

## Theory of Planetary Wave-Zonal Flow Interaction

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### ABSTRACT

Small amplitude planetary waves are superimposed on a mean zonal flow with arbitrary horizontal and vertical shears. An expression is derived for the change of the zonal wind and temperature field forced by statistically stationary eddies satisfying a source-free planetary wave equation. This result depends on the existence of singular lines, where the phase speed of an elementary wave is equal to the mean zonal wind speed, or on the presence of a Newtonian cooling process. Second-order interactions vanish when both of these phenomena are absent. The planetary wave-zonal flow interaction is discussed in terms of the eddy transport of potential vorticity. The theory provides a partial interpretation of the maintenance of atmospheric zonal flows, such as that of the wintertime stratosphere, by planetary waves propagating from some other region of the atmosphere.

### 1. Introduction

Horizontal eddy transports of momentum and heat provide the basis for much of our present interpretation of the climatology of large-scale dynamical processes in the earth's atmosphere. The eddy momentum transports are generally poleward and act to augment the existing currents (Starr, 1948, 1968; Newell, 1963), while the heat transports are also usually poleward and act to maintain large departures of the mean temperature field from radiative equilibrium (White, 1951, 1954).

The large number of previous theoretical investigations seeking to explain various aspects of the energetic interaction of eddies with the zonal flow include the nonlinear tendency studies (Platzman, 1952; Charney, 1951; Kuo, 1953; Lorenz, 1953; Lipps, 1966) as well as the stability theory studies (Kuo, 1951; Pedlosky, 1964). Realistic numerical models are now reasonably successful in simulating the observed mechanics of eddy-zonal flow interaction (Smagorinsky *et al.*, 1965). For a theoretical description of the time-averaged state of the atmosphere corresponding to that obtained from observations, it would appear that the tendency and stability theories are of somewhat limited value. More generally, any monotonically growing or decaying solution for eddy motions is by itself an inadequate description of atmospheric eddy processes, since over a long time period atmospheric motions are essentially statistically stationary in time. There must necessarily be compensating processes to balance any such monotonic growth or decay. Without including some description of such restoring forces, it can never be known

whether these forces over a long term will more than cancel the effects deduced from the theory.

This paper is concerned with the question of the manner in which statistically stationary planetary waves determine eddy transports, maintaining mean zonal winds and temperatures. Over a long time period the mechanisms for energy input into eddies, such as baroclinic instability or lower boundary processes, are assumed to act as a large scale "stirring" of the atmosphere. The net energy input into the eddies must equal the energy loss by interaction with the zonal flow; thus, neglecting climatic fluctuations, the eddy wind and temperature fields are random stationary time series at a given point in space.

The present investigation is restricted to the analysis of the interaction of eddies with a zonal wind in a domain *away from the source region*, no effort being made to describe statistically the sources. Eddy motions are assumed governed by a source-free equation for quasi-geostrophic motions occurring in a middle latitude,  $\beta$ -plane geometry and of sufficiently small amplitude that a linearized wave model can be employed. One way to relate such a theoretical model to actual atmospheric statistics is by the specification of observed statistics on some internal boundary. Then, provided there are no sources beyond this boundary, solutions obtained using these statistics as boundary conditions should approximately describe the eddy motions in the region beyond the boundary. For example, Mak (1969) has developed within such a framework a two-layer, primitive equation model to study the horizontal propagation of large-scale disturbances into the tropics. Another problem of considerable interest is the construction, from observations at 10 mb or below, of the statistics of planetary waves at higher levels in the wintertime stratosphere

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where no hemispheric observations are presently available.

Observational statistics on the motions in the lower stratosphere are interpreted as showing that the zonal wind and the zonal temperature fields in this region are maintained against dissipation over a long time period through work done on this region by the troposphere below (Oort, 1964; Newell, 1964; Miller, 1967). This maintenance of the zonal flow in the lower stratosphere is generally considered to be a consequence of planetary wave propagation out of the troposphere. Likewise, the zonal flow at higher levels in the stratosphere and mesosphere is thought to be greatly influenced by such vertically propagating planetary waves. Thus, it would be expected that the theoretically computed forcing of stratospheric motions by planetary waves should describe the observed maintenance of these motions.

However, the *nonlinear perturbation analysis* of Charney and Drazin (1961) shows that forcing of zonal motions by time-independent, small amplitude planetary waves in a vertical shear flow vanishes. This apparent contradiction between observations and the perturbation theory is clarified by the present study. Two important features of planetary wave theory as now understood were neglected by Charney and Drazin. First, singular lines occur wherever the phase speed of a given elementary planetary wave equals the speed of the mean zonal wind, giving *absorption* of planetary waves, with jumps in the eddy momentum and heat transports (Dickinson, 1968a). Second, *damping* of planetary waves in the stratosphere by photochemical-radiative relaxation is significant (Dickinson, 1968b). When these two processes are neglected, the forcing by planetary waves is also absent for transient but statistically stationary planetary waves, and for mean flows with horizontal as well as vertical shear. A net forcing of mean flows by eddies depends on the eddy transport of potential vorticity, small amplitude planetary waves giving such a transport only at singular lines or in the presence of diabatic relaxation effects.

