The Thermodynamic Equation in Cumulus Dynamics

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ABSTRACT

Neither the pseudo-adiabatic nor the saturation-adiabatic form of the thermodynamic equation used in meteorological practice is suitable for the study of cumulus dynamics inasmuch as this form ignores the microphysics of condensation and precipitation. A thermodynamic equation has been derived treating the cloud as a mixture of dry air, water vapor, and liquid water distributed into droplets, drops and/or other centers of condensation and evaporation. The equation implicitly includes the effect of fallout of the centers from their parent parcels of air and is explicitly supplemented by an equation of continuity for the centers. A simple way has been indicated for extending the basic equation which has been derived for condensation-evaporation centers of uniform mass (and fall velocity relative to air) to clouds having a population of centers of varying mass.

1. Introduction

The classical moist adiabatic (saturation-adiabatic and pseudo-adiabatic) forms of the thermodynamic equation have long been considered inadequate for explaining the observed thermal structure of cumulus clouds. The temperatures in cumuli have been found to be generally closer to those in the environment than they predict and, following conditions in the environment, the lapse rates inside cumuli were found to lie *between* the dry and moist adiabatic rather than at the predicted moist adiabatic value. Moreover, liquid-water contents measured in cumuli have shown values lower than those predicted by moist adiabatic formulas. These observations were regarded as indicative of mixing of environmental air into the cloud and were so explained by Stommel (1947) who devised the so-called entraining model of the cumulus. It should be noted that Stommel's model took cumulus thermodynamics outside the adiabatic framework and did not question the validity of a basic assumption of the moist adiabatic equation; namely, that air containing water in the condensed phase will always be saturated and, conversely, any vapor in excess of saturation must condense. Later studies in cloud physics and cumulus dynamics have indicated that a reexamination of this saturation assumption is certainly in order.

The first indication of the failure of the saturation assumption was provided by the phenomenon of the unsaturated downdraft observed by the Thunderstorm Project (Byers and Braham, 1949) as a "humidity dip" at the ground under a thunderstorm. Byers and Braham considered the phenomenon rather paradoxical since it is at the epoch of the heaviest rainfall that the surface relative humidity decreases from near saturation to values as low as 60–70%. In other words, the downward flowing air becomes unsaturated as it descends even in the presence of large concentrations of liquid water indicated by the accompanying rain. Analyzing a suggestion of these authors, Das and Subba Rao (1968) showed that this lack of saturation in the downdraft could arise from the inability of water drops carried in them to evaporate sufficiently to maintain saturation. As a concomitant of this picture they found that lapse rates in unsaturated downdrafts would lie between their dry and moist adiabatic values. Similar conclusions have been reached independently by Kamburova and Ludlam (1966).

Another challenge to the saturation assumption has been posed by the computations of Howell (1949), Mordy (1959) and Neiburger and Chien (1960) on the growth of a population of droplets in an updraft carrying condensation nuclei. These authors invariably come up with a supersaturated state in some part of the life of the updraft. A dramatic conceptual consequence of this supersaturation has been discussed by McDonald (1962) who shows that it is possible to have lapse rates in updrafts drastically different from their moist adiabatic values.

The failures of the saturation assumption mentioned above are due to the fact that condensation and evaporation in the atmosphere can occur only through the intervention of *centers*—a general term we shall use to include condensation nuclei, water droplets, ice crystals and the precipitation hydrometeors such as water drops, *graupel*, hailstones, