A Numerical Experiment on the Growth and Feedback Mechanisms of Hailstones in a One-Dimensional Steady-State Model Cloud

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ABSTRACT

Calculations are made on the growth of hailstone embryos of given size and concentration which are injected into a one-dimensional steady-state updraft, and grow while ascending, the updraft obeying the condition that ρV_z is a constant. The growth was found to have a considerable effect on the free water content of the cloud due to depletion by the growing particles. The hailstones of this model generally reach biggest sizes if their concentration is low and if the embryos are as big as possible. Embryos of 5 mm diameter can grow to 2.5–3.0 cm in diameter within 8–12 min if the conditions are right.

It is further shown that thermal feedback is of great importance in calculating the cloud temperature since it greatly affects buoyancy and icing conditions; in this case, the frictional heating of the falling hydrometeors has to be included along with the heat of fusion. The buoyancy is investigated because it is necessary to decide which set of input parameters for the growth curves and the free water contents distributions is reasonable. For those hailclouds where hailstones grow while ascending, it may be concluded that the biggest updrafts do not necessarily produce the biggest hailstones. The icing conditions of the growing particles turned out to be such that the outermost layers of the biggest stones always grow non-spongy.

1. Introduction

A big step forward in the theoretical study of the growth of hailstones by Iribarne and de Pena (1962) remained nearly unnoticed by the scientific community. However, application of the same type of conservation equations for water substances in rain clouds by Kessler (1967) made it imperative to investigate the depletion of the liquid water content in a hailcloud by the growing hailstones. It seemed necessary, however, to extend Iribarne and de Pena's range of parameters, and to consider the variation of temperature and pressure vs height and the replenishing of liquid water by condensation according to pseudo-adiabatic lifting. Further, the scope of the previous work was considerably enlarged by studies of thermal feedback and buoyancy effects. Growth time and icing conditions at various growth states are also included in this study.

Since the Toronto cloud physics group is mainly laboratory oriented, the microphysics is stressed rather than rigorous dynamic modeling of storms. The work was done in order to demonstrate that the growth of typical hailstones can be explained without complex assumptions such as hailstone oscillation or rain accumulation.

Although the model as such is intended to be fairly simple, it may be useful for recognizing the important parameters and the simplifying assumptions which will be used in more complex models.

The calculations are based on the assumption that the effects of an ensemble of hailstones are equal to the sum of the effects of its single particles, i.e., that there is no interference between neighboring hailstones by way of wakes. This is not unreasonable as has been shown by List and Hand (1969) for raindrops, unless we consider conditions near or within balance levels where the concentration of hailstones increases considerably.

2. The cloud model

The selection of a proper hail-producing model cloud is rather difficult and somewhat arbitrary since reliable field observations on hail growth are not available. From the mathematical point of view it is easiest to assume a steady-state model, i.e., a model with all variables constant at any point in space. It is also felt that before three-dimensional studies are undertaken certain basic problems should be investigated in depth; for instance, the aerodynamic feedback of the precipitation particles to the updraft (List and Lozowski, 1968) and the concepts of entrainment. Therefore, further simplification is introduced by treating the cloud as one-dimensional; that is, all parameters are functions of height z only. Hence, the updraft is represented by the continuity equation for air, i.e.,

$$\rho V_z = \text{constant}, \qquad (1)$$

where ρ is the air density and V_z the updraft speed at the level z. Eq. (1) implies that the updraft is increasing with height as the density decreases.

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From investigations of natural hailstones it is known that a big portion of these hydrometeors contains graupel as embryos. It is assumed now that, in nature, graupel or frozen raindrops are formed outside the main updraft column; they are, therefore, injected into the model updraft at a certain level. This is quite convenient since difficulties involved in formulating hailstone-embryo growth are avoided. We further assume, for reasons of simplicity, that the hailstones grow only while ascending. If they reach a level where their terminal speed equals the updraft speed, the hailstones will be removed. Other models could certainly also be justified. No claim is made that the one described here is representative of all hailstorms, but it could closely represent clouds with tilted updrafts where the embryo particles are sucked into the updraft at a low level and where the grown hailstones fall out at the balance level; it could also represent vertical updrafts where graupel or frozen drops are collected in converging regions and are thrown out of the funnel as hailstones in upper regions where there is high divergence.

It is also assumed that the graupel source is located at the OC level and that it is dispersing embryos of only one size at an initial concentration given by N_0 (particles m^{-3}). In this one-dimensional model the number flux of hailstones must be constant at all levels; this implies that

$$N(V_z - V_t) = \text{constant}, \qquad (2)$$

where V_t is the terminal speed of the particles.

The cloud in which the hailstones grow is based on Beckwith's (1960) soundings taken on days with hail in the Denver area. Environment temperature T_E (°K) is related to pressure (List *et al.*, 1965) according to

$$T_E = 53 \ln(0.284P), \qquad (3)$$

where P is the pressure (mb), and is given as a function of height by

$$H = 2.76\times10^{-4}(9.0066\times10^4 - T_E^2), \qquad (4)$$

where H is the height in km above the ground (Fig. 1). The cloud base is assumed to be at the 5C level and at 670.8 mb, the pressure in the cloud being assumed equal to the pressure of the environment.

Because the hailstones grow inside the cloud where temperature and density are different from the values in the environment, the pseudo-adiabatic cloud temperature will be calculated as a first approximation. Starting with a parcel of saturated air containing 1 gm dry air and r_w gm water vapor at a given pressure, the parcel is lifted a small height interval, resulting in a change of the total pressure of ΔP(dyn cm^{-2}), a change in temperature ΔT(°K) and an amount of water Δr_w (gm) condensing out, the latent heat release being $-L_v\Delta r_w$ heat units. With the assumption of no heat exchange with the surroundings, the first law of thermodynamics leads to

$$-L_v\Delta r_w = C_p\Delta T - RT\Delta\ln(P-e_w) + r_wC_{pv}\Delta T - r_wR_wT\Delta\ln e_w, \qquad (5)$$

where L_v is the latent heat of condensation per gram of water vapor; C_p and C_{pv} are the specific heats of dry air and water vapor, respectively; R and R_w the universal gas constants for dry air and water vapor, respectively; e_w (dyn cm^{-2}) the saturation water vapor pressure at temperature T; and r_w the saturation mixing ratio.

Employing the Clausius-Clapeyron equation and replacing r_w by $RR_w^{-1}e_w(P-e_w)^{-1}$, the expression (5) can be rewritten as

$$\Delta T = \frac{[RT(P-e_w)^{-1} + RR_w^{-1}L_ve_w(P-e_w)^{-2}]\Delta P}{C_p + RR_w^{-1}e_w(P-e_w)^{-1}C_{pv} + RR_w^{-1}L_v^2e_w[(P-e_w)R_wT^2]^{-1}[1+e_w(P-e_w)^{-1}]}. \qquad (6)$$

A step-by-step integration of Eq. (6) based on height steps of 0.1 km is now performed by making use of (3) and (4) to obtain the pseudo-adiabatic cloud temperature; the calculations are started at the environment freezing level, which was found to correspond to a cloud temperature of 0.80C and are carried out up to a height of 5 km above the freezing level.

The increase in pseudo-adiabatic free water r_{ps} [(gm water) (gm cloud air)$^{-1}$] of the cloud parcel is equal the decrease in r_w since the free water is assumed to ascend at the speed of the updraft.

The pseudo-adiabatic temperature thus obtained is shown as a function of height above the environment freezing level in Fig. 1. In these calculations the heat of fusion of freezing cloud water is not considered since droplet freezing is assumed to start at the −32C level [in accordance with Das (1962)], 4.65 km above the freezing level. The effect is quite small since the calculations are stopped at the 5-km level.