Our development assumes as given functions of latitude and pressure the mean zonal winds and temperatures as well as certain eddy statistics. The rate of zonal momentum increase is determined by the difference between the eddy momentum convergence and the Coriolis torque momentum loss by horizontal mean meridional motion, and the rate of zonal temperature increase by the difference between eddy heat transport convergence and adiabatic cooling by vertical mean meridional motions. Horizontal and vertical mean meridional motions are coupled by the continuity equation. The zonal momentum and temperature equations are combined to obtain the mean northward and vertical velocities and, hence, the net changes of zonal momentum and temperature in terms of the eddy momentum and heat convergences (Eliassen, 1951; Kuo, 1956).

The relationship between these eddy transports and the mechanisms of singular line interaction and Newtonian cooling is established first for eddy transports of a single frequency, then for arbitrary eddy motions with time stationary statistics. The latter relationship is combined with the expressions for the forcing of the zonal flow to describe the manner in which statistically stationary planetary waves act to maintain departures of the mean zonal flows from the configuration that would be observed in the absence of the waves.

## 2. Formulation

A middle latitude  $\beta$ -plane model is used, the Coriolis parameter  $f$  being given by  $f=f_0+\beta y$  with  $y=a\varphi$ , where  $a$  is the radius of the earth and  $\varphi$  latitude. A zero subscript will indicate a constant value of a quantity. Further notation is  $x=a\lambda \cos \varphi_0$ , with  $\lambda$  as longitude, as eastward coordinate;  $z=\log(p_0/p)$  as a vertical coordinate;  $u=\dot{x}$ ,  $v=\dot{y}$  are horizontal velocities;  $w=\dot{z}$ ;  $\rho_s=e^{-z}$ ;  $T$  is temperature,  $R$  the gas constant,  $c_p$  specific heat, and  $\kappa=R/c_p$ . The zonal average of a quantity is indicated by a bar and deviations from the zonal average (eddies) by primes. Diabatic heating is assumed to restore the temperature to a zonal radiative equilibrium temperature  $\bar{T}_e$ , the rate of heating being given by

$$Q = -\alpha c_p (T - \bar{T}_e). \quad (1)$$

The Newtonian cooling coefficient  $\alpha$  is assumed independent of longitude and time.

The zonal momentum, temperature, continuity and thermal wind equations are written as follows:

$$\frac{\partial \bar{u}}{\partial t} + d\bar{u} - f_0 \bar{v} + \frac{\partial}{\partial y} (\overline{u'v'}) = 0, \quad (2a)$$

$$c_p \left[ \frac{\partial \bar{T}}{\partial t} + \alpha (\bar{T} - \bar{T}_e) + \frac{\partial}{\partial y} (\overline{v'T'}) \right] + \frac{wS}{\kappa} = 0, \quad (2b)$$

$$\rho_s \frac{\partial \bar{v}}{\partial y} + \frac{\partial}{\partial z} (\rho_s \bar{w}) = 0, \quad (2c)$$

$$R \frac{\partial \bar{T}}{\partial y} + f \frac{\partial \bar{u}}{\partial z} = 0. \quad (2d)$$

The function  $T_e(z)$  is a mean reference temperature and the static stability  $S$  is given by

$$S = R \left[ \frac{\partial T_e}{\partial z} + \kappa T_e \right].$$

The Rayleigh friction  $d\bar{u}$  and Newtonian cooling  $c_p(\bar{T} - \bar{T}_e)$  insure the absence of a long-term increase of the mean wind and temperature fields. The following discussion is simplified by assuming that  $\alpha$  and  $d$  are constant and equal in (2), and that  $\bar{T}_e$  depends only on  $z$ .

Reducing (2) to an equation in  $\bar{w}$  or  $\bar{v}$  alone by elimination of the other dependent variables then gives

$$\left[ \left( f_0^2 \frac{\partial}{\partial z} (1/\rho_s) \frac{\partial}{\partial z} \right) + S \frac{\partial^2}{\partial y^2} \right] \bar{w} = - \left[ R \frac{\partial^3}{\partial y^3} (\bar{v}' T') + f_0 \frac{\partial^3}{\partial y^2 \partial z} (\bar{u}' v') \right], \quad (3a)$$

$$\left[ (f_0^2/\rho_s) \frac{\partial}{\partial z} \frac{\rho_s}{S} \frac{\partial}{\partial z} + \frac{\partial^2}{\partial y^2} \right] \bar{v} = \left[ (R/\rho_s) \frac{\partial^3}{\partial y^2 \partial z} \left\{ \frac{\rho_s}{S} (\bar{v}' T') \right\} + (f_0/\rho_s) \frac{\partial^2}{\partial y \partial z} \left\{ \frac{\rho_s}{S} \frac{\partial}{\partial z} (\bar{u}' v') \right\} \right]. \quad (3b)$$

The dissipation coefficients  $\alpha$  and  $d$ , being taken equal, do not enter these equations for the meridional circulation.

