Reply

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I agree with Veronis' main point, i.e., that the neglect of the $2\,\Omega\cos\varphi$ Coriolis terms is not justified solely by the shallowness condition $|r-a|\ll a$, but the physical situation must also be considered. I regret any confusion my note may have caused and hope that the present Correspondence will clarify matters.

The example of viscous Stewartson layers which Veronis cites can occur in a homogeneous fluid. However, the ocean and atmosphere are for the most part stably stratified; the potential density ρ^* decreases with height, rapidly enough that the buoyancy frequency

$$N=\left(-\frac{g}{\rho}\frac{d\rho^*}{dz}\right)^{\frac{1}{2}}$$

is typically much larger than the angular velocity Ω of the earth. This has considerable influence on the relative importance of $2\Omega\cos\varphi$ and $2\Omega\sin\varphi$. If the motion is treated as quasi-incompressible, plane-wave solutions of the form $\exp[i(\alpha x+\beta y+\gamma z-\omega t)]$, with x eastward and y northward, superimposed on a resting ocean or atmosphere, obey the frequency relation (O. Phillips, 1966, p. 193)

$$\omega^2=\frac{4\Omega^2(\gamma\sin\varphi+\beta\cos\varphi)^2+N^2(\alpha^2+\beta^2)}{\alpha^2+\beta^2+\gamma^2}. \qquad (6)$$

This results from the (linearized) form of Eq. (2) of Veronis and the other hydrodynamic equations. If (2a) were used, the $\cos\varphi$ term would be absent. The formula is to be interpreted only as a WKBJ approximation to the complete partial differential equation, since the latter has variable coefficients. When $N^2 \gg 4\Omega^2$, it is difficult to have $\Omega \cos\varphi$ play a dominant role (at least when the wavenumbers α, β, γ are restricted to real values) since this requires large β^2, and then N^2 dominates. On the other hand, $\Omega \sin\varphi$ can be made dominant by simply choosing (α/γ) and (β/γ) to be sufficiently small, whereupon the so-called "inertia waves" result, in which $\omega \sim 2\Omega \sin\varphi$. [See also Bretherton (1964).]

Although this argument shows that it is difficult to ascribe any *dominant* importance to the $2\Omega \cos\varphi$ term when $N^2 \gg 4\Omega^2$, this term can contribute in more subtle ways. For example, Backus (1962) has shown that although the theoretical effect of the Coriolis force on the propagation of oceanic surface waves is very small, the small Coriolis effect enters primarily through the $2\Omega \times \cos\varphi$ term rather than through $2\Omega \sin\varphi$. A second example in which $2\Omega \cos\varphi$ may enter significantly *in the finer details* occurs in the "inertia waves" referred to above. To see this, it is convenient to rewrite (6) as

$$\omega^2 = \frac{4\Omega^2 \sin^2\varphi + N^2\left(\frac{\alpha^2+\beta^2}{\gamma^2}\right) + 8\Omega^2 \sin\varphi \cos\varphi\left(\frac{\beta}{\gamma}\right) + 4\Omega^2 \cos^2\varphi\left(\frac{\beta}{\gamma}\right)^2}{1+(\alpha^2+\beta^2)/\gamma^2}. \tag{7}$$

The third and fourth terms in the numerator are the $2\Omega \cos\varphi$ terms. Smallness of α/γ and β/γ gives $\omega \sim 2\Omega \times \sin\varphi$, as mentioned earlier. But the *deviation* of ω from $2\Omega \sin\varphi$ due to a nonzero value of (β/γ) can be influenced by the third term in the numerator as much as by the $N^2\beta^2/\gamma^2$ term if (β/γ) is small enough. A more careful examination of this point requires a solution of the complete partial differential equation, since (6) is a large wavenumber approximation. As noted by Eckart (1960, p. 101 and pp. 130–135), this solution is not a trivial matter because that partial differential equation is nonseparable in z and φ. [Some further remarks, unfortunately also inconclusive, can be found in an unpublished thesis by B. Hughes (1964) from Cambridge University, and in a paper by W. Munk and myself scheduled for the November 1968 issue of *Reviews of Geophysics*.]