Herewith, the basic assumptions concerning the model hailcloud are more or less completely described. Before the growth of hailstones is considered, however, some implications about maximum diameters of hailstones might be discussed. Based on the pseudo-adiabatic cloud temperature and the pressure-vs-height equations, diameters of hailstones at the balance level are calculated for different updraft speeds at the 0C level and extended to higher levels by the continuity equation (1). The results, displayed in Fig. 2, show how the maximum diameter which can be accommodated by the model increases with height. In this particular steady-state model the growth of big stones can there-

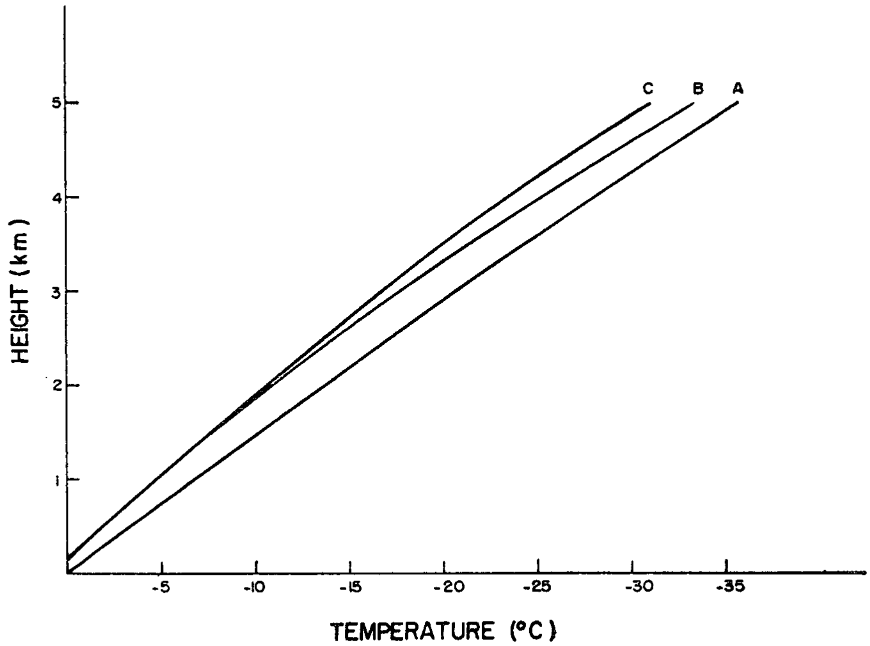

FIG. 1. Environment (A) and cloud (B, C) temperature variations with height, curve B being calculated solely on the basis of pseudo-adiabatic lifting of an air parcel, while C includes heat sources induced by hailstones.

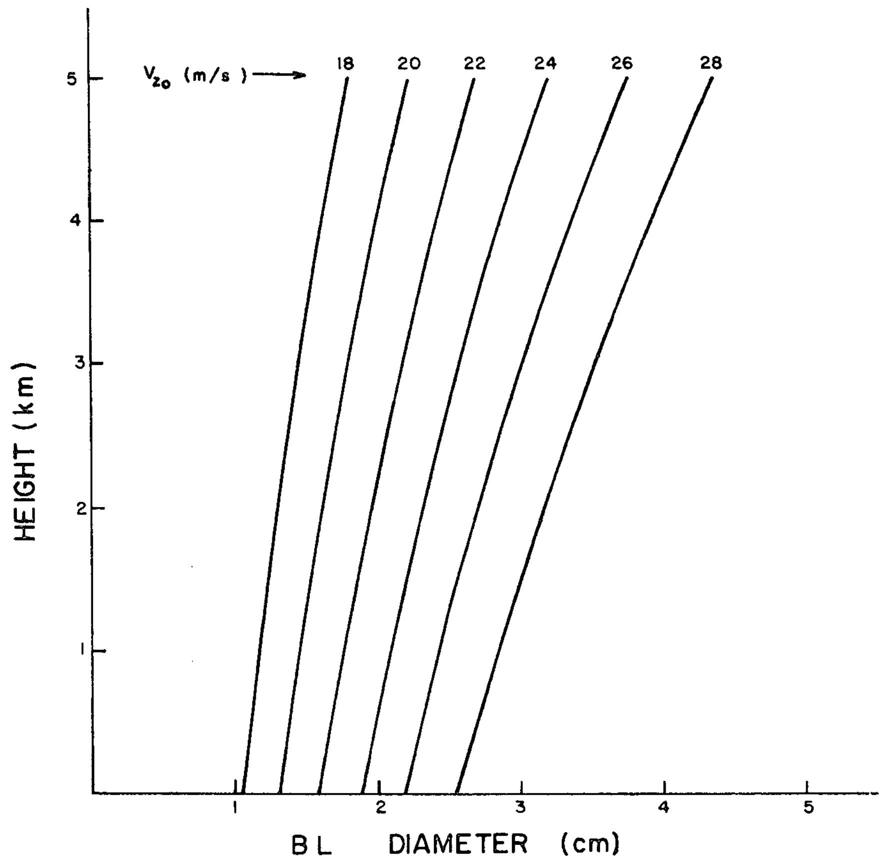

FIG. 2. Diameter of hailstones having terminal speeds equal to updrafts (balance level conditions) as function of height, for different 0C level updraft speeds V_{z0}.