For small quasi-geostrophic perturbations and in the absence of sources, the eddy stream function  $\psi'$  is governed by the linearized potential vorticity equation

$$\frac{\delta}{\delta t} \bar{q}' + \frac{\partial \bar{q}'}{\partial y} \frac{\partial \psi'}{\partial x} = -\rho_s^{-1} f_0^2 \left( \frac{\rho_s \alpha}{\partial z} \right) \frac{\partial \psi'}{\partial z}, \quad (4)$$

using the definitions  $\delta/\delta t = \partial/\partial t + \bar{u} \partial/\partial x$  for the linearized substantial derivative,

$$q' = \rho_s^{-1} f_0^2 \partial/\partial z [(\rho_s/S) \partial \psi'/\partial z] + (\partial^2/\partial x^2 + \partial^2/\partial y^2) \psi'$$

for the eddy potential vorticity, and

$$\bar{q} = (f - \partial \bar{u}/\partial y) + \rho_s^{-1} \partial/\partial z (\rho_s f_0 R \bar{T}'/S)$$

for the zonal averaged potential vorticity. The eddy horizontal velocities  $u'$  and  $v'$ , eddy geopotential  $\phi'$ , and temperature  $T'$  are determined from  $\psi'$  using

$$\left. \begin{aligned} u' &= -\frac{\partial \psi'}{\partial y}, & v' &= \frac{\partial \psi'}{\partial x} \\ \phi' &= f_0 \psi', & RT' &= f_0 \frac{\partial \psi'}{\partial z} \end{aligned} \right\} \quad (5)$$

Eddy motions are to be cyclic continuous in  $x$ . The  $x$ -dependence of solutions is separated out by the assumption that  $\psi'(x, y, z, t)$  is a sum of terms of the form

$$\psi' = \psi(y, z, t, k) e^{ikx} + \psi(y, z, t, -k) e^{-ikx}. \quad (6)$$

Since the reduced stream function  $\psi$  is to be identified with a complex Fourier transform in the  $x$  variable, it is complex and its reflection about the origin in  $k$  space must equal its complex conjugate,

$$\psi(-k) = \psi^*(k), \quad (7)$$

where a star indicates the complex conjugate of a quantity.

The relations (5), (6) and (7) allow us to evaluate the meridional eddy momentum and heat transports, the meridional velocity and temperature variances, and the temperature-geopotential covariance, respectively, as

$$\left. \begin{aligned} \overline{u'v'} &= \frac{1}{2} ik (\psi^* \partial \psi / \partial y - \psi \partial \psi^* / \partial y) \\ \overline{v'T'} &= -\frac{1}{2} i (k f_0 / R) (\psi^* \partial \psi / \partial z - \psi \partial \psi^* / \partial z) \\ \overline{v'^2} &= k^2 |\psi|^2 \\ \overline{T'^2} &= (f_0 / R)^2 |\partial \psi / \partial z|^2 \\ \overline{T'\phi'} &= \frac{1}{2} (f_0^2 / R) \partial / \partial z |\psi|^2 \end{aligned} \right\} \quad (8)$$

### 3. A Lagrange identity for elementary planetary waves in a shear flow

The analysis of this section assumes planetary waves have been excited by an unspecified source starting at some large but finite time in the past,  $t = -\tau$ . This source excites motions which, after the initial transients have died out, have no systematic increase or decrease of amplitude for time less than  $\tau$ . For times greater than  $\tau$  the sources are turned off and the motions rapidly go to zero. The time dependence is separated out by use of a complex Fourier integral representation, evaluated in the same fashion as an inverse Laplace transform. That is, the eddy stream function  $\psi$  is given by

$$\psi = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\nu t} \Psi(\nu, y, z) d\nu, \quad (9)$$

where  $\nu = \nu_r + i\nu_i$  is a complex variable. The path of integration in (9) appropriate to inverting the Laplace transform is in the  $\nu_i > 0$  plane. For  $\nu_i > 0$ , the integrand is analytic in  $\nu$ .

A relation is now determined between various Fourier component covariance and variance variables for  $\nu_i > 0$ , then the limit  $\nu_i \rightarrow 0$  is taken. Variables entering this relation are the single frequency eddy momentum and heat transports, meridional velocity and temperature variance, and the temperature-geopotential covariance, denoted, respectively, by  $\hat{M}(\nu)$ ,  $\hat{H}(\nu)$ ,  $\hat{V}(\nu)$ ,  $\hat{\Theta}(\nu)$  and  $\hat{\Phi}(\nu)$ . Following the notation of (8), they are written as follows:

$$\left. \begin{aligned} \hat{M}(\nu) &= \frac{1}{2} ik \left( \Psi^* \frac{\partial \Psi}{\partial y} - \Psi \frac{\partial \Psi^*}{\partial y} \right) \\ \hat{H}(\nu) &= -\frac{1}{2} i \left( \frac{k f_0}{R} \right) \left( \Psi^* \frac{\partial \Psi}{\partial z} - \Psi \frac{\partial \Psi^*}{\partial z} \right) \\ \hat{V}(\nu) &= k^2 |\Psi|^2 \\ \hat{\Theta}(\nu) &= (f_0 / R)^2 \left| \frac{\partial \Psi}{\partial z} \right|^2 \\ \hat{\Phi}(\nu) &= \frac{1}{2} (f_0^2 / R) \frac{\partial}{\partial z} |\Psi|^2 \end{aligned} \right\} \quad (10)$$

These variables are functions of  $y$  and  $z$ . Attention is now confined to free motions away from the source region so that the homogeneous equation (4) applies. The potential vorticity equation has time-independent coefficients provided the zonal wind and temperature do not change with time. This being assumed, the Fourier decompositions (6) and (9) allow (4) to be written in terms of the spectral amplitude function  $\Psi$  as

$$\left[ f_0^2 \frac{\partial}{\partial z} \left( \frac{\rho_s}{S} \right) \frac{\partial}{\partial z} + \rho_s \left( \frac{\partial^2}{\partial y^2} - k^2 \right) \right] \Psi - \rho_s \frac{\partial \bar{q}}{\partial y} - k \Psi + i f_0^2 \frac{\partial}{\partial z} \left( \frac{\rho_s \alpha}{S} \right) \frac{\partial \Psi}{\partial z} = \frac{\partial \hat{M}}{\partial y} \quad (11)$$

