Asymptotic Similarity in Neutral Barotropic Planetary Boundary Layers[1]

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ABSTRACT

New similarity expressions for turbulent Ekman layers have recently appeared, the theoretical foundation of which is analyzed in this paper. It is shown that the flow in Ekman layers has to be solved by singular perturbation methods. The similarity laws given by Gill and Csanady are, in first approximation, independent of the surface Rossby number if it is sufficiently large.

1. Nature of the problem

The subject of our analysis is the turbulent boundary layer in a steady-state adiabatic atmosphere with barotropic conditions. Quite recently, a new description of the turbulent Ekman layer has been proposed by Gill[2] and Csanady (1967). The approach taken by these investigators follows very closely the empirical studies of similarity in turbulent shear flows developed in the last decade (e.g., Clauser, 1956; Coles, 1956). Csanady and Gill found that the new similarity laws compare favorably with experimental data. In this paper we shall concentrate on the fundamental properties and the logical foundations of the new similarity laws, since we feel that Gill and Csanady were not fully aware of the elegance of their propositions.

The flow within the barotropic planetary boundary layer under neutral steady-state conditions is a function only of the independent variable z (height) and three external parameters G (geostrophic wind speed), f (Coriolis parameter) and z_0 (surface roughness length). Out of these four variables one can form, at most, two independent dimensionless ratios, the selection of which is rather arbitrary. We may take, for example, the ratios z/z_0 and what has been named the surface Rossby number, $\mathrm{Ro}=G/fz_0$, as the two independent ratios, but these may not be the ones best suited for scaling the flow and the equations. Other parameters of the flow related to G, f and z_0 enter into the solution in an important way, as, for example, a scale height h of the boundary layer and the friction velocity u_* related to the stress at the surface. One could use these parameters to formulate other pairs of independent ratios, as for example z/z_0 and h/z_0, or z/z_0 and z/h. The problem may be solved satisfactorily by describing suitably scaled dependent variables of the flow in terms of any two of the independent ratios, and by finding the relations that exist between these independent ratios and those formed out of z and the external parameters.

We are interested in the properties of the planetary (Ekman) layer at very large values of h/z_0. We have to recognize that large-scale features of the flow may be lost if z_0 is used as a length scale, while small-scale features of the flow that may occur close to the surface cannot be adequately described when the whole depth h of the planetary layer is used as a length scale. We are thus led to consider separately the flow in the surface layer and the flow in the outer portion of the boundary layer. For convenience we shall call the part of the planetary layer above the surface layer the "outer Ekman layer."

In the atmosphere the ratio h/z_0 is very large. Typical values are 10^4–10^5. The ratio does not involve z, and so must be a unique function of surface Rossby number Ro, which is also typically very large (10^5–10^9). Our analysis rests on the proposition that when correctly scaled the nondimensional description of the flow in the atmospheric turbulent boundary layer should be well behaved (that is, should approach neither zero nor infinity) in the limit as h/z_0 approaches infinity. The virtues of this asymptotic approach will become evident in the course of the analysis. The technique that we will employ is based on a branch of mathematical analysis known as singular perturbation methods (Van Dyke, 1964).

2. Equations of the flow

We concern ourselves with steady-state horizontal flows above an infinite homogeneous surface, and we neglect variations of density within the boundary layer. The appropriate equations of motion are [see, for

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[2] Gill, M. E., 1967: The turbulent Ekman layer. Dept. Applied Mathematics and Theoretical Physics, Univ. of Cambridge, England (unpublished manuscript).

example, Haltiner and Martin (1957, p. 220)]:

$$f(v-v_g)+\frac{d}{dz}\left(\frac{\tau_x}{\rho}\right)=0, \tag{1}$$

$$-f(u-u_g)+\frac{d}{dz}\left(\frac{\tau_y}{\rho}\right)=0. \tag{2}$$

In (1) and (2), u and v are the x and y components, respectively, of the mean wind velocity, u_g and v_g the same for the geostrophic free stream velocity, τ_x and τ_y the same for the Reynolds stress, f is the Coriolis parameter, and ρ the density. The coordinate system will be oriented so that at the surface

$$\tau_x=\rho u_*^2, \quad \tau_y=0. \tag{3}$$

The modulus of the geostrophic wind G is given by

$$G^2=u_g^2+v_g^2. \tag{4}$$

The zero point of the height coordinate is defined so that the wind speed vanishes at a height $z=z_0$, Eq. (3) being assumed to apply at this level. At infinite heights the stress must vanish and the flow becomes geostrophic.