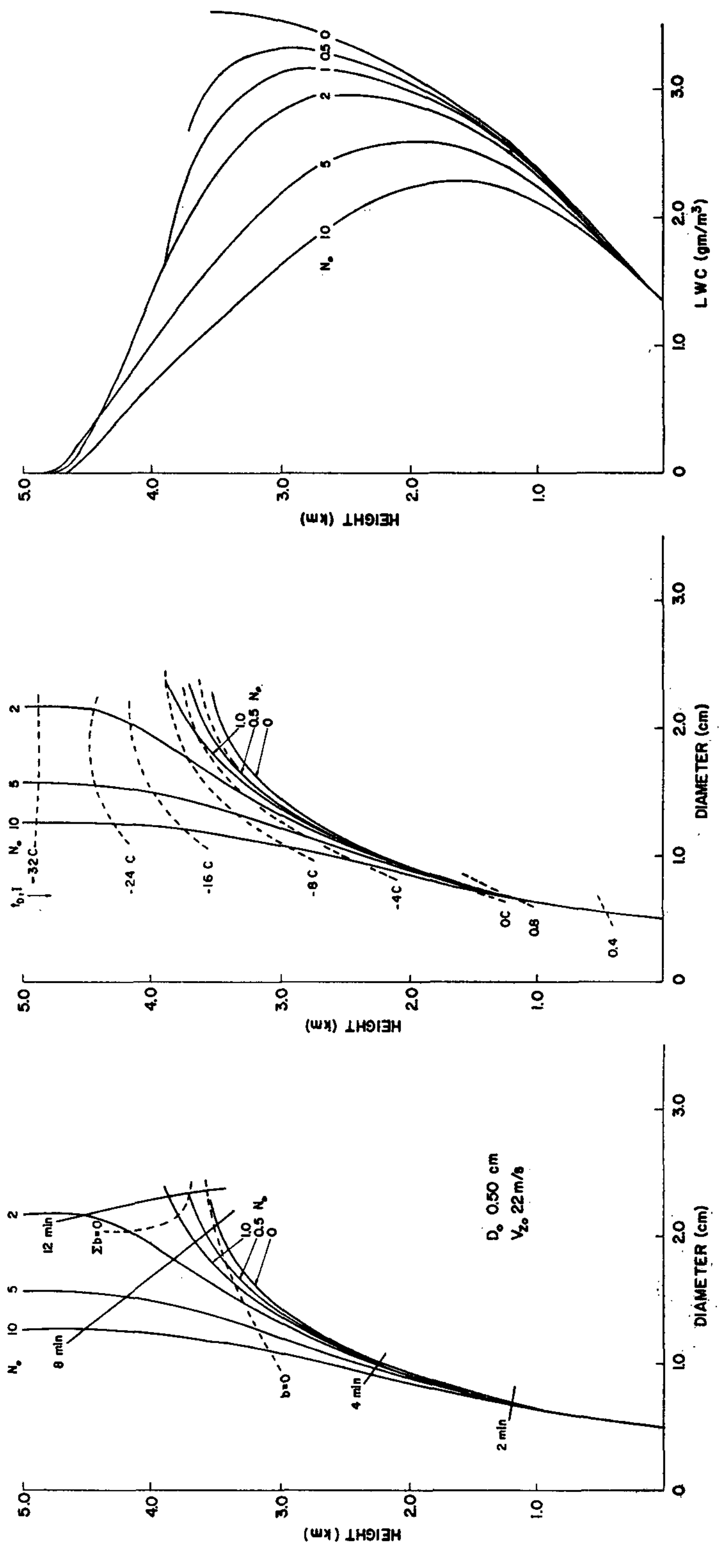

FIG. 3. Growth curves (a, b) and liquid water distributions (c) for 0.5-cm hailstones injected at the OC level at various concentrations N_0 into an updraft of 22 m sec^{-1}. Isolines are also shown for buoyancy b and cumulative buoyancy Σb in (a), for growth time in (a), and for icing conditions t_D, I in (b).

fore only be obtained if the updrafts are high enough. No statement can be made as yet about what input parameters, such as embryo size and embryo concentration, will lead to such big stones.

3. The growth of hailstones and their effects on liquid water content

As already mentioned, the main feature of these calculations will be the particle growth and the depletion of the liquid water content by the growing stones. The free water content at the 0C level, the level of the embryo source, is known. An input embryo diameter D_0 (cm) can be assumed as well as the number concentration N_0 (m^{-3}). The embryos and the hailstones shall be spherical with a drag coefficient of 0.5. These conditions yield a terminal speed V_t of

$$V_t = 48.8(D/\rho)^{\frac{1}{2}} \text{ [cgs units]}, \qquad (7)$$

where D is the diameter.

The growth of the hailstones as they rise in the updraft from one level to another is given by

$$\frac{\Delta D}{\Delta z} = \frac{V_t r \rho}{2\rho_I (V_z - V_t)}. \qquad (8)$$

The difference $(V_z - V_t)$ between updraft and particle speed appears here because the time spent by the hailstones in a layer of thickness Δz is inversely proportional to its speed relative to the ground.

This growth equation assumes a collection efficiency of unity; that is, all of the liquid water swept out by the hailstone is accreted. The hailstone density ρ_I is assumed to be 0.915 gm cm^{-3}.

The expression for the conservation of liquid water r [(gm water) (gm air)$^{-1}$] in an updraft parcel is given by

$$\frac{\Delta r}{\Delta z} = \frac{\Delta r_{ps}}{\Delta z} - \frac{\pi D^2 V_t r N}{4 V_z}, \qquad (9)$$

where Δr_{ps} is the change of the liquid water content due to condensation. The liquid water, which is swept out by the N hailstones per unit volume as the air ascends, is given by the second term on the right side of the equation. The depletion term is proportional to V_z^{-1}; that is, to the time spent by the air in traveling from the lower to the upper boundary of a height interval.

Using Eqs. (1)–(7) and the equation of state for the cloud air converts Eqs. (8) and (9) into two first-order simultaneous differential equations in D, r and z. They were solved for various values of the input parameters N_0, V_{z0} and D_0 using finite height steps Δz of 100 m. The solution for D and r as functions of z was accomplished by using second-order approximations to all variables. Values for diameter and free water have a computational error of less than 5% after integration has been carried out to the top of the cloud (5 km above the base).

Growth curves and free water contents were calculated for embryo diameters of 0.25, 0.50, 0.75 and 1.0 cm and for velocities varying in steps of 1 m sec^{-1} for a range of 18–28 m sec^{-1}. A selection of these calculations is shown in Figs. 3–5. The hailstone number density was chosen in such a manner that the buoyancy at the input level was big enough to sustain the weight of the hailstones at this level. This criterion led to conditions represented in Fig. 6, which displays input number density vs diameter for zero buoyancy. This criterion turned out quite reasonable as will be shown in Section 5.

Figs. 3–5 show the existence of two types of growth curves. The first is a sigmoid type with an inflection point between the 0C and the 5-km level and describes the growth of hailstones which are thrown out on top of the cloud, whereas the second type is representative of hailstones reaching a balance level at a height <5 km. According to the model, the hailstones are removed at such a level and the calculations are stopped. The free water content is seen to drop off rapidly just below the balance level because the number density increases rapidly in this region of the model.

Figs 3–5 show one very important fact: In order to obtain big hailstones, given the embryo diameter and the updraft speed, the input number concentration has to be quite low and of the order of 0.5–2 m^{-3}.

The conditions which would be obtained if depletion is not considered are shown by the curves for number concentrations $N_0 = 0$. No depletion results in the same free water content distributions, whereas, depending on input parameters, depletion leads to considerable variation of free water profiles. This finding is of great importance because icing conditions which are dependent on free water content could change very easily at one level and help to explain the shell growth of hailstones without resorting to theories involving recycling of hailstones or accumulation of raindrops. In this case the authors are mainly thinking of variations in updraft speed.

The surprisingly big effect of the updraft velocity on the hailstone growth is demonstrated in Fig. 7 for $D = 0.5$ cm and $N_0 = 2$ m^{-3}. It can also be concluded from Fig. 7 that it is not the biggest updraft which produces the biggest hailstones. If the updraft is big, the growing hailstones don't stay in the cloud long enough to grow considerably; on the other hand, if the updraft is relatively small, the hailstones grow fast and reach only a low balance level. The most favorable updrafts are the intermediary ones which lift the hailstones to the levels where the highest terminal velocities coincide with the highest updraft speeds. This statement holds for all cases where hailstones grow while ascending.

The depletion works in two ways: (i) increased competition due to higher particle concentration leads to smaller growth rates; and (ii) the lower growth rates

