_4)=7.15\ \text{m},\\ w&=1.25,\ \text{giving}\ \bar{w}'(\Delta z_4)=19.9\ \text{m sec}^{-1},\\ \Delta\gamma&=1.22,\ \text{giving}\ \overline{\Delta\gamma}'(\Delta z_4)=1.025,\end{aligned}$$

and

$$F_0^{-2}(\Delta z_4)=0.9,\quad M(\Delta z_4)=-1.92.$$

We have 4.4 m to go to complete range no. 2. The value of $t_{\min}$ computed on that height is 0.056. We take $t=0.056$; then $\sigma=1.053$, $a=0.95$, $x=0.0595$ and $\Delta z_5=4.45$ m, which is close enough. At the top of range

TABLE 1. Results of sample problem.

Height above the ground z (m)	Height increments Δz (m)	Range	Physical quantities: Plume radius $\bar{b}'(z)$ (m)	Physical quantities: Vertical velocity $\bar{w}'(z)$ (m sec^{-1})	Physical quantities: Buoyancy $\underline{\Delta\gamma}'(z)$	Physical quantities: Froude number F_0^{-2}	Physical quantities: Stratification parameter M	Side calculations: Phase space height t	Side calculations: Nondimensional physical height x	Side calculations: Normalized flux wb^2	Side calculations: Normalized core radius b	Side calculations: Normalized velocity w	Side calculations: Normalized buoyancy $\Delta\gamma$
0		1	10	3.13	0.6	30	-5.55×10^{-2}						
	6.3							2.0					
6.3		1	6.24	8.82	0.61	2.4	-0.425		0.344	1.09	0.624	2.82	1.02
	39							2.0					
45.3		1	5.75	17.8	0.65	0.58	-3.82		0.938	1.74	0.92	2.02	1.065
45.3		2	5.75	17.8	0.65	0.58	-3.82						
	9.25							0.1					
54.25		2	7.1	15.9	0.842	1.15	-1.82		0.122	1.36	1.24	0.89	1.3
	16.4							0.3					
70.6		2	7.15	19.9	1.025	0.9	-1.92		0.248	1.27	1.005	1.25	1.22
	4.45							0.056					
75		2	7.25	20.7	1.517	1.24	0		0.0595	1.065	1.012	1.04	1.48
75		3	7.25	20.7	1.517	1.24	0						
	66							2.0					
141		3	10.5	24.6	0.595	0.462	0		1.125	2.515	1.45	1.19	0.398
	210							2.0					
351		3	22.3	20	0.162	0.442	0		1.36	3.68	2.12	0.815	0.272
	181							0.5					
532		3	32.7	18	0.0846	0.42	0		0.54	1.915	1.465	0.904	0.522
	4.4							0.056					
576		3	35.3	17.5	0.0512	0.246	0.238		0.024	1.14	1.08	0.975	0.98
576		4	35.3	17.5	0.0512	0.246	0.238						
	1050							2.0					
1626		4	106	9.5	0.0055				1.48	4.91	3.01	0.542	0.107

2 we have [using (32)] $wb^2=1.065$, with

$$b=1.012,\ \text{giving}\ \bar{b}'(\Delta z_5)=7.25\ \text{m},$$
$$w=1.04,\ \text{giving}\ \bar{w}'(\Delta z_5)=20.7\ \text{m sec}^{-1},$$
$$\Delta\gamma=1.48,\ \text{giving}\ \overline{\Delta\gamma}'(\Delta z_5)=1.517.$$

c. Range 3: Neutral surroundings. The starting conditions are now $F_0^{-2}=1.24$, $M=0$. Using $t=2$ we have $\sigma=1.73$, $a=0.365$, $x=1.125$ and $\Delta z_6=66$ m. At the height $z=\sum_{i=1}^{6}\Delta z_i=141$ m, the mass flux [using (43)] is $wb^2=2.515$, with

$$b=1.45,\ \text{yielding}\ \bar{b}'(\Delta z_6)=10.5\ \text{m}$$
$$w=1.19,\ \text{yielding}\ \bar{w}'(\Delta z_6)=24.6\ \text{m sec}^{-1},$$
$$\Delta\gamma=0.398,\ \text{yielding}\ \overline{\Delta\gamma}'(\Delta z_6)=0.595,$$

with

$$F_0^{-2}(\Delta z_6)=0.462\quad\text{and}\quad M=0.$$

Table 1 summarizes the above calculations, the numerical results being stopped at a height of 1625 m. Fig. 6 shows the plume radius and the vertical velocity. The above example was programmed and solved on a computer and the results are displayed by the full curves. The crosses indicate the results of hand calculations.

