The Response of the Planetary Boundary Layer to Time Varying Pressure Gradient Force[1]

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ABSTRACT

The equations for horizontally homogeneous planetary boundary layer flow with constant eddy viscosity are integrated in time and height. The special case for which the direction of the pressure gradient force is a periodic function of time is studied in detail. The nondimensional number $F=z(4Kt)^{-\frac{1}{2}}$ is seen to be the proper scale which describes the flow response to the boundary layer.

1. Introduction

As a first step towards improving our understanding of the time dependent planetary boundary layer (PBL), a simple model assuming horizontal uniformity and a constant eddy viscosity has been investigated. The steady-state solution of this case is the well-known Ekman spiral (Ekman, 1905; Taylor, 1915). Although more realistic steady-state models have been developed (Lettau, 1962; Blackadar 1962), including baroclinic effects (Blackadar, 1965), and stability effects (Blackadar and Ching, 1965), the simpler time dependent model has been studied because analytic solutions of the equations of motion can be given. An understanding of this model hopefully will provide a basis from which a more realistic model of the PBL may be constructed, including height and stability dependence of the eddy viscosity, and horizontal convergence.

Previous studies of the time dependent PBL have been carried out by Buajitti and Blackadar (1957), Ooyama (1957), Estoque (1963), and Pandolfo and Brown (1967). The studies by Buajitti and Blackadar, Ooyama, and Estoque all consider time variations in the eddy viscosity but keep the large-scale synoptic conditions constant.

Pandolfo and Brown (1967) are the first to consider time variations in the pressure gradient force. They give special analytical solutions of the boundary layer with the pressure gradient force both as a periodic function of time and as a step function. In this note a more general analytic solution of a similar problem and a physical interpretation of the results is given.

2. Analytical study

For a two-dimensional, incompressible and horizontally homogeneous mean flow with constant eddy viscosity K, the vector equation of momentum written in complex notation takes the form

$$\frac{\partial \mathbf{V}}{\partial t}+if\mathbf{V}=if\mathbf{V}_g+K\frac{\partial^2 \mathbf{V}}{\partial z^2}, \tag{1}$$

where $\mathbf{V}=u+iv$, $\mathbf{V}_g=i(\rho f)^{-1}(\partial p/\partial x+i\partial p/\partial y)$, f is the Coriolis parameter ($2\Omega\sin\phi$), Ω the angular velocity of earth, ϕ latitude, ρ the density of dry air (constant), p pressure, and i the square root of -1. Multiplying (1) by e^{ift} gives

$$\frac{\partial \mathbf{W}}{\partial t}=K\frac{\partial^2 \mathbf{W}}{\partial z^2}+if\mathbf{V}_g e^{ift}, \tag{2}$$

where

$$\mathbf{W}=\mathbf{V}e^{ift}. \tag{3}$$

Eq. (2) is the diffusion equation with a source term

$$\mathbf{P}(z,t)=if\mathbf{V}_g e^{ift}. \tag{4}$$

Sobolev (1964) discussed the solution to (2) for all space ($-\infty\leq z\leq\infty$). Using the "Method of Images," the general solution collapses into the half-space ($0\leq z\leq\infty$) which is the relevant region of interest here. Thus,

$$\mathbf{W}(z,t)=(4\pi Kt)^{-\frac{1}{2}}\int_0^\infty \mathbf{W}(z,0)\{\exp[-(s-z)^2/(4Kt)]-\exp[-(s+z)^2/(4Kt)]\}ds+(4\pi K)^{-\frac{1}{2}}\int_0^t(t-\zeta)^{-\frac{1}{2}}$$
$$\times\left\{\int_0^\infty \mathbf{P}(s,\zeta)\{\exp[-(s-z)^2/4K(t-\zeta)]-\exp[-(s+z)^2/4K(t-\zeta)]\}ds\right\}d\zeta. \tag{5}$$