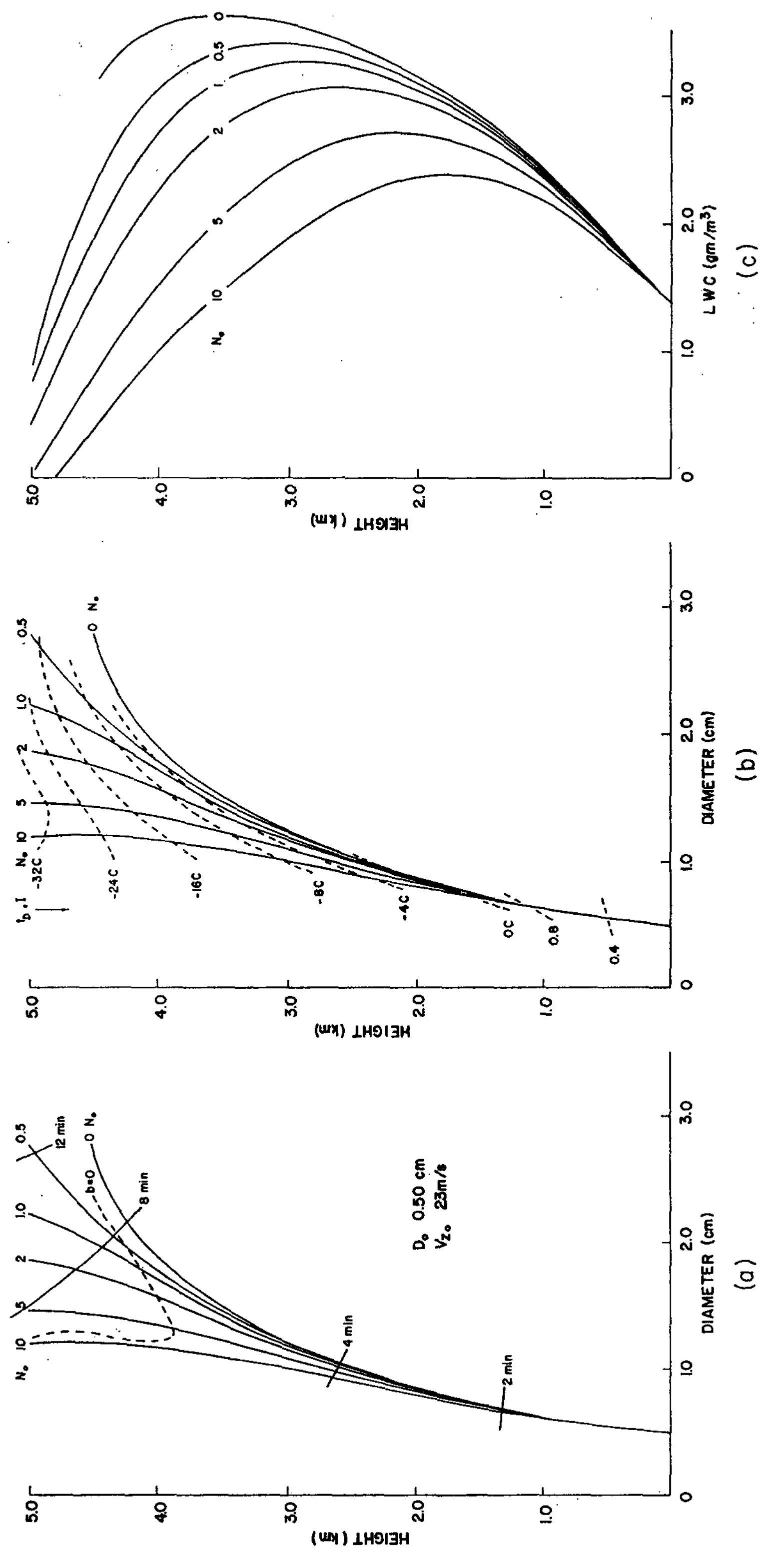

FIG. 4. Same as Fig. 3 except for updraft of 23 m sec^{-1}.

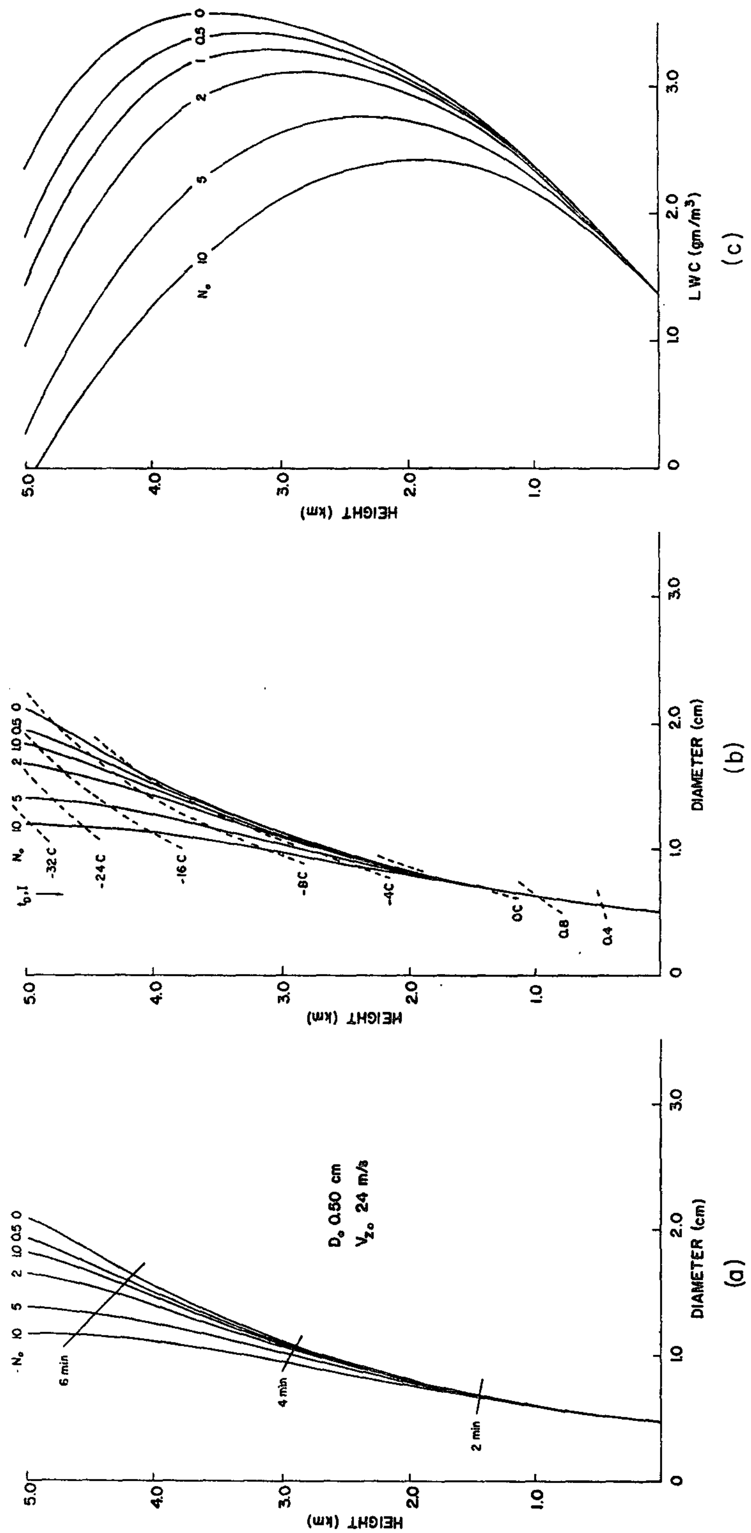

FIG. 5. Same as Fig. 3 except for updraft of 24 m sec^{-1}.