Multiplying (11) by  $k\Psi^*$ , equating the imaginary parts, and writing the resulting expression in terms of the definitions (10), we have

$$R f_0 \rho_s^{-1} \frac{\partial}{\partial z} \left[ \left( \frac{\rho_s}{S} \right) \hat{H} \right] - \frac{\partial \hat{M}}{\partial y} = \Sigma_1 + \Sigma_2, \quad (12)$$

where  $\Sigma_1$  and  $\Sigma_2$ , evaluated in the limit as  $\nu_i \rightarrow 0$ , are

$$\left. \begin{aligned} \Sigma_1 &= \lim_{\nu_i \rightarrow 0} \frac{-\nu_i \frac{\partial \bar{q}}{\partial y} \hat{V}}{(k\bar{u} - \nu_r)^2 + \nu_i^2} = -\pi \frac{\partial \bar{q}}{\partial y} \hat{V} \delta(\nu - k\bar{u}) \\ \Sigma_2 &= (\bar{u} - \nu/k)^{-1} R \left\{ \rho_s^{-1} \frac{\partial}{\partial z} \left[ \left( \frac{\rho_s \alpha}{S} \right) \hat{\Phi} \right] - \left( \frac{R\alpha}{S} \right) \hat{\Theta} \right\} \end{aligned} \right\} \quad (13)$$

Eq. (12) generalizes for quasi-geostrophic motions a result of Eliassen and Palm [1961, cf. their (10.8)] which assumes adiabatic motions and the absence of singular lines. The expression (12) relates the horizontal eddy momentum and heat transports for a single spectral component to sources of potential vorticity occurring at singular lines or in the presence of dissipation.

**4. Statistics of transient eddies in a shear flow**

The analysis of the previous section gives the relationship (12) between the various eddy variance and covariance variables associated with a single frequency of motion. Observed disturbances have a continuous frequency spectrum. The spectral synthesis necessary to obtain the relationship corresponding to (12) for arbitrary eddy motions is derived in this section. Consider the term  $\psi \partial \psi^* / \partial \eta$  (where  $\eta$  is  $y$  or  $z$ ), which is written as

$$\psi \frac{\partial \psi^*}{\partial \eta} = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \int_{-\infty}^{\infty} \frac{d\nu'}{2\pi} \left[ e^{i(\nu - \nu')t} \Psi(\nu) \frac{\partial \Psi^*}{\partial \eta}(\nu') \right]. \quad (14)$$

It is now assumed that  $\psi$ ,  $\partial \psi / \partial y$  and  $\partial \psi / \partial z$  are random with spectra and cross spectra that are stationary in time, wedge brackets being used to denote an average over all time, i.e.,

$$\langle (\ ) \rangle = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \int_{-\tau}^{\tau} (\ ) dt. \quad (15)$$

Since the spectral representation for the delta function  $\delta(\nu - \nu')$  is

$$\delta(\nu - \nu') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\nu - \nu')t} dt, \quad (16)$$

the averaging of (14) over time and integration in  $\nu'$  establishes the identity

$$\left\langle \psi \frac{\partial \psi^*}{\partial \eta} \right\rangle = \lim_{\tau \rightarrow \infty} \frac{1}{2\tau} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ \Psi(\nu) \frac{\partial \Psi^*(\nu)}{\partial \eta} \right] d\nu, \quad (17)$$

the dependence of  $\Psi(\nu)$  and its derivatives on  $\tau$  entering through the assumption that these functions are Fourier transforms of time-dependent functions which are nonzero only between  $-\tau$  and  $\tau$ . Usual definitions of power spectra and cospectra of the random time series are now obtained according to (17) by dividing the relations (10) by  $2\tau$  and taking the limit as  $\tau \rightarrow \infty$ . The same symbols as in (10) but without the carets are used to denote these quantities. In particular, the eddy momentum transport and heat transport cospectra and northwind spectra are respectively denoted  $M(\nu)$ ,  $H(\nu)$  and  $V(\nu)$ , so that the time-averaged statistics for the random eddy motions are Fourier analyzed as

$$\left. \begin{aligned} \langle \overline{u'v'} \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} M(\nu) d\nu \\ \langle \overline{v'T'} \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} H(\nu) d\nu \\ \langle \overline{v'^2} \rangle &= \frac{1}{2\pi} \int_{-\infty}^{\infty} V(\nu) d\nu \end{aligned} \right\} \quad (18)$$

There is no further need to refer to the individual spectral amplitude functions  $\Psi$ ,  $\partial \Psi / \partial y$  and  $\partial \Psi / \partial z$ , which do not exist for the now-infinite interval of time. The integration of (12) over frequency, using (18) and (10), now gives

$$R f_0 \rho_s^{-1} \frac{\partial}{\partial z} \left( \frac{\rho_s \langle \overline{v'T'} \rangle}{S} \right) - \frac{\partial}{\partial y} \langle \overline{u'v'} \rangle = \sigma_1 + \sigma_2. \quad (19)$$