We must now attempt to find scales for the variables in the equations of motion such that the nondimensionalized terms of the equation are well behaved (i.e., neither degenerate to zero nor become infinite) in the asymptotic limit.

In view of the boundary condition at the surface it is clear that τ_x and τ_y should be scaled by ρu_*^2, but we do not immediately know what velocity and length scales W and h to use in order to complete the nondimensionalizing of the equations. For example, W might be u_* or G, but this leads to a dilemma since, if the terms of the equations of motion are well-behaved at infinite h/z_0 when the velocity defects are scaled by G, they cannot be well-behaved when scaled by u_*, for the ratio u_*/G approaches zero at infinite h/z_0.

While the equations of motion above are incapable of indicating what velocity to use for scaling, they do provide a relation between the length scale h and the unknown velocity scale W. If we define

$$F_x\equiv\frac{u-u_g}{W}, \quad F_y\equiv\frac{v-v_g}{W}, \tag{5}$$

$$t_x\equiv\tau_x/\rho u_*^2, \quad t_y\equiv\tau_y/\rho u_*^2, \tag{6}$$

the equations of motion become formally independent of the ratio z_0/h only if

$$h=u_*^2/fW. \tag{7}$$

This can be verified by substituting (5), (6) and (7) into the equations of motion. Thus, we have

$$F_y+\frac{hdt_x}{dz}=0, \tag{8}$$

$$-F_x+\frac{hdt_y}{dz}=0. \tag{9}$$

Integrating (8) and (9) and employing the boundary conditions, we obtain

$$\int_0^\infty F_y\frac{d(z-z_0)}{h}=1, \tag{10}$$

$$\int_0^\infty F_x d\left(\frac{z-z_0}{h}\right)=0. \tag{11}$$

The length scale h is thus seen to be an integral height scale of the nondimensional velocity defects F_x and F_y.

The flows we must consider are those for which z_0/h approaches zero. In this limit the scaled velocity defects F_x and F_y must be unique functions of z/h in the domain that excludes the surface layer (where z is of the same order as z_0). Eqs. (8) and (9) then show that the scaled stresses must similarly be functions only of z/h. The equations of motion have thus been nondimensionalized in such a way that the terms are capable of the postulated asymptotic behavior, but the equations are incapable by themselves of indicating how the velocity scale should be selected.

3. Surface layer

The surface layer is an integral part of the entire Ekman layer. However, there are special problems to be faced here because the universal velocity-defect functions F_x and F_y that describe the flow throughout the bulk of the Ekman layer are not capable of describing the flow in the vicinity of the surface, where all of the length scales of the flow must be scaled by z_0.

If we suppose that the scaling velocity can be correctly chosen so that the terms of the equation of motion are asymptotically well-behaved in the outer Ekman layer, these terms cannot be well-behaved when z_0 is used as a height scale. This result may be obtained by rewriting (8) and (9) as follows:

$$\frac{(v-v_g)}{W}\frac{z_0}{h}+\frac{d(\tau_x/\rho u_*^2)}{d(z/z_0)}=0, \tag{12}$$

$$-\left(\frac{u-u_g}{W}\right)\frac{z_0}{h}+\frac{d(\tau_y/\rho u_*^2)}{d(z/z_0)}=0. \tag{13}$$

Since the velocity defect terms are well-behaved, it follows that as $h/z_0 \to \infty$,

$$\frac{d(\tau_x/\rho u_*^2)}{d(z/z_0)} = O\left(\frac{z_0}{h}\right), \quad \frac{d(\tau_y/\rho u_*^2)}{d(z/z_0)} = O\left(\frac{z_0}{h}\right). \qquad (14)$$

Thus, as h/z_0 approaches infinity, the surface layer becomes a constant-stress layer, and the stress within this layer must be written

$$\tau_x = \rho u_*^2, \quad \tau_y = 0. \qquad (15)$$

The equations of motion then degenerate and provide no basis for establishing the correct identification of the scaling velocity W.