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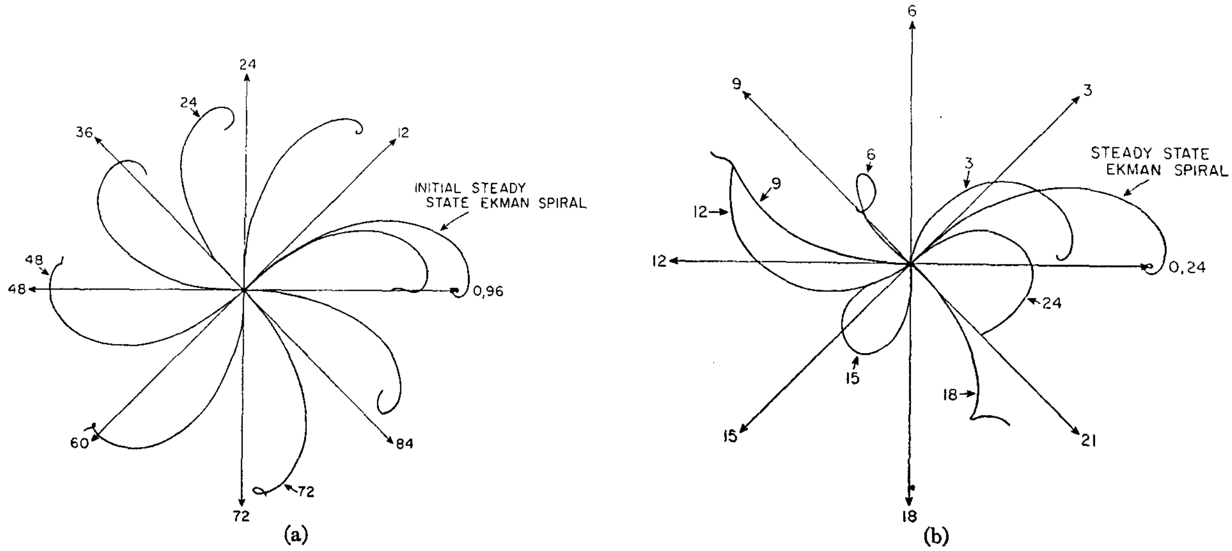

FIG. 1. Theoretical wind hodographs for a rotating pressure gradient with a periodic oscillation of 4 days (a), and 1 day (b). See text for values of parameters.

This is a general solution depending only on the initial condition $\mathbf{W}(z,0)$ and the source term (4) which can be a function of both time and height. Note that the boundary condition at the surface

$$\mathbf{W}(0,t)=0 \qquad (6)$$

is satisfied for all time.

Since the source term chosen for this study varies periodically in direction as a function of time only, i.e.,

$$\mathbf{V}_g(t)=u_{g0}e^{i\alpha t}, \qquad (7)$$

we have

$$\mathbf{P}(t)=ifu_{g0}e^{i\alpha t}e^{ift}. \qquad (8)$$

For the initial condition, we chose the steady-state Ekman distribution of wind for a pressure gradient force initially oriented in the north-south direction, i.e.,

$$\mathbf{W}(z,0)=u_{g0}\{1-\exp[-(1+i)\beta z]\}, \qquad (9)$$

where

$$\beta^2=f/2K. \qquad (10)$$

Useful references in evaluating the integrals of (5) are Gautschi (1964) and Gradshteyn and Ryzhik (1965). After performing the integrations, (5) becomes

$$\begin{aligned}\frac{\mathbf{V}(z,t)}{u_{g0}}=\frac{\mathbf{W}\exp(-ift)}{u_{g0}}&=\alpha(f+\alpha)^{-1}\operatorname{erf}[z(4Kt)^{-\frac{1}{2}}]\exp(-ift)+f(f+\alpha)^{-1}\exp(i\alpha t)\\&+0.5\exp[(1+i)\beta z]\{\operatorname{erfc}[z(4Kt)^{-\frac{1}{2}}+(1+i)(ft/2)^{\frac{1}{2}}]\}-0.5\exp[-(1+i)\beta z]\{1+\operatorname{erf}[z(4Kt)^{-\frac{1}{2}}-(1+i)(ft/2)^{\frac{1}{2}}]\}\\&-0.5f(f+\alpha)^{-1}\exp[i\alpha t+(1+i)\beta' z]\{\operatorname{erfc}[z(4Kt)^{-\frac{1}{2}}+(1+i)(0.5(f+\alpha)t)^{\frac{1}{2}}]\}\\&-0.5f(f+\alpha)^{-1}\exp[i\alpha t-(1+i)\beta' z]\{\operatorname{erfc}[z(4Kt)^{-\frac{1}{2}}-(1+i)(0.5(f+\alpha)t)^{\frac{1}{2}}]\},\end{aligned} \qquad (11)$$

where $\operatorname{erf}(z)=2\pi^{-\frac{1}{2}}\int_0^z\exp(-r^2)dr$, $\operatorname{erfc}(z)=1-\operatorname{erf}(z)$, and $\beta'^2=(f+\alpha)/2K$.