The left-hand side of (19), interpreted as the transport of quasi-geostrophic potential vorticity (Bretherton, 1966a), depends on the terms  $\sigma_1$  and  $\sigma_2$ , referred to as the singular line and Newtonian cooling interaction

terms, respectively, and defined by

$$\sigma_1 = -\frac{\partial \bar{q}}{\partial y} V(k\bar{u}), \tag{20}$$

$$\sigma_2 = R \left[ g - \frac{R\alpha}{\bar{u}S} \{T'^2\} \right], \tag{21}$$

where

$$\left. \begin{aligned} g &= \frac{1}{2\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\partial [(\rho_s \alpha / S) \Phi(\nu)] / \partial z}{\rho_s (\bar{u} - \nu / k)} d\nu \\ \{T'^2\} &= \frac{\bar{u}}{\partial \pi} \int_{-\infty}^{\infty} \frac{\Theta(\nu)}{(\bar{u} - \nu / k)} d\nu \end{aligned} \right\}, \tag{22}$$

$\mathcal{P}$  denoting the principal value.

If the eddy motions have a time-independent component, the spectra and cospectra have delta function peaks at zero frequency. The eddies associated with the time-independent component of the motion satisfy (12) directly with  $\nu=0$ , and using (8) rather than (10) for the definition of the eddy statistics.

### 5. Nonlinear forcing of the zonal flow by planetary waves

The equations for forced meridional motion (3) are now considered, taking into account the kinematic constraint (19). Using (19) to eliminate the momentum transport in (3a) and the eddy heat transport in (3b), the resulting equations for  $\bar{w}$  and  $\bar{v}$  are

$$\mathcal{L}\bar{w} = -\mathcal{L} \left[ (R/S) \frac{\partial}{\partial y} \langle \bar{v}'T' \rangle \right] + (f_0/S) \frac{\partial^2}{\partial y \partial z} (\sigma_1 + \sigma_2), \tag{23a}$$

$$\hat{\mathcal{L}}\bar{v} = \hat{\mathcal{L}} \left[ f_0^{-1} \frac{\partial}{\partial y} \langle \bar{u}'v' \rangle \right] + f_0^{-1} \frac{\partial^2}{\partial y^2} (\sigma_1 + \sigma_2), \tag{23b}$$

where  $\mathcal{L}$  and  $\hat{\mathcal{L}}$  are the elliptic operators

$$\left. \begin{aligned} \mathcal{L} &= (f_0^2/S) \left( -\frac{\partial}{\partial z} \rho_s^{-1} \frac{\partial}{\partial z} - \rho_s \right) + \frac{\partial^2}{\partial y^2} \\ \hat{\mathcal{L}} &= f_0^2 \rho_s^{-1} \frac{\partial}{\partial z} \left( \frac{\rho_s}{S} \right) \frac{\partial}{\partial z} + \frac{\partial^2}{\partial y^2} \end{aligned} \right\}, \tag{24}$$

with  $\sigma_1$  and  $\sigma_2$  defined by (20) and (21), respectively.

The formal solutions to (23) are

$$\bar{w} = - (R/S) \frac{\partial}{\partial y} \langle \bar{v}'T' \rangle + \mathcal{L}^{-1} \left[ (f_0/S) \frac{\partial^2}{\partial y \partial z} (\sigma_1 + \sigma_2) \right], \tag{25a}$$

$$\bar{v} = f_0^{-1} \frac{\partial}{\partial y} \langle \bar{u}'v' \rangle + f_0^{-1} \hat{\mathcal{L}}^{-1} \left[ \frac{\partial^2}{\partial y^2} (\sigma_1 + \sigma_2) \right], \tag{25b}$$

where  $\mathcal{L}^{-1}$  and  $\hat{\mathcal{L}}^{-1}$  are inverses of the operators  $\mathcal{L}$  and  $\hat{\mathcal{L}}$ . We assume a region far enough from all boundaries that the zonal wind and temperature are essentially independent of their values at any boundary. The geometry then can be considered unbounded for the purpose of inverting the operators defined by (24). If we assume for simplicity that  $S(z)$  is constant, then the inverse operation  $\mathcal{L}^{-1}X$  is given by

$$\mathcal{L}^{-1}X(y,z) = \rho_s^{-\frac{1}{2}}(z) \int_{-\infty}^{\infty} \int G(y-y', z-z') \times [\rho_s(z')S/f_0^2]^{\frac{1}{2}} X(y',z') dy' dz'. \tag{26}$$

The Green's function  $G(y,z)$  is

$$G(y,z) = (2\pi)^{-1} K_0 \left[ \frac{1}{2} (y^2 + Sz^2 / f_0^2)^{\frac{1}{2}} \right],$$

where  $K_0(\zeta)$  is the usual Bessel function with imaginary argument that decays exponentially away from  $\zeta=0$ , and asymptotes to  $-\log \zeta$  for small  $\zeta$ .