A similar restraining influence of small $4\Omega^2/N^2$ on the importance of $2\Omega \cos\varphi$ *vis-à-vis* $2\Omega \sin\varphi$ occurs in the case of Rossby waves. To show this, it is necessary, however, to derive the Rossby frequency relation in a different manner than is to be found in the literature, since the usual derivations ignore the effect in question. A convenient starting point is the homogeneous form of the linear equations given by Eckart (1960, p. 56). The WKBJ formalism, in which variables are represented in the form $\boldsymbol{u} = \mathrm{Re}[U \exp(i\psi)]$ and in which the common phase ψ is normally taken to vary rapidly in both space and time, is applied to these equations, but with special treatment of the frequency $\omega = -\partial\psi/\partial t$ compared to the wavenumber $\mathbf{k} = (\alpha, \beta, \gamma) = \nabla\psi$. For example, $\partial u/\partial t = e^{i\psi}(\partial U/\partial t - i\omega U)$ and $\nabla u = e^{i\psi}(\nabla U + i\mathbf{k}U)$ are expanded as follows, with subscripts and brackets indicating the successive approximations:

$$\frac{\partial U}{\partial t} - i\omega U = [\text{zero}] - [i\omega_1 U_0] + \left[\frac{\partial U_0}{\partial t} - i\omega_2 U_0 - i\omega_1 U_1\right] + \cdots$$

$$\nabla U + i\mathbf{k}U = [i\mathbf{k}_0 U_0] + [i\mathbf{k}_0 U_1 + i\mathbf{k}_1 U_0 + \nabla U_0] + \cdots.$$

In other words, ω_0 is set equal to zero, but the dominant time dependence is still due to ω. The result is most readily obtained by considering the zero- and first-order expansions of the five equations given by Eckart; thus,

$$\omega_1 = \frac{-\left(\frac{2\Omega \cos\varphi}{r}\right)\alpha_0}{\alpha_0^2+\beta_0^2+\frac{4\Omega^2}{N^2}\left[\left(\frac{N^2}{c_0^2}+\Gamma^2\right)\sin^2\varphi + (\gamma_0 \sin\varphi + \beta_0 \cos\varphi)^2\right]}, \tag{8}$$

where c_0 is the speed of sound in the undisturbed state and Γ is Eckart's compressibility parameter, $(g/c_0^2)+\frac{1}{2}d \times [\ln(\rho_0 c_0)]/dr$, in the undisturbed state ($\Gamma^2 + N^2/c_0^2$, however, is normally small compared to any reasonable value of γ_0^2). The Coriolis term $2\Omega \sin\varphi$ in the horizontal equations of motion is responsible for the $\sin\varphi$ terms in the denominator and, through its variation with latitude, for the numerator. The Coriolis term $2\Omega \cos\varphi$ introduces only the $\beta_0 \cos\varphi$ term in the denominator. Smallness of $4\Omega^2/N^2$ again prevents this term from contributing in any significant manner.

Veronis shows that the equations (2) possess an angular momentum principle *if one is allowed to re-introduce approximations of order* $(r-a)/a$. The point of view expressed in my note was that it is preferable to work with a self-contained system of equations in which further approximations are no longer considered. For example, the type of manipulations Veronis used to derive (5) from (2) would not be invoked by a computer which was programmed to solve numerically an initial-value problem using (2). Angular momentum is not unique in this respect; similar questions can also be expected with respect to conservation of mass, circulation and vorticity theorems, and conservation of potential vorticity, whenever, as has been done in deriving (2), scale factors are approximated in a manner which destroys the validity of Stokes' and Gauss' theorems. My suggestion that the scale factors be approximated initially can be thought of as producing a *geometric model* which is free of the above defects, and which—thanks to the large values of N^2/Ω^2—seems to be capable of describing all major effects of Ω on our atmosphere and ocean.

A minor point of interpretation. In the paragraph following (5) Veronis writes that (3) is tantamount to requiring that particles remain on the surface $r=a$. Since (3) follows from (2a) without any restriction that $w=0$, I prefer to interpret (3) as stating simply that the angular momentum of a particle is to be evaluated *as if* the particle were located at $r=a$.

I take this opportunity to acknowledge the receipt of some stimulating correspondence earlier on this matter from several people, especially from Mr. John Olsen. My first exposure to the effect of vertical stratification on the importance of $2\Omega\cos\varphi$ was in a paper by Veronis (1963).

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