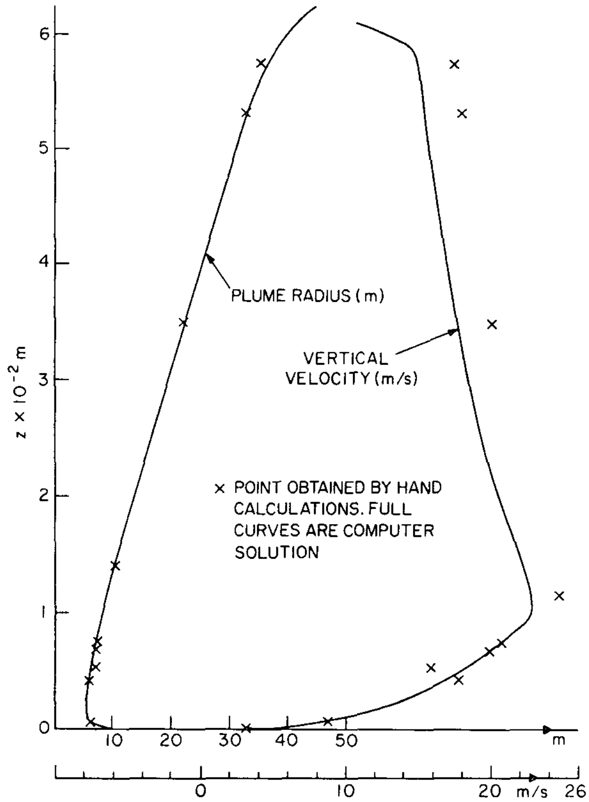

FIG. 6. Plot of the plume radius and vertical velocity vs height.

9. Conclusions

We first consider the behavior of a buoyant plume in neutral surroundings, and by considering the behavior of the system in the phase plane we were able to ascertain some of the characteristics of the flow in physical space. In particular, for a base Froude number smaller than unity there is first a pinch in the plume diameter and then its radius increases with height. Also, for a base Froude number $<\sqrt{2}$, the vertical velocity increases at first before becoming a monotonic decreasing function of height. The same holds true for stable surroundings; as could have been anticipated, the local behavior of the plume near its base is essentially controlled by the base Froude number. When the surroundings are stably stratified the plume rises to a finite height; in so doing entrainment is present over the entire height of the plume. The model considered in this paper was derived using the assumption that vertical gradients are small compared to radial gradients. In the case of stably stratified surroundings the core radius increases to very large values as the top of the plume is approached, and as a result this assumption breaks down at some height below the top of the plume. Also, in cases where the core radius decreases with height near the base, care must be exercised to ensure that radial gradients are indeed much larger than vertical gradients.

By considering the behavior of the system in a phase space where the vertical direction is a measure of the flux integral with height and the other two coordinates are the flux and its derivative with height, we were able obtain closed expressions in t for all the relevant physical quantities that characterize a buoyant plume rising in nonrotating surroundings. As pointed out previously, the logarithmic derivative with height of the potential temperature can be considered constant within slices of the atmosphere, and the problem must be formulated *ab-initio* at the base of each slice and continued into the next. By using an approximate expression relating t to x, we are able to evaluate the relevant physical quantities in physical space.