In order to determine the zonal flow forcing given by the solutions for  $\bar{w}$  and  $\bar{v}$ , we substitute (25b) and (25a) into (2a) and (2b), respectively, giving

$$\frac{\partial \bar{u}}{\partial t} + d\bar{u} = \hat{\mathcal{L}}^{-1} \left[ \frac{\partial^2}{\partial y^2} (\sigma_1 + \sigma_2) \right], \tag{27}$$

$$-R \left[ \frac{\partial \bar{T}}{\partial t} + \alpha (\bar{T} - \bar{T}_e) \right] = S \mathcal{L}^{-1} \left[ (f_0/S) \frac{\partial^2}{\partial y \partial z} (\sigma_1 + \sigma_2) \right]. \tag{28}$$

Eqs. (27) and (28) describe the forcing of zonal winds and temperatures in terms of the *singular line interaction term*  $\sigma_1$  and the *Newtonian cooling interaction term*  $\sigma_2$ . When these interactions are absent ( $\sigma_1 = \sigma_2 = 0$ ), the *nonlinear forcing of the zonal flow vanishes to second order*. This statement generalizes the theorem on perturbation forcing of a zonal flow given by Charney and Drazin (1961).

### 6. Interpretation as potential vorticity mixing by planetary waves

The forcing of zonal flow by singular line interaction obtained in the previous section is simply interpreted in terms of potential vorticity transport. For this purpose adiabatic motions are assumed. Letting  $q$  be any conservative quantity,

$$\frac{dq}{dt} = 0. \tag{29}$$

For small amplitude eddy motions,

$$\frac{\delta q'}{\delta t} + v' \frac{\partial \bar{q}}{\partial y} + w' \frac{\partial \bar{q}}{\partial z} = 0, \tag{30}$$

$q'$  being eddy  $q$  and  $\bar{q}$  a zonal average  $q$ . Assuming geostrophic scaling, the term  $w'\partial\bar{q}/\partial z$  may be neglected to a first approximation. Also, if  $\eta'$  is the small amplitude displacement of a particle from its initial position,

$$v' \simeq \frac{\delta\eta'}{\delta t} \tag{31}$$

Consequently, neglecting the vertical transport term, (30) can be integrated to the relation

$$q' = -\eta' \frac{\partial\bar{q}}{\partial y} \tag{32}$$

expressing the fact that at any time a parcel retains its initial value of  $q' + \bar{q}$ .

Combining (31) and (32) to evaluate  $\overline{v'q'}$ , the transport of  $q$  is (Taylor, 1915; Bretherton, 1966a)

$$\overline{v'q'} = -\frac{\partial\bar{q}}{\partial y} \frac{1}{2} \frac{\delta}{\delta t} \overline{\eta'^2} \tag{33}$$

The argument up to this point is but a variation of Taylor's well-known derivation, the transport of  $q$  being given by the product of the mean gradient of  $q$  and the rate of particle dispersion. Bretherton (1966b) gives an interesting interpretation of baroclinic instability in terms of the relation (33) as applied to potential vorticity transport.

Now we relate the dispersion of particles by random stationary motions to the energy of meridional motion along singular lines. [Bretherton (1966a) derives a similar expression for a single frequency component.] Note that

$$\frac{1}{2} \frac{\delta}{\delta t} \overline{\eta'^2} = \overline{\eta'v'} \tag{34}$$

Again as in (6), disturbances are proportional to  $e^{ikx}$  and, as in (9), are Fourier analyzed in time. For elementary Fourier components proportional to  $e^{-iv't}$  it follows from (31) that  $\eta' = iv' / (\nu - \bar{u}k)$ . Consequently, further derivation along the lines used to obtain (18) gives

$$\overline{\eta'v'} = \frac{1}{2\pi} \int_{-\infty}^{\infty} N(\nu) d\nu \tag{35}$$

where

$$N(\nu) = \pi V(\nu) \delta(\nu - k\bar{u}) \tag{36}$$

Consequently,

$$\frac{1}{2} \left\langle \frac{\delta\eta'^2}{\delta t} \right\rangle = \frac{1}{2} V(k\bar{u}) \tag{37}$$

the rate of particle dispersion being proportional to the variance of meridional eddy motion with frequency equal to  $k\bar{u}$ , i.e., the critical frequency for a singular

line. The deduced relation (37) together with (33) gives the meridional transport by eddies of any conservative quantity  $q$  as determined by random, source-free planetary waves.

For the remainder of our discussion,  $q$  represents the quasi-geostrophic potential vorticity. The equation for zonal potential vorticity  $\bar{q}$  in the absence of dissipation is written as

$$-\frac{\partial^2\bar{q}}{\partial t\partial y} = \frac{\partial^2}{\partial y^2} \overline{v'q'} \tag{38}$$

where  $-\partial\bar{q}/\partial y$  is related to  $\bar{u}$  by

$$-\partial\bar{q}/\partial y = \hat{\mathcal{L}}\bar{u} \tag{39}$$

with  $\hat{\mathcal{L}}$  given by (24). In the absence of the singular line interaction term given by the right-hand side of (37), it follows from (33) and (34) that the potential vorticity transport vanishes, i.e.,  $\overline{v'q'} = 0$ , and hence from (38) and (39),  $\hat{\mathcal{L}}\bar{u}$  does not change in time; inversion of  $\hat{\mathcal{L}}$  shows that  $\bar{u}$  does not change either. A similar argument shows that for adiabatic motions  $\bar{T}$  cannot change with time when the singular line interaction term vanishes. More elaborate linear equations in  $\bar{u}$ , which include various momentum and heat dissipation processes, may easily be obtained to generalize the left-hand side of (38); likewise, the vorticity transport on the right-hand side of (38) can result from forms of dissipation other than those considered in this study.