The maintenance of the stress within the surface boundary layer requires the maintenance of turbulence energy, and it is to be expected that the turbulent energy and the Reynolds stress will be related to each other in a way that does not depend on the surface Rossby number or on the ratio h/z_0 (which is a function of Ro). The rate of production of turbulent energy per unit mass within the surface layer follows from (15) to be $u_*^2\, du/dz$, while the rate of dissipation is equal to $3q^3/2\ell$, where $3q^2/2$ is the mean kinetic energy and ℓ is the integral length scale of the turbulence (Batchelor, 1953). Now u_* is the only velocity scale that will make the turbulence intensity q independent of h/z_0 (and of Ro) as $h/z_0 \to \infty$. Thus,

$$q/u_* = Q(z/z_0). \qquad (16)$$

Moreover, ℓ must scale with z_0 like all lengths in the surface layer. Hence, we have

$$\ell/z_0 = L(z/z_0). \qquad (17)$$

If we now equate the production and dissipation rates of turbulent energy within the surface layer, we obtain

$$\frac{d(u/u_*)}{d(z/z_0)} = \frac{Q^3}{L}. \qquad (18)$$

Since Q and L are functions of z/z_0 only in the limit as $h/z_0 \to \infty$, we can now assert that u/u_* is a universal function of z/z_0 within the surface layer. Formally,

$$\frac{u}{u_*} = f_x\left(\frac{z}{z_0}\right). \qquad (19)$$

We can obtain an estimate for the rate of change of v with respect to height by differentiating (1) and nondimensionalizing. We will use u_* as a velocity scale, as suggested by (19), since we expect v to be at most of the same order of magnitude as u. Using also (7), we obtain

$$\frac{d(v/u_*)}{d(z/z_0)} = \frac{W}{u_*}\left(\frac{z_0}{h}\right)^2 \frac{d^2 t_x}{d(z/h)^2}.$$

Since the second derivative of t_x with respect to z/h is finite, due to the postulated behavior of (8), we conclude that v/u_* is independent of height within the surface layer in the limit as $h/z_0 \to \infty$, even if W/u_* increases with Ro as rapidly as h/z_0. Since $v=0$ at the surface, we have, for the entire surface layer,

$$v/u_* = 0. \qquad (20)$$

Eqs. (19) and (20) indicate that we expect the flow in the surface layer, when scaled with u_* and z_0, to be independent of the surface Rossby number in the limit as $h/z_0 \to \infty$. This is one form of what may be called "Rossby-number similarity."

It is noted that the solution (15), (19) and (20) is designed to meet the boundary conditions of velocity and stress at the surface. However, (15), (19) and (20) are unable to satisfy the boundary conditions at the top of the Ekman layer. This sacrifice is inherent in our asymptotic approach to the problem. We will recover the lost boundary conditions in Section 5.

4. The velocity scale in the outer Ekman layer

Previous investigators have assumed that the velocity scale for the outer layer is u_*, but no very compelling argument has been given for this assumption. In anticipation of the next section, it is to be expected that the problem of matching the inner and outer solutions is greatly simplified when the same velocity scales are used for both layers. In the absence of a compelling reason for identifying W with u_*, the ratio W/u_* must be considered to depend on the surface Rossby number. It can be shown that matching of the solutions for the inner and outer layers is possible whenever W has the form

$$W = u_*\left(\frac{z_0 f}{u_*}\right)^K, \qquad (21)$$

where K is any constant .The functional forms of f_x, u_g/u_*, and v_g/u_* vary with the form of K, and it is then only by appeal to observed forms of the wind profile near the surface that one can really deduce that K has the value zero. Thus, the ability to match solutions does not in itself provide the basis for a logical assignment of the scaling velocity W.

The selection of u_* as the proper velocity scale for the outer layer follows rather naturally by applying the principle of asymptotic similarity to the turbulent energy equation. If we neglect the vertical diffusion of turbulent energy, this equation takes the form

$$\frac{\epsilon h}{W u_*^2} = (t_x F_x' + t_y F_y') = D\left(\frac{z}{h}\right), \qquad (22)$$

where primes indicate differentiation with respect to z/h and where ϵ again satisfies

$$\epsilon = \tfrac{3}{2} q^3/\ell. \qquad (23)$$

We expect that the turbulence characteristics scale in the same way as their counterparts in the mean flow.