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APPENDIX

Mass Flux Calculations

a. Stable surroundings (M>0): For a given value of t, the flux wb^2 can be expressed as

$$(wb^2)^2=1+2F_0^2\int_0^t(1+4\xi-2M\xi^2)^{\frac{1}{2}}d\xi. \qquad (1a)$$

We consider the fourth power of the integrand found in (1a), i.e.,

$$1+4\xi-2M\xi^2=2M\left(-\xi^2+\frac{2}{M}\xi+\frac{1}{2M}\right)$$
$$=\left(\frac{M+2}{M}\right)\left\{1-\left[\frac{M\xi-1}{(1+M/2)^{\frac{1}{2}}}\right]^2\right\}. \quad (2a)$$

Let

$$s=1\Big/\left(1+\frac{M}{2}\right), \quad (3a)$$

and when $Mt<1$ define

$$\zeta=\frac{1-M\xi}{(1+M/2)^{\frac{1}{2}}}, \quad (4a)$$

so that (1a) becomes

$$(wb^2)^2=1+2\mathrm{F}_0{}^2\frac{(M+2)^{\frac{3}{4}}}{\sqrt{2}M^{5/4}}\int_{(1-Mt)s^{\frac{1}{2}}}^{s} d\zeta(1-\zeta^2)^{\frac{1}{4}}. \quad (5a)$$

If $Mt>1$, let

$$r=\frac{M\xi-1}{(1+M/2)^{\frac{1}{2}}},$$

and write (1a) as

$$(wb^2)^2=1+2\mathrm{F}_0{}^2(I_1+I_2), \quad (6a)$$

where

$$I_1=\int_0^{1/M} d\xi(1+4\xi-2M\xi^2)^{\frac{1}{4}}$$
$$=\frac{(2+M)^{\frac{3}{4}}}{\sqrt{2}M^{5/4}}\int_0^{s^{\frac{1}{2}}} d\zeta(1-\zeta^2)^{\frac{1}{4}}, \quad (7a)$$

and

$$I_2=\int_{1/M}^{t} d\xi(1+4\xi-2M\xi^2)^{\frac{1}{4}}$$
$$=\frac{(2+M)^{\frac{3}{4}}}{\sqrt{2}M^{5/4}}\int_0^{(Mt-1)s^{\frac{1}{2}}} dr(1-r^2)^{\frac{1}{4}}. \quad (8a)$$

Now, let $\zeta^2=x$ in (5a) and (7a) and $r^2=p$ in (8a); we then obtain

$$(wb^2)^2=\begin{cases} 1+\mathrm{F}_0{}^2\dfrac{(2+M)^{\frac{3}{4}}}{M^{5/4}}\left\{\displaystyle\int_0^s \frac{dx}{x^{\frac{1}{2}}}(1-x)^{\frac{1}{4}} - \int_0^{s(1-Mt)^2}\frac{dx}{x^{\frac{1}{2}}}(1-x)^{\frac{1}{4}}\right\}, & \text{for } Mt\le 1, \quad (9a)\\[2ex] 1+\dfrac{\mathrm{F}_0{}^2(2+M)^{\frac{3}{4}}}{M^{5/4}}\left\{\displaystyle\int_0^s\frac{dx}{x}(1-x)^{\frac{1}{4}} + \int_0^{s(Mt-1)^2}\frac{dp}{p}(1-p)^{\frac{1}{4}}\right\}, & \text{for } Mt>1. \quad (10a)\end{cases}$$

We now can combine (9a) and (10a) into a single equation that reads

$$(wb^2)^2=1+\frac{\mathrm{F}_0{}^2(2+M)^{\frac{3}{4}}}{M^{5/4}}\left\{\int_0^s x^{-\frac{1}{2}}dx(1-x)^{\frac{1}{4}} -\frac{1-Mt}{|1-Mt|}\int_0^{s(Mt-1)^2} x^{-\frac{1}{2}}dx(1-x)^{\frac{1}{4}}\right\}. \quad (11a)$$

b. Unstable surroundings ($M<0$). Let

$$\mu=-M. \quad (12a)$$

Then,

$$1+4\xi-2M\xi^2=\frac{2}{\mu}\left[(\mu\xi+1)^2+\left(\frac{\mu}{2}-1\right)\right]. \quad (13a)$$

We must now consider three cases.

CASE 1. $[(\mu/2)-1]<0$.