To summarize, *it is necessary to transport potential vorticity by the eddies in order to force a zonal flow.* Except at singular lines, an adiabatic wave motion superimposed on a zonal flow will merely slosh back and forth the conservative quantity with no mean transport occurring. However, where the phase speed of the wave equals the zonal flow velocity, the distance of a particle from its origin will monotonically increase according to the linear theory (37), and dispersion occurs. A brief discussion of this effect was given by Rossby (1942). Lighthill's (1962) presentation of Miles' water wave theory indicates the extension of singular line dispersion to finite amplitude waves.

Without going into any details it should be noted that a relation for the rate of particle dispersion can be obtained in spherical coordinates and generalized to include vertical transports and summation over wavenumber, giving a theoretical basis for the calculation of turbulent diffusion by continuous movement (Taylor, 1921) of small amplitude eddy motions in a zonal flow. The theory would apply, for example, to the calculation of exchange coefficients in terms of atmospheric wind statistics in the stratosphere for the purpose of determining the transport of ozone or radioactive debris. Such transport is, evidently, related to the presence of singular line interactions between planetary waves and the stratospheric zonal flow.

The interaction between eddies and the zonal flow depending on the Newtonian cooling term indicates

more generally that diabatic or dissipative effects, relaxing the constraint of potential vorticity conservation in the eddy equations, allows the zonal flow to be driven by eddies even in the absence of singular line interaction.

**7. Maintenance of zonal flows**

The question is often raised as to how observed zonal flow configurations are *maintained* against the various processes that would tend to dissipate them. Considering the atmosphere as a whole, one can argue that since dissipative processes usually destroy energy of the zonal flow, the generation of energy of the zonal flow, for example by upgradient momentum transports, acts to maintain the flow. This sort of reasoning, while giving some understanding of the overall motions of the troposphere, which contains most of the mass of the atmosphere, may not provide a very satisfactory basis for consideration of an open subsystem such as the stratosphere. The discussion of this section indicates how difficulties may arise in interpreting the energetics of vertically propagating planetary waves interacting with a zonal flow.

Since mean meridional motions are usually much weaker and more difficult to observe than the eddies, open subsystems are often characterized by the direction of eddy transfers from one kind of energy to another. Consider the following conceptual model for the dynamics of such an open region between two constant pressure surfaces. Assuming a  $\beta$ -plane geometry, we take the wind in the region under consideration to be westerly, depending only on the pressure coordinate and having a single maximum value somewhere between the two pressure levels. Examine now adiabatic, source-free, stationary planetary waves, incident from below, propagating through the region. Assuming the linear theory to apply, we can use expressions derived by Eliassen and Palm (1961) to describe the energetic interaction between the various kinds of eddy and zonal flow energies.

The wind maximum divides the region into two subregions (Fig. 1): the lower region, denoted I, where the wind increases with height, and the upper region, denoted II, where the wind decreases with height. Assuming for simplicity a constant value of  $s$  and  $\rho_0$ , we have from Eliassen and Palm that a) the northward heat transport by the planetary wave is independent of height [a specialization of (12) for the stated conditions], and b) the upward eddy energy flux by the wave is proportional to the product of  $\bar{u}(z)$  and the northward heat flux. Consequently, there is a divergence of eddy energy flux  $\tau_E$  in I, a convergence in II.

In order to maintain the eddy kinetic energy  $K_E$  in I, there must be a gain from eddy available potential energy  $A_E$  equal to the loss of  $K_E$  necessary to give the divergence of eddy energy flux. In turn the  $A_E$  must be maintained by an equal conversion from zonal available

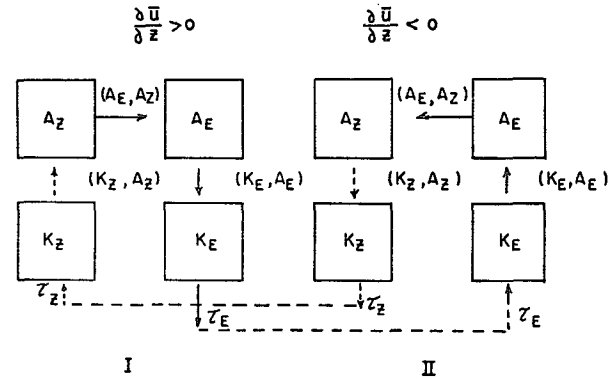


FIG. 1. Depiction of the energetics of a vertically propagating adiabatic Rossby wave in a vertical shear flow and in the absence of singular line interaction. The energy cycles between the eddies and mean flow without any net transfer to or from the mean flow.

potential energy  $A_Z$ . The equality of all these conversions is evident from the assumption that the eddy amplitudes are independent of time, and is easily formally demonstrated from the relevant equations.

Now referring to the Charney-Drazin theorem that no change of the zonal flow occurs, we also see that the  $A_Z$  must be maintained by an equal conversion from zonal kinetic energy  $K_Z$ , which in turn must gain the same amount of energy by convergence of the mean flow energy flux  $\tau_z$ . Thus, for the hypothetical example considered, energy flows in a loop around the baroclinic part of the usual atmospheric general circulation energy cycle, as indicated in Fig. 1, and there is no net loss or gain of any kind of energy. The divergence of eddy energy flux is compensated by an equal convergence of mean motion energy flux so that the planetary waves, while forcing meridional circulations, do not actually transfer energy from or to the mean zonal flow. Such transfer of energy can only occur in the presence of eddy singular line interaction or dissipation. In region II, the reverse cycle occurs with the same ultimate conclusion retained.