Thus, we expect the functions q^3/W^3 and ℓ/h to depend only on z/h and to be independent of Ro as $h/z_0 \rightarrow \infty$. From (22) and (23), we then find

$$\left(\frac{W}{u_*}\right)^2 = \frac{2}{3}\left(\frac{q}{W}\right)^{-3}\left(\frac{l}{h}\right)D\left(\frac{z}{h}\right). \tag{24}$$

It follows that W/u_* is independent of surface Rossby number. Since it is also independent of z/h, the ratio must be a constant, and we incur no loss of generality by setting the constant equal to unity.

We must thus replace Eqs. (5) and (7) by

$$\frac{u-u_g}{u_*} = F_x\left(\frac{z}{h}\right), \quad \frac{v-v_g}{u_*} = F_y\left(\frac{z}{h}\right), \tag{25}$$

with

$$h = u_*/f. \tag{26}$$

This result is crucial to the success of the analysis. It has been obtained by using the turbulent energy budget to demonstrate that if the turbulence scales are to be determined in the same way as the scales of the velocity defects, the requirements of asymptotic similarity demand that W must be of the same order of magnitude as u_*. Both in the surface layer and in the outer Ekman layer we had to face the perennial closure problem of turbulence theory. We feel that the asymptotic analysis of the energy budget is the most rational way of solving this problem.

The scale height of the Ekman layer is u_*/f by (26). A fairly accurate estimate for the depth δ of the Ekman layer is $\delta = u_*/4f$. The ratio of the scale height h to the surface roughness length z_0 is the fundamental parameter in our analysis. From (26), we find that $h/z_0 = u_*/fz_0 = u_*/G\text{Ro}$. We are tempted to call u_*/fz_0 the "surface-friction Rossby number."

The problems associated with the boundary conditions on the equations of motion (8) and (9) and the similarity relations (25) are germane to the analysis. Since $u_*/G \rightarrow 0$ as $h/z_0 \rightarrow \infty$, an asymptotic formulation based on (25) cannot meet the boundary conditions at the surface without introducing singularities. Also, if the boundary conditions at the surface were imposed on (8) and (9), the surface Rossby number (or h/z_0) would enter through these conditons (which involve z_0). Therefore, we sacrifice the boundary conditions at the surface as far as the outer Ekman layer is concerned, in order to obtain well-behaved equations and well-behaved similarity laws that are independent of the surface Rossby number in the asymptotic limit. Rossby-number similarity as found by Gill and Csanady thus is not merely a convenient assumption, but an immediate result of asymptotic analysis. This point cannot be overemphasized, since our approach affords extrapolation of field data to arbitrarily large surface Rossby numbers, a feature that can never be taken for granted without careful investigation of the limit process involved. Also, the asymptotic approach makes the similarity relations (25) truly universal, so that a single graph will suffice for all values of Ro that are encountered in practice.

Our analysis leaves us with similarity laws for the outer Ekman layer that cannot describe the flow in the surface layer and, conversely, with similarity laws for the surface layer that cannot describe the flow in the bulk of the Ekman layer. This problem is solved by matching (Van Dyke, 1964). Matching is essentially a generalized method of applying boundary conditions to singular perturbation problems. One might visualize matching as a dual process by which the flow at the top of the surface layer provides the missing boundary conditions at the bottom of the outer Ekman layer and vice versa.

5. Results of matching

Singular perturbation methods make use of the premise that there exists a narrow layer in which the descriptions for the large-scale flow and for the small-scale flow are valid simultaneously. This layer, which is situated at the bottom of the outer Ekman layer, and at the top of the surface layer, is called the "matched layer"; its existence is based on Kaplun's extension theorem (Van Dyke, 1964). Specifically, in the matched layer the value of u/u_*, together with its derivative in the vertical, must be equal when expressed by either of the two similarity functions f_x and F_x.