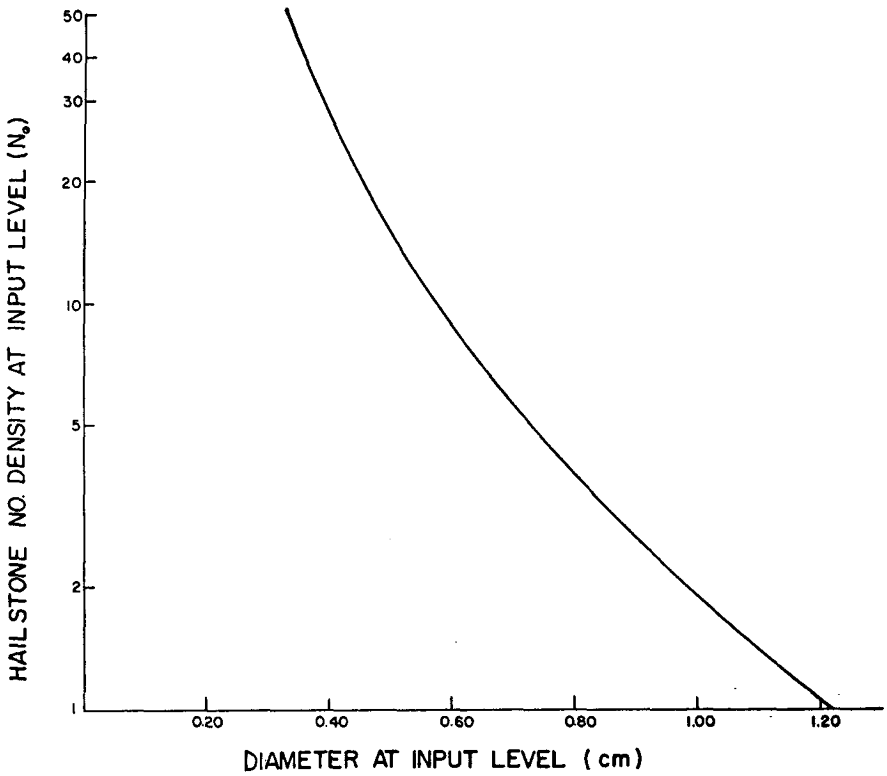

FIG. 6. Relationship between number density and hailstone diameter for zero buoyancy at input level.

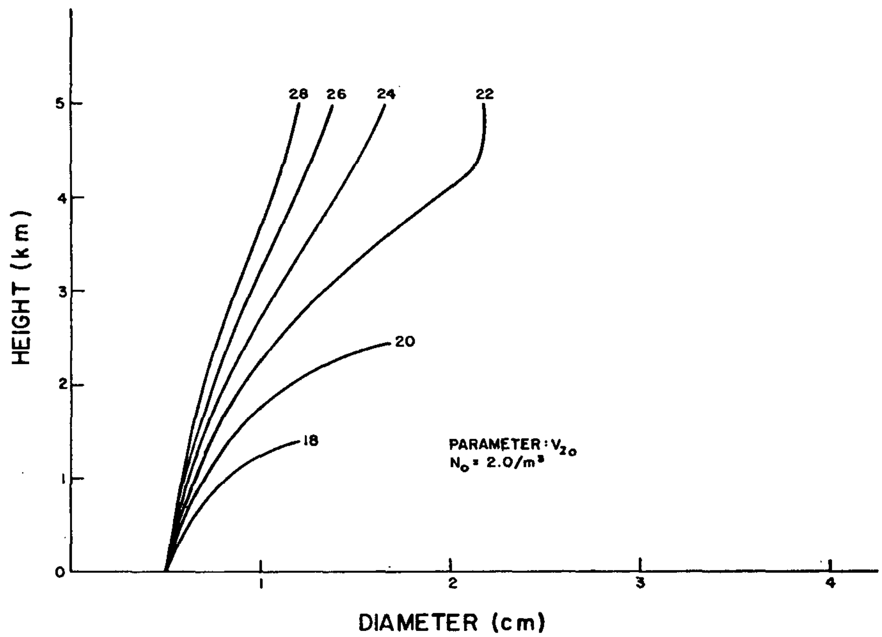

FIG. 7. Variation of the growth curves with updraft speed.

lead to larger absolute upward speeds of the particles and, as a result, the hailstones pass more quickly through the cloud—an additional reason for lesser growth.

The variation of the number density with height is displayed in Fig. 8. High number densities can be attributed to balance levels or to conditions which nearly lead to balance levels, a balance level being defined by the level for which the terminal particle speed equals the updraft velocity (Atlas, 1966).

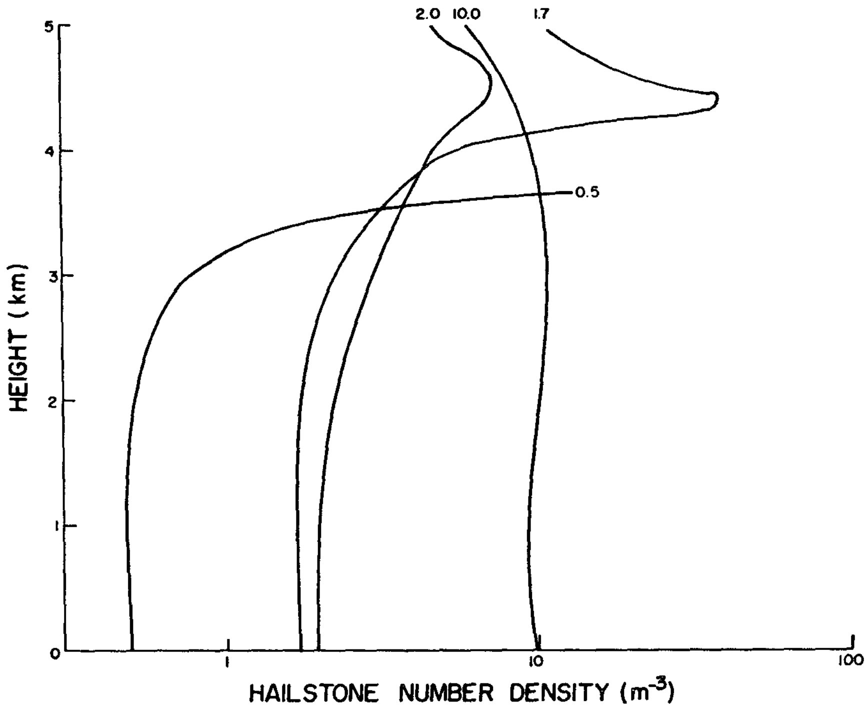

FIG. 8. Variation of the hailstone number density with height, using a 0C level updraft speed of 22 m sec^{-1} and particle diameter of 0.5 cm, for various number densities N_0 (m^{-3}) of injected hailstones.

4. Thermal feedback

In Section 3 it was shown that hailstone feedback in form of depletion of the liquid water content can no longer remain neglected. This is reason enough to search for other feedback mechanisms. The most important one for a one-dimensional model seems to be the thermal feedback. Growing hailstones represent a heat source due to the release of latent heat of fusion L_f according to

$$Q_F^* = (\pi/4)D^2 V_t E w_f L_f I, \tag{10}$$

where Q_F^* is the rate of heat liberated by the accreted and partially or completely frozen water droplets, E the collection efficiency (=1), and $w_f = r\rho$ the free water content (gm cm^{-3}). The fraction I of accreted water which freezes is determined by the heat balance equation (List, 1963), while L_f is a function of the hailstone surface temperature t_D. As described more completely in Section 6, I can assume values between 0 and 1 if t_D=0C. It is further assumed that all the heat released on the hailstone surface is given off to the surrounding air and is not used for heating or cooling of the hailstone.

There is also a second heat source: frictional heating of the air by the falling hailstones, a factor never previously considered, but necessary on the basis of physical arguments. The rate at which heat is produced by one hydrometeor is given by the product of particle weight W and terminal speed V_t. (This term is also applicable to models involving raindrops or rain formation.)