On the basis of eddy heat transports or conversions between  $K_E$  and  $A_E$ , the lower region I would be classified as a thermodynamically direct *heat engine* region, and II a thermodynamically indirect *refrigerator* region. Such classification is clearly misleading if interpreted as signifying the the planetary waves are generated from the zonal flow in the heat engine region and act to maintain the zonal flow in the refrigerator region.

A more satisfactory approach to the concept of maintenance of zonal wind systems in an open region is based on the idea that in the absence of large-scale eddies the zonal flow would relax back to some symmetric equilibrium state depending only on the processes of radiative-photochemical heating and on possible eddy viscosity and heat conduction by small-scale motions. Departures from such a state and its concomitant

distribution of potential vorticity will be maintained by large-scale eddy transports.

The present study shows that small amplitude, transient planetary waves propagating through an otherwise equilibrium region and with accompanying singular line interactions will redistribute the potential vorticity distribution by down-gradient transport. For example, one would expect that in the absence of planetary waves the mean westerly zonal winds of the winter stratosphere would resemble closely, except for sign, the easterly summer stratospheric zonal winds. Such antisymmetry of stratospheric winds is indicated by the models of Leovy (1964). In the upper stratosphere, except possibly in high latitudes, the mean potential vorticity gradient, dominated by the planetary vorticity gradient, is positive. Planetary wave mixing should thus give equatorward eddy transport of potential vorticity. Assuming the eddy transport of vorticity is of the same sign as the potential vorticity transport, there should occur a divergence of eddy momentum transport in middle and subtropical latitudes, tending to shift the westerly jet poleward of its equilibrium value. This mixing of potential vorticity by wintertime planetary waves, forcing large departures from the dynamic and thermodynamic equilibrium state, could be the ultimate explanation for the well-known large incursions of the summer easterly stratospheric jet into the subtropics in the winter hemisphere.

The possible importance for zonal flow structure of eddy and zonal-mean dissipation of temperature and horizontal winds should not be overlooked. The outstanding difficulty is the general ignorance of the appropriate forms of dissipation, other than Newtonian cooling, acting in the stratosphere and above. The interaction term depending on Newtonian cooling as given by (22) depends on the power spectra of eddy temperature and the cospectra of eddy geopotential and temperature, but no simple interpretation of this term is available.

While, as indicated by the model discussed above, it is not possible to ascertain from the sign of eddy energy transformations alone whether the eddies are sources or sinks for  $K_z$  or  $A_z$ , it is clear that if energy is being lost by non-eddy processes, the eddy conversions must make up the difference. Thus, assuming only radiative losses are significant, a region can be classified as thermodynamically "direct" or "indirect" according to whether  $A_z$  is created or destroyed by diabatic processes. It is still not possible to say, however, that eddies maintain an indirect region so defined but are themselves generated in a direct region. The zonal flow in either region, if representing a large departure from the flow that would exist in the absence of the eddies, can be considered maintained by the eddies. This interpretation, of course, presupposes the eddies to be generated outside of the region in question. Energetic considerations probably will not resolve the question as

to whether the eddies are internally or externally generated. Externally generated eddies can be identified by their dependence on conditions outside the region considered, and by the fact that they would be absent if the region could be isolated from the rest of the atmosphere by containing walls.

## 8. Concluding remarks

We have shown that the presence of singular lines or eddy diabatic damping is necessary for the forcing of zonal atmospheric motions by small amplitude, quasi-geostrophic eddy motions. No reference is necessary in the analysis to the detailed theory of planetary waves in the neighborhood of singular lines (Dickinson, 1968a) or to Newtonian cooling (Dickinson, 1968b). Such theories can be used for the theoretical calculation of the eddy wave statistics, assumed here known.

Several restrictions in our analysis, introduced primarily for increased lucidity in the derivations, can be removed, if necessary, for further applications of the theory. In particular, the derivation can be generalized to spherical coordinates with variable Coriolis parameter using the approximate model of Dickinson (1968c). The assumptions that the zonal flow  $\bar{u}$  and temperature  $\bar{T}$  are independent of time and that the eddy statistics are stationary in time were introduced to facilitate the Fourier analysis in the time variables. Unless friction and diabatic damping exactly cancel the forcing of the zonal flow, these assumptions cannot be strictly valid and so need further clarification. The analysis can be made rigorous by the introduction of *two time scales*, a short time scale characterizing rapid eddy fluctuations and a long time scale characterizing slow secular changes of the zonal flow and of the eddy statistics. Equivalently, there should be a *spectral gap* between rapid eddy fluctuations and slow fluctuations of the eddy statistics and the zonal flow. Inasmuch as the zonal flow forcing is quadratic in the eddy amplitudes, this spectral gap is insured in the present theory by the assumption of small amplitude eddy motions.

When eddy amplitudes are as large as the zonal flow amplitude, the linearization breaks down. Furthermore, there is no justification for the assumption that the zonal flow evolves on a time scale long compared to that of eddy fluctuations. Since this situation often occurs in the atmosphere, numerical studies of finite amplitude disturbances should be pursued to determine the limitations of the present linear theory.

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