It will be helpful to introduce the following notation:

$$\zeta = z/z_0, \quad \frac{z_0 f}{u_*} = \phi(\text{Ro}), \quad \frac{zf}{u_*} = \zeta\phi. \tag{27}$$

The result of matching the two functions for u/u_* is

$$F_x(\zeta\phi) = f_x(\zeta) - \frac{u_g}{u_*}. \tag{28}$$

The independent variables occurring in this equation are the two dimensionless ratios Ro and ζ. We can thus proceed to differentiate this equation with respect to each of the independent variables while holding the other constant. Using a prime to denote the derivative of a function with respect to its argument, we obtain

$$\phi F_x'(\zeta\phi) = f_x'(\zeta), \tag{29}$$

$$\zeta \frac{d\phi}{d\text{Ro}} F_x'(\zeta\phi) = -\frac{d}{d\text{Ro}}\left(\frac{u_g}{u_*}\right). \tag{30}$$

From these F_x' can be eliminated algebraically, with the result

$$\zeta f_x'(\zeta) = -\left(\frac{d\ln\phi}{d\text{Ro}}\right)^{-1} \frac{d}{d\text{Ro}}\left(\frac{u_g}{u_*}\right). \tag{31}$$

The left-hand side of this equation is a function of ζ only, while the right-hand side is a function of Ro only.

Thus, each side is a constant, whose value we denote as $1/k$. Integration of both sides of Eq. (31) then gives

$$u/u_*=\frac{1}{k}\ln\frac{z}{z_0}, \tag{32}$$

$$u_g/u_*=\frac{1}{k}\left[\ln\left(\frac{u_*}{fz_0}\right)-A\right]. \tag{33}$$

Eq. (32) is the well-known solution of Prandtl. The arbitrary constant A in Eq. (33) has been chosen to conform to Gill's notation.

It should be noted here that the matched layer, mathematically speaking, occurs when $z/z_0\rightarrow\infty$ and $zf/u_*\rightarrow 0$ simultaneously. For this reason it is formally incorrect to force Eq. (32) to satisfy the boundary condition at the surface. However, since in meteorological practice z_0 is always chosen to conform with Eq. (32), this issue is of minor importance.

In a similar way, the two expressions for v/u_* can be matched. This is quite simple, since v/u_* is zero in the surface layer. The function F_y in Eq. (25) has to agree with this for $z/h\rightarrow 0$, which yields

$$v_g/u_*=-B/k, \tag{34}$$

where the arbitrary constant B, like A in Eq. (33), has been chosen to conform with Gill's notation.

Eqs. (33) and (34) provide a relation between the geostrophic drag coefficient u_*/G and Ro and a relation between Ro and the angle α subtended by the surface wind direction and the direction of the isobars. These relations are most useful in the implicit forms

$$\ln \mathrm{Ro}=A-\ln\frac{u_*}{G}+\left(\frac{k^2G^2}{u_*^2}-B^2\right)^{\frac{1}{2}}, \tag{35}$$

$$\sin\alpha=\frac{B}{k}\frac{u_*}{G}. \tag{36}$$

The first of these equations was derived by Kazanski and Monin (1961) from the equations of motion by using different arguments, employing assumptions about the properties of the exchange coefficient close to the surface. Eqs. (35) and (36), or their equivalents, have also been given by Gill (*loc. cit.*) and Csanady (1967). We want to reiterate that our analysis has been carried out without any assumptions, except those related to the asymptotic behavior of the functions involved. We also want to point out that the logarithmic wind profile in the surface layer and the corresponding logarithmic wind profile in the lower part of the outer Ekman layer—which has not been derived here, but can be found easily by subtracting (33) from (32)—are based on the asymptotic properties of the matched layer. In this way our analysis emphasizes the often overlooked dependence of the surface layer on the properties of the entire Ekman layer.

6. Asymptotic similarity

We have found that Rossby-number similarity is a consequence of the asymptotic analysis, and that it is clearly analogous to Reynolds-number similarity as exhibited by turbulent boundary layers on smooth surfaces (Townsend, 1956). In both cases, it is an asymptotic feature that occurs in first approximation as the independent parameter tends to infinity. A general expression for this kind of behavior would be "asymptotic similarity." Asymptotic similarity may be defined as scale-ratio independence of the large-scale description and of the small-scale description in nondimensional forms, chosen such that all functions remain finite as the independent parameter increases beyond limit.