To see what effect these additional heat sources have on the cloud temperature which is given by the pseudo-adiabatic values, the above mentioned two terms must be introduced into Eq. (6). The heat given off per unit time by thermal feedback to $(1+r_w)$ gm of cloud air is represented by $N\rho^{-1}(Q_F^*+WV_t)(1+r_w)$. When this expression is included in (6), one obtains

$$\Delta T = \frac{N\rho^{-1}V_z^{-1}(Q_F^*+WV_t)(1+r_w)\Delta z + [RT(P-e_w)^{-1} + RR_w^{-1}e_w L_v (P-e_w)^{-2}]\Delta P}{C_p + RR_w^{-1}e_w(P-e_w)^{-1}C_{pv} + RR_w^{-1}L_v^2 e_w[(P-e_w)R_w T^2]^{-1}[1+e_w(P-e_w)^{-1}]}. \tag{11}$$

This equation was solved for temperature as function of height using values of D and w_f obtained from different combinations of input parameters. One of the cases which showed the most marked effect of thermal feedback is illustrated in curve C of Fig. 1, for V_{z0}=25 m sec^{-1}, D_0=1.0 cm and N_0=2.0 m^{-3}. For this case there is very little difference between the thermal feedback temperature T_{TF} and the pseudo-adiabatic cloud temperature T_{AD} in the lower part of the cloud, but at the 5-km level the difference between T_{TF} and

T_{AD} is about the same (~2.2C) as the difference between T_{AD} and T_E, the temperature of the environment.

The ratio $(T_{TF}-T_{AD})/(T_{AD}-T_E)$ for the small embryo sizes (D=0.25 cm) increases with increasing concentration N_0; however, for large D_0 this ratio shows decreasing tendencies. The above ratio increases for bigger particles (2.06 at the 5-km level for N_0=1.5 m^{-3}, D_0=1 cm), where the heating by thermal feedback might even be twice as high as the effect due to pseudo-adiabatic lifting.

Since the sum of two terms, Q_F^* and WV_t, is contributing to the thermal feedback, the relative importance of the single terms will be looked into by forming their ratio, Q_F^*/WV_t. If Q_F^* is replaced by $D^2w_fV_t$ and WV_t by D^3V_t, neglecting constants, the ratio becomes w_fD^{-1}. Since w_f generally decreases at the higher levels while D increases, the frictional heat term WV_t is of greater relative importance at these heights than at lower levels. In the cases examined WV_t was found to be one order of magnitude smaller than Q_f^* in the lowest levels, but became equal to or greater than Q_F^* higher up as the free water fell off.

Including the thermal feedback term, the first term in the numerator of Eq. (11), in the pseudoadiabatic equation (6) considerably increases the computational chores. Growth curves and free water content distributions were therefore evaluated on the basis of the pseudo-adiabatic cloud temperature. The effect of thermal feedback was assumed to be quite small in this respect as an air density change in (7) and (8) of 2% would affect the growth rates and free water profiles by about the same amount.

Performing computations of some growth curves and w_f distributions including thermal feedback demonstrated, indeed, that growth and depletion are insenitive to changes in air density introduced by feedback. The biggest effects on $D=D(z)$ and $w_f=w_f(z)$ were less than about 3%. However, thermal feedback will produce considerable effects if buoyancy is treated because buoyancy depends upon density differences.

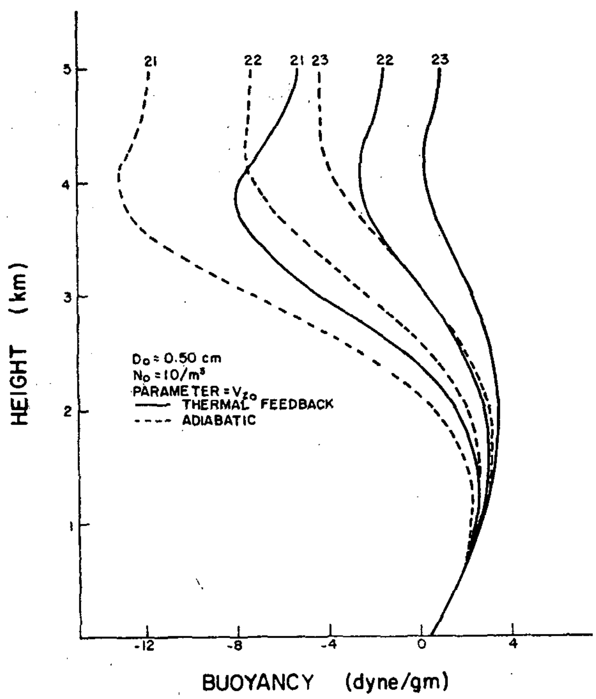

FIG. 9. Buoyancy of growing hailstones as a function of height, at different updraft speeds, calculated on the bases of pseudo-adiabatic lifting, dotted lines, and pseudo-adiabatic lifting plus heating by hailstones (thermal feedback), solid lines.

5. Buoyancy

In the previous sections no attempt was made to link the updraft with forces exerted and created by the growing hailstones. One of the necessary steps toward a dynamic treatment of a model cloud containing growing hailstones is the assessment of buoyancy. Therefore, properties of our one-dimensional updraft model will be compared with the environment which is described by expressions (3) and (4). As defined in this paper, buoyancy consists of two parts, the first contribution resulting from the difference in density between the environment and the cloud, and the second a negative contribution due to the downward drag forces exerted on the air parcel by the water droplets and the hailstones. Since terminal speed is assumed, the drag forces are represented by the particle weights. Such a treatment is generally applied by others (e.g., Srivastava, 1967).

The buoyancy force per unit volume due to air density difference alone is given by

$$b'=g(\rho_E-\rho), \tag{12}$$

where g is the acceleration due to gravity; the subscript E stands for environment. Since the pressures in the cloud and the environment are assumed to be the same at a given level, (12) can be rewritten as

$$b'=g\rho(TT_E^{-1}-1). \tag{13}$$

Because the cloud temperature is higher than that of the environment, b' is always positive. Negative contributions are represented by the weight of the cloud droplets, w_fg, and the weight of the hailstones, NW. Thus, the buoyancy force per volume becomes

$$b=g\rho(TT_E^{-1}-1)-(NW+w_fg). \tag{14}$$

To demonstrate the large effect of thermal feedback, the next two figures illustrate buoyancies both with and without thermal feedback. Fig. 9 gives a good indication of how these buoyancies vary with height and updraft speed for hailstones which do not reach balance level. The trend for the buoyancy to increase just above the freezing level and to decrease at higher levels is quite general. The first increase is due to the fact that the first term in (13) becomes bigger with height because

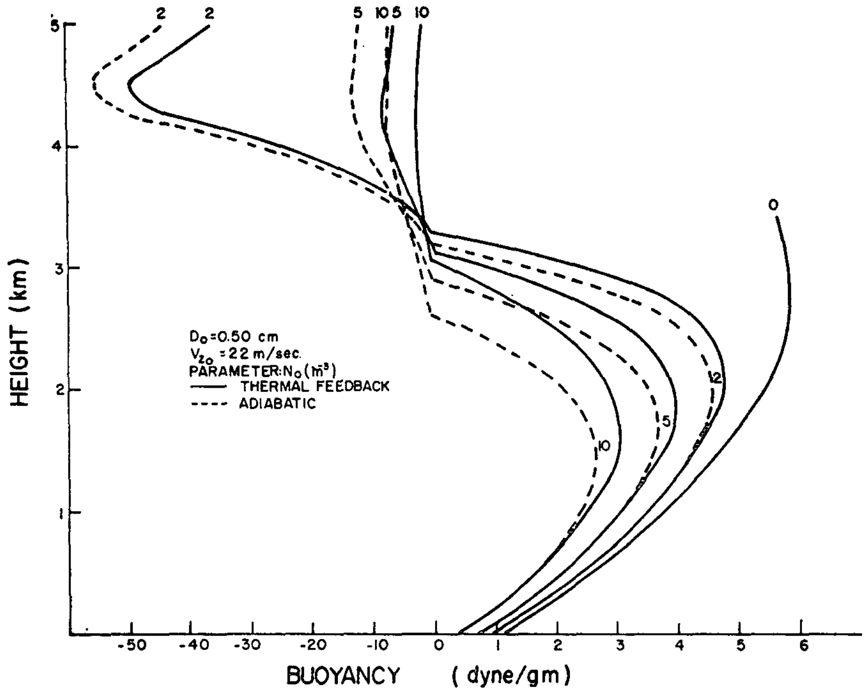

FIG. 10. Variation of buoyancy as a function of height, for different number densities at one 0C level updraft speed, calculated on the bases of pseudo-adiabatic lifting, dotted lines, and pseudo-adiabatic lifting plus heating by hailstones (thermal feedback), solid lines.

of the increasing difference between the cloud and the environment temperature; however, above a certain level the second term of the same equation becomes more important due to the increasing weight of the hailstones and the cloud droplets. This latter effect diminishes again in most cases as the upper regions of the updraft are reached.