Then (13a) factors as

$$1+4\xi-2M\xi^2=\left(\frac{2-\mu}{\mu}\right)\left\{\left[\frac{\mu\xi+1}{(1-\mu/2)^{\frac{1}{2}}}\right]^2-1\right\}. \quad (14a)$$

Again, let

$$\zeta=\frac{\mu\xi+1}{(1-\mu/2)^{\frac{1}{2}}}.$$

Then,

$$\int_0^t (1+4\xi-2M\xi^2)^{\frac{1}{4}}d\xi$$
$$=\frac{2^{\frac{1}{4}}(1-\mu/2)^{\frac{3}{4}}}{\mu^{5/4}}\int_{1/(1-\mu/2)^{\frac{1}{2}}}^{(\mu\xi+1)/(1-\mu/2)^{\frac{1}{2}}} d\zeta(\zeta^2-1)^{\frac{1}{4}}. \quad (15a)$$

Letting $\zeta^2=1/\tau$ in (15a), we find that

$$\int_{1/(1-\mu/2)^{\frac{1}{2}}}^{(\mu\xi+1)/(1-\mu/2)^{\frac{1}{2}}} d\zeta(\zeta^2-1)^{\frac{1}{4}}$$
$$=\frac{1}{2}\int_{(1-\mu/2)/(\mu t+1)^2}^{1-\mu/2} \tau^{-7/4}(1-\tau)^{\frac{1}{4}}d\tau. \quad (16a)$$

By substituting (16a) into (15a) and the resulting equation into (1a), we obtain Eq. (32) of the text.

CASE 2. $[(\mu/2)-1]>0$.

In this case (13a) is

$$1+4\xi-2M\xi^2$$
$$=\frac{2}{\mu}\left[(\mu/2)-1\right]\left\{\left(\frac{\mu\xi+1}{[(\mu/2)-1]^{\frac{1}{2}}}\right)^2+1\right\}, \quad (17a)$$

and

$$\int_0^t d\xi(1+4\xi-2M\xi^2)^{\frac{1}{4}}$$
$$=2^{\frac{1}{4}}\frac{[(\mu/2)-1]}{\mu^{5/4}}\int_{1/[(\mu/2)-1]^{\frac{1}{2}}}^{(\mu t+1)/[(\mu/2)-1]^{\frac{1}{2}}} d\zeta(\zeta^2+1)^{\frac{1}{4}}. \quad (18a)$$

Then, let

$$1+\zeta^2=\frac{1}{\rho}, \quad (19a)$$

i.e.,

$$d\zeta=-\frac{1}{2}\left(\frac{\rho}{1-\rho}\right)^{\frac{1}{2}}\frac{d\rho}{\rho^2}. \quad (20a)$$

Then,

$$(1+\zeta^2)^{\frac{1}{4}}d\zeta=-\tfrac{1}{2}\rho^{-7/4}(1-\rho)^{-\frac{1}{2}}d\rho, \quad (21a)$$

and (18a) becomes

$$\int_{1/[(\mu/2)-1]^{\frac{1}{2}}}^{(\mu t+1)/[(\mu/2)-1]^{\frac{1}{2}}} d\zeta(\zeta^2+1)^{\frac{1}{4}}$$
$$=\frac{1}{2}\int_{(1-2/\mu)/(1+4t+2\mu t^2)}^{1-2/\mu}\rho^{-7/4}(1-\rho)^{-\frac{1}{2}}d\rho. \quad (22a)$$

By substituting (22a) into (18a) and the resulting equation into (1a) we obtain Eq. (33) of the text.

CASE 3. $[(\mu/2)-1]=0$, $(\mu=2)$.

In this case (13a) reduces to

$$1+4\xi-2M\xi^2=2/\mu(\mu\xi+1)^2=(2\xi+1)^2=\zeta^2,$$

and therefore

$$\int_0^t d\xi(1+4\xi-2M\xi^2)^{\frac{1}{4}}$$
$$=\frac{1}{2}\int_1^{(2t+1)}\zeta^{\frac{1}{2}}d\zeta=\tfrac{1}{3}[(2t+1)^{\frac{3}{2}}-1],$$

i.e.,

$$(wb^2)^2=1-\frac{2F_0{}^2}{3}[(2t+1)^{\frac{3}{2}}-1]. \quad (23a)$$

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