Eq. (11) has been numerically evaluated for a number of cases, two of which are listed below. The complex error function was evaluated using the approximation (Gautschi, 1964)

$$\begin{aligned}\operatorname{erf}(x+iy)=\operatorname{erf}(x)&+(2\pi x)^{-1}\\&\times\exp(-x^2)(1-\cos 2xy+i\sin 2xy)\\&+2\pi^{-1}\exp(-x^2)\sum_{n=1}^{\infty}(n^2+4x^2)^{-1}\\&\times\exp(-0.25n^2)[f_n(x,y)+ig_n(x,y)]+\epsilon(x,y),\end{aligned}$$

where

$$\begin{aligned}f_n(x,y)&=2x-2x\cosh ny\cos 2xy+n\sinh ny\sin 2xy,\\g_n(x,y&=2x\cosh ny\sin 2xy+n\sinh ny\cos 2xy,\\&|\epsilon(x,y)|\approx 10^{-16}|\operatorname{erf}(x+iy)|.\end{aligned}$$

The value of n was limited to 18 in order to stay within the limits of the computer. The value of $n=6$ was also tried and the relative difference between $n=6$ and $n=18$ was of the order of 1% or less, implying that the series approximation given above converges rapidly with increasing n.

Figs. 1a and 1b are sample wind hodographs resulting from $f=0.0001$ sec^{-1}, $u_{g0}=10^3$ cm sec^{-1}, and $K=5\times 10^4$

$cm^2\ sec^{-1}$ for *case a*, where $\alpha = 1.81805 \times 10^{-5}\ sec^{-1}$, $T = 2\pi/\alpha = 4$ days; and for *case b*, where $\alpha = 7.2722 \times 10^{-5}\ sec^{-1}$, $T = 2\pi/\alpha = 1$ day. Figs. 2a and 2b are the same hodographs plotted relative to the orientation of the pressure gradient.

From Eq. (11) it is clear that the dimensionless quantity,[2] $z(4Kt)^{-\frac{1}{2}} = F$, is the characteristic parameter that indicates to what extent the flow has been adjusted to the surface boundary. We can distinguish three regimes:

a. $F \ll 1$. For this condition, (11) becomes

$$V(z,t) \rightarrow f u_{g0}(\alpha + f)^{-1} \times \exp(i\alpha t)\{1 - \exp[-(1+i)\beta' z]\}. \quad (12)$$

By multiplying (12) by $\exp(-i\alpha t)$, the flow is normalized to the initial direction of the pressure gradient force. Aside from a slightly different constant, it is clear that the only difference in the functional form of (12) and the steady-state Ekman solution is that β is replaced by β'.

We see here, as is also illustrated in Figs. 2a, 2b and 3 that the transient terms have become negligible.

b. $F = O(1)$. The transient terms are important and the full Eq. (11) should be used.

c. $F \ll 1$. In this case (11) reduces to

$$V(z,t) \rightarrow u_{g0}(\alpha + f)^{-1}[\alpha \exp(-ift) + f \exp(i\alpha t)]. \quad (13)$$

Here, the flow has not yet been affected by the presence of the boundary and describes the initial inertial oscillation. This oscillation is shown in Figs. 2a and 2b by connecting the end points of the hodographs.

Thus, $(4Kt)^{\frac{1}{2}}$ is a scale height below which the flow tends toward the steady state, while above, the flow is governed more by the inertial oscillation and is less affected by the presence of the boundary.

However, significant deviations remain even after very long times and at low elevations or small F as Fig. 3 illustrates. A similar effect had been earlier noted by Ekman (1905) for a constant pressure gradient but unbalanced initial condition. He states: "...the periodical deviations from the average velocity, at a given depth, abate only slowly (inversely as the square root of time)..." The implication is that long after a pressure system decays or advects out of the area, the inertial oscillations it introduced can still be significant; this indicates that the usual steady-state distribution is needed to obtain a better estimate of the real magnitude of this effect of inertial oscillations.