We will illustrate this concept with an example from the statistical theory of turbulence. A large-scale description of the turbulent energy spectrum may be written as

$$E\kappa^{5/3}/\epsilon^{\frac{2}{3}}=\Phi(\kappa L), \tag{37}$$

in which κ is the modulus of the wavenumber vector, ϵ the dissipation rate, E the three-dimensional energy spectrum function, and L some characteristic integral scale of the turbulence. This large-scale description is supposedly independent of viscosity (Batchelor, 1953). A convenient small-scale description reads, according to the universal equilibrium theory,

$$E\kappa^{5/3}/\epsilon^{\frac{2}{3}}=\phi(\kappa\eta), \tag{38}$$

where η is the Kolmogorov microscale. The small-scale description is independent of L. As the ratio L/η of the two length scales approaches infinity, the expressions (37) and (38) can be maintained only, if in a matched range of wavenumbers, as $L/\eta\rightarrow\infty$,

$$E\kappa^{5/3}/\epsilon^{\frac{2}{3}}=C. \tag{39}$$

This is the well-known expression for the inertial subrange. The close analogy between this derivation of the $\kappa^{-5/3}$ law and the derivation of the logarithmic wind profile given earlier leads us to propose the name "inertial sublayer" for the logarithmically matched layer in turbulent boundary-layer flows.

7. Discussion

We conclude the analysis by briefly commenting on some of the quantitative aspects associated with the similarity laws.

The values of the constants A and B in (33) and (34) have been determined empirically by Gill to lie within a range centered near 1.7 and 4.7, respectively. This is all that can be said without making additional assumptions. Exchange coefficients afford a convenient way of proceeding somewhat further. Using relations between the exchange coefficient and the kinetic energy of the turbulence derived earlier by Monin, Kazanski and Monin (1961) deduced values for A and B of 1.7 and

1.8, respectively, but their results did not fit the data as well as Gill's results.

The solution that has been obtained by matching gives the forms of the functions $(u-u_g)/u_*$ and $(v-v_g)/u_*$ in the matched layer, close to the surface, but fails to prescribe what these functions should be in the main portion of the outer Ekman layer. At this point it is necessary to invoke some more specific postulate such as exchange-coefficient relations of the form $\tau_x=\rho\kappa_m(du/dz)$, $\tau_y=\rho\kappa_m(dv/dz)$. The results achieved by matching show that in the constant-stress layer, K_m is given by

$$K_m=ku_*z. \tag{40}$$

However, as is widely known, K_m soon departs from this expression, reaching a maximum within 100–200 m from the ground and decreasing with further increase in height.

Many models for the wind structure in the Ekman layer have been advanced. The majority are two-layer models, notably those of Rossby and Montgomery (1935), Yudin and Shvetz (1940) and Estoque (1963). In these two-layer models the logarithmic solution for the surface layer is combined with an Ekman or similar solution for the outer layer through the use of appropriate boundary conditions at the interface, a process that has been described as "patching." The height chosen for the patching plays a critical role in the solution; a self-consistent postulate is needed if accurate relations for the dependence of u_*/G and α on Ro are to be obtained. In the past, the specification of the patching height z_p has never been done in a way that is consistent with the requirements of Rossby-number similarity. However, this can be done easily, by specifying a relation of the form

$$z_p=bh=bu^*/f. \tag{41}$$

When this relation is combined with the other boundary conditions, all of the parameters of the Ekman solution can be expressed in terms of the two dimensionless ratios and the constant b; furthermore, the two Gill constants A and B can be expressed in terms of the single constant b. By comparing observed values of u_*/G and α with predictions based on different assumed values of b, it was determined that the best value of b is about 0.01. Gill's constants are then found to be 0 and 4.5, respectively.

In recent years a number of one-layer models have been solved by Blackadar (1962), Lettau (1962), Appleby and Ohmstede (1964) and Bobileva *et al.* (1965). In its own way, each of these models is based on some relation between K_m and the wind shear so as to be consistent with the relations prevailing in the surface layer. We shall not discuss these models in detail, but rather will make a few observations concerning their properties. Most of these models conform exactly or reasonably closely to the requirements of Rossby-number similarity. In every case, though, there is an important constant that cannot be predicted, but must be justified or determined experimentally. Given the value of this constant, the solutions yield the shapes of the functions F_x and F_y, and completely determine the values of the Gill constants A and B. In general, the predictions agree so closely with one another that it would be futile to argue the merits of the various models on this basis.

Extension of the asymptotic similarity concepts to the flow in diabatic boundary layers is a challenging prospect. We have initiated further investigations into this problem; at this time, however, we can only state that the presence of yet another length scale (the Monin-Oboukhov length L) seems to make the analysis extremely complicated.

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