It can be observed in Fig. 9 that the inclusion of thermal feedback does not alter the buoyancy in the first 1.5 km; however, between levels 3 and 5 km, its effect (as could be expected from Fig. 1) is quite pronounced. For an updraft of 23 m sec^{-1}, inclusion of thermal feedback keeps the buoyancy positive throughout the whole height range, whereas considerable negative buoyancy results without the direct and indirect heating of the cloud by the hailstones.

Fig. 10 shows similar buoyancy-height relationships for three different values of N_0. Again the effect of feedback is considerable, particularly at the higher levels where it might not be so obvious due to the different scale of the negative part of the abscissa.

It is obvious from Figs. 9 and 10 that the thermal feedback has to be considered in any buoyancy calculations. All the subsequent calculations, therefore, do include heating effects by release of latent heat of fusion and by friction. If buoyancy is mentioned later on, it includes thermal feedback.

For zero particle concentration the buoyancy is always positive (Fig. 10). If the input concentration is 2 m^{-3} with $D_0=0.5$ cm and $V_{z0}=22$ m sec^{-1}, the buoyancy reaches very large negative values, caused by high hailstone concentrations or small absolute upward speeds of these hydrometeors. Higher embryo input concentrations reduce the maximum positive and negative buoyancies considerably.

The variation of buoyancy with updraft velocity is further explored in Fig. 11, where single cases correspond to both types of growth curves, the sigmoid and the balance level. The break in the trend of the curves while passing through zero buoyancy is caused by the compression of the negative scale. Higher zero-level updrafts lead to greater buoyancies for two reasons: strong updrafts cause lesser hailstone accumulation or smaller concentrations, and faster particle growth at low levels in weak updrafts increases the magnitude of the negative buoyancy term.

Since the cloud model treated in this paper is principally one-dimensional and since no entrainment is considered, the buoyancy force of one layer is acting upon the neighboring layers. If one assumes no vertical mixing or compression, caused by a nonhydrostatic pressure force on the updraft due to the weight of the hailstones, a new expression, the *cumulative buoyancy*, can be introduced. The cumulative buoyancy, Σb, is defined as the sum of the buoyancies of the layers starting from the 0C level. The argument for the introduction of Σb is that the motion of the layer is influenced by the buoyant forces above and below, and air parcels with negative buoyancy may, as a result, be forced to move upward. In a one-dimensional model the cumulative

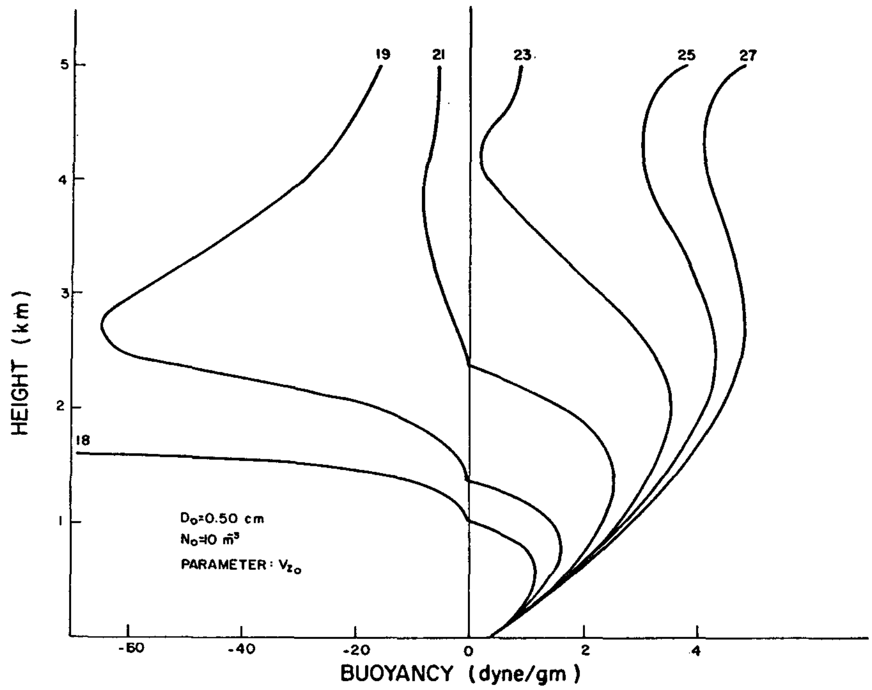

FIG. 11. Buoyancy as function of height for different updraft speeds (heating due to hailstones included).

buoyancy represents a driving force which should be, in the end, closely related to the updraft strength. The level at which this sum becomes zero has some special significance because at this point the net force upon a vertical air column between these levels is zero. If this situation occurs at the cloud top, we have a realistic situation for a steady-state hail cloud because the net force on a column of air within the whole of the cloud will be zero; thus, with no friction and other counteractive forces in the flow system, no accelerations or decelerations occur. On the other hand, if this level of zero cumulative buoyancy occurs lower down in the cloud, then in order to obtain a non-negative net force, Σb has to be compensated by a non-hydrostatic vertical pressure gradient; such a situation is unrealistic for steady-state hail clouds unless some strong external mechanism (high upper winds) exists for their development. The combination of parameters giving rise to strong negative Σb may therefore be rejected as far as this model is concerned. On the other hand, if the cumulative buoyancy is positive at the top of the cloud and no other reactive forces could be accounted for, the updraft would have to increase, i.e., the model would not be steady-state.

The level at which the cumulative buoyancy Σb becomes zero was determined for all the combinations of parameters given in Section 3. The selection of Figs. 3–5 was particularly based on realistic values for buoyancy and cumulative buoyancy. Figs. 3a, 4a, and 5a contain isolines of $\Sigma b=0$ as well as isolines at which the buoyancy changes from positive to negative values, i.e., $b=0$. For all the figures shown the spread between the two isolines becomes smaller with decreasing input embryo concentration N_0. This is in accordance with Fig. 10 which shows the rapid drop of the buoyancy with height for the smaller N_0. The successive Figs. 3a, 4a and 5a for $D_0=0.5$ cm demonstrate how the height of both isolines rises for increasing V_{z0}. Furthermore, this rise in height is more pronounced for the higher N_0, causing the gradual change of slope of both lines with increasing V_{z0}. The greatest change in the $\Sigma b=0$ level for $D_0=0.5$ cm occurs between V_{z0} values of 21 and 22 m sec^{-1}. Here, for N_0 values of 5 and 10 m^{-3}, this level changes from about 3.5 km to somewhere above 5 km, while for lower values of N_0 the change is fairly small. For $V_{z0}=23$ m sec^{-1} (Fig. 4a) the $\Sigma b=0$ isoline has already moved above the 5-km level; the same is true for $V_{z0}=24$ m sec^{-1}.