In Fig. 4 the variations of the cross-isobaric angle ψ at a height of 1 m has been plotted as a function of α and t; while in Fig. 5 the square root of the geostrophic drag coefficient,[3] $\sqrt{C_g(1)} = u^*/u_{g0}$, where u^* is the friction velocity, has been plotted as a function of α and t. The deviations from the steady-state values are large and increase with increasing rotation rate α. Note that the steady state value of $\sqrt{C_g}$ decreases with increasing rotation rate α. This is a consequence of the fact that the wind at the top of the friction layer is less than geostrophic because of the addition of the isallobaric wind arising out of the pressure gradient change.

The assumption of a constant K is rather unrealistic near the surface because it predicts a linear wind profile whereas the observations indicate a logarithmic profile.

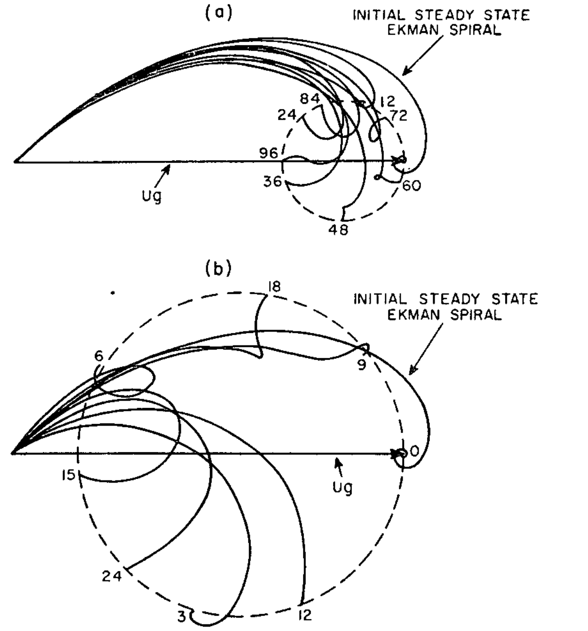

Fig. 2. Same hodographs as in Fig. 1 in a coordinate system rotating with the pressure gradient, for period of 4 days (a), and 1 day (b).

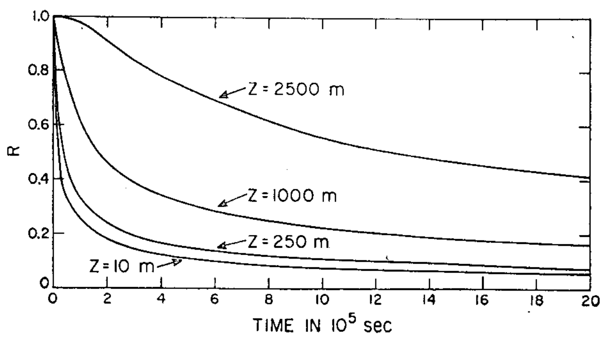

Fig. 3. Time variation of normalized deviation of velocity from equilibrium, $R = |V - V_s|/|V - V_{s0}|$, where V_s is the modified Ekman spiral as given by Eq. (12) and V_{s0} is the initial Ekman spiral.

[2] In the theory of heat conduction the dimensionless number kt/z^2 is called the Fourier number; k is the thermal diffusivity which plays a similar role as the eddy viscosity K in our problem. The number F is therefore something like the inverse root of an eddy Fourier number.

[3] In much of the recent literature on the planetary boundary layer the coefficient u_*/u_{g0} has been named the geostrophic drag coefficient. This is inconsistent with the customary use of drag coefficient in fluid dynamics which is u_*^2/u^2.