The remarkable fact of these considerations is that buoyancy and cumulative buoyancy are extremely sensitive to changes in the updraft speed. In other words, dynamic relationships between updraft speed and buoyancy have to be quite accurate ($\pm5\%$ in velocity) to predict the right cloud conditions. No attempt in this direction will be made now.

6. Icing conditions

In the previous sections we explored the growth of hailstones, the depletion of the liquid water content they cause, the heat source they represent, and the buoyancy which can be attributed to the updraft air. In order to make the picture more complete, we now investigate the icing conditions of the hailstones, i.e., their surface temperature t_D or the sponginess of the deposits growing at certain levels, the latter represented by the fraction I of accreted water which freezes.

The surface temperature and the sponginess can be obtained from a heat balance equation (List, 1963) in which the sum of the heat transferred to a growing hailstone by conduction and convection, by evaporation or sublimation, and by accretion of supercooled cloud droplets is used to freeze the accreted water either partially ($I<1$; $t_D=0C$) or completely ($I=1$; $t_D\leq 0C$). This balance equation can be solved for either t_D or I; thus,

$$t_D=\frac{Y[L_f(t_D)I+\bar{C}_w t_A]+1.68kt_A-[e_{SH}(t_D)-e_W(t_A)]C_{1,2}D_{wa}T_A^{-1}}{1.68k+\bar{C}_w Y}, \tag{15}$$

$$I=\frac{-1.68kt_A+[e_{SH}(\mathrm{OC})-e_W(t_A)]C_{12}D_{wa}T_A^{-1}-Y\bar{C}_w t_A}{YL_f(\mathrm{OC})}, \tag{16}$$

with

$$Y=5.48\nu^{\frac{1}{2}}\rho^{-\frac{1}{2}}E\theta^{-1}w_f D^{\frac{3}{2}}, \tag{17}$$

where t_D (°C) is the hailstone temperature, t_A (°C) and T_A (°K) the air temperature, $\bar{C}_w$ the specific heat of water averaged over the temperature range (t_D,t_A), k the thermal conductivity of air, $e_{SH}(t_D)$ the saturation vapor pressure over the hailstone surface, and $e_W(t_A)$ the saturation vapor pressure over water at air temperature; $C_{1,2}$ has a value of 0.235 cal (°C) cm^{-3} mb^{-1} at −20C or lower and varies linearly with temperature from −20C to a value of 0.207 cal (°C) cm^{-3} mb^{-1} at 0C; D_{wa} is the diffusivity of water vapor in air, L_f the latent heat of fusion, ν the kinematic viscosity, E the collection efficiency (=1), and θ a roughness factor; $E\theta^{-1}$ is 0.675 for conditions of spongy growth explored by List in 1960. Eq. (15) was solved by an iteration process leading to a computation accuracy of t_D of ±0.02C. Values for t_D or I were obtained in this way at intervals of 0.2 km for all the cases for which the buoyancies were determined.

The results are shown in Figs. 3b, 4b, and 5b in the form of dashed isolines, giving the hailstone surface temperature t_D or, I, the fraction of the accreted water which freezes. In this calculation the heat stored in the hailstones was neglected, i.e., their temperature was assumed to be the equilibrium temperature of deposits growing at a given level. As long as spongy ice is growing, no error is introduced by such a procedure; at $t_D<0C$, the hailstone surface temperatures would, for our model, be somewhat higher than the calculated values due to heat flow from the warmer interior of the rising hailstone.

In most cases the isolines of t_D and I slope from the lower left to the upper right. At constant height, t_D increases and I decreases with higher values of w_f and D [see Eqs. (15) and (16)]. Since the liquid water content and the diameter at a given height usually increase with decreasing N_0, the sloping of the isolines is sufficiently explained. Exceptions which can be observed, but not in the data shown here, can be explained by close examination of the liquid water content distributions.

The most important conclusion from the calculated icing conditions is that the biggest hailstones usually have a deposit or surface temperature slightly below 0C, i.e., their growth is usually not spongy in the outer shells as might be expected on the basis of laboratory measurements. The reason for this result can be understood by looking at the growth curves and the velocity profile. Fast growth normally means spongy growth; however, in our model fast growth leads to low balance levels and dropout at smaller sizes. It is interesting to learn from a thorough investigation of two samples of hailstones (List *et al.*, 1969) that many outer shells of 4-cm hydrometeors were definitely grown non-spongy. Whether or not these stones grew in a cloud similar to our model is naturally still an open question.

The idea, mentioned in Section 3, that shells could grow upon hailstones by changes in input parameters, particularly of updraft speed (or the level of the cloud base) becomes even more attractive when considering the remarkable dependence of the icing conditions on these factors. However, extensive studies would have to be undertaken to clarify this point beyond any doubt.

7. Time of growth

Last, but not least, the time taken by the hailstones to reach a certain level in the cloud, as they are carried upward by the updraft, were obtained from

$$t=\int_0^z \frac{1}{(V_z-V_t)}dz, \tag{18}$$

where z is the vertical distance measured from the input level. For the growth curves that terminate at a balance level, the calculation was stopped at the preceding level.

The growth times obtained are shown in Figs. 3a, 4a, and 5a as solid isolines intersecting the growth curves. For a given updraft and at a given height, smaller concentrations N_0 normally mean larger diameters which, in turn, results in a higher terminal velocity and thus a longer growth time. The longest growth times at the 5-km level occur for the smallest concentration of particles which reach this level.

In general, it can be seen that the growth times of relatively big stones, i.e., stones which reach a balance level below or at the 5-km level, are of the order of 8–12 min. These values are consistent with radar observations.

8. Discussion

The authors recognize that the simplifications of the model studied in this paper are sometimes quite artificial and that more thorough studies are needed to improve and expand the model or to make it more realistic. Nevertheless, this study was quite successful in demonstrating that "normal" spherical hailstones can grow to 2–4 cm in size without having to resort to recycling or water accumulation zones. They may well exist, but they do not seem to be necessary.

If ellipsoidal particles are considered, they may grow to even larger sizes in a steady-state model. Bigger spherical particles could grow in updrafts which increase in strength with time.

A one-dimensional model is only a first step toward a three-dimensional one. However, the available concepts of entrainment often defy realism and three-dimensional aerodynamic feedback (List and Lozowski, 1968) is so completely unexplored that more basic studies have to be undertaken to make the step to three dimensions safer. It has to be realized that non-hydrostatic pressure terms are essential and that buoyancy can be partly compensated or over-compensated by such terms and not only by momentum changes. The magnitude of the resistance to a thunderstorm circulation should also be better known so that it might be compared to the driving forces already mentioned.

In summary, it can be said that the study of a one-dimensional model of the growth of hailstones has improved our insight into the mechanics of hail formation and many conclusions might, with caution, also be applied to certain natural hailstorms. Of particular importance is the conclusion that the degree and the variation of free water depletion as functions of updraft variations and particle parameters can account for the growth of hailstone shells.

Hail prevention. The calculations on hailstone growth might also be of some importance for preventing hail fall from clouds for which this model is representative. First it is seen that relatively big stones can only be expected for particle concentrations at the 0C level of 0.5–2.0 m^{-3}, a range which might also be representative in nature. If the concentrations are increased to about 10 m^{-3} or higher, the final size is such that complete or nearly complete melting will occur while the hailstones fall toward the ground. Hail damage might therefore be prevented by increasing the embryo concentrations to about 10 m^{-3}.

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