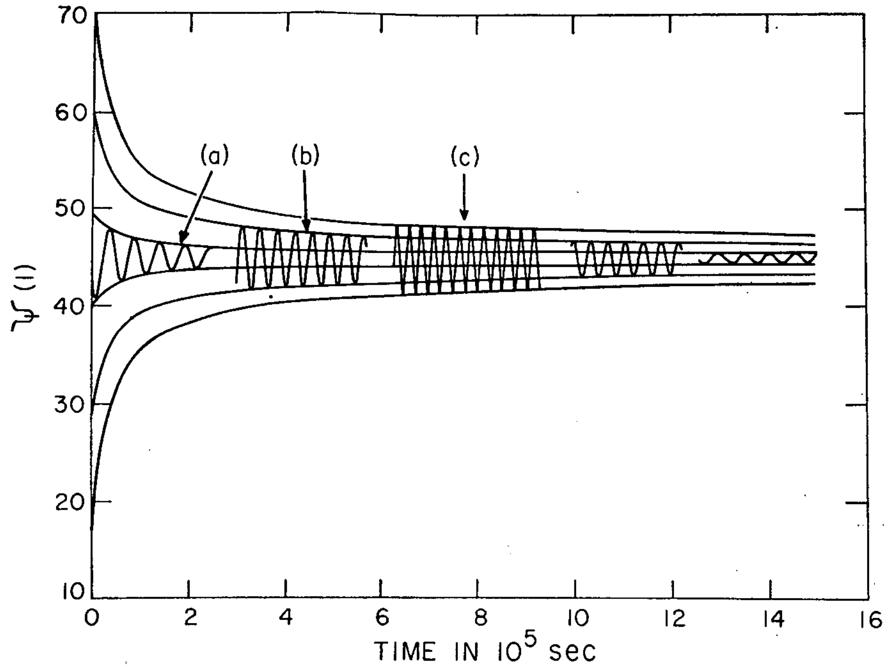

FIG. 4. Cross-isobar angle ψ at 1 m vs time for different rotation rates for periods of 4 days (a), 1 day (b), and $\frac{1}{2}$ day (c).

Also, it predicts $\psi(0)=45°$, whereas the observed angle is usually less than 45°. The assumption of horizontal uniformity of the flow combined with a rotating pressure field is probably even more extreme. Little attention, therefore, should be given to the inertial oscillations which are rather artificial. It is not difficult to model the eddy viscosity to give a much better description of the conditions near the surface. However, the generalization to nonuniform horizontal conditions will require considerably more effort and imagination. It is along these

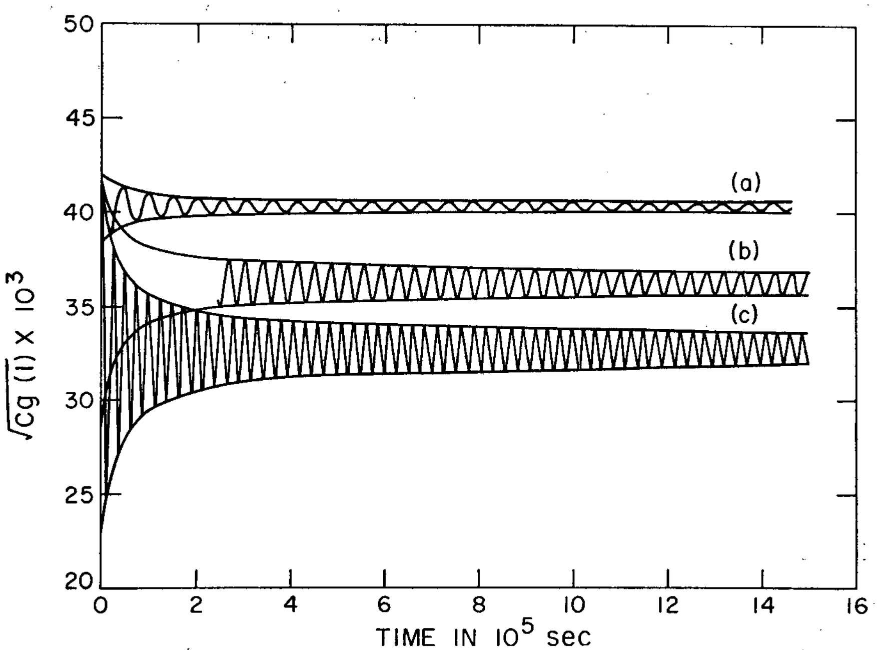

FIG. 5. The square root of the geostropic drag coefficient $\sqrt{C_g(1)}=u_*/u_{g0}$, u^* at 1 m, vs time for different rotation rates for periods of 4 days (a), 1 day (b), and $\frac{1}{2}$ day (c).

lines that future work will be directed with continued emphasis on the large-scale synoptic changes and their effects on the planetary boundary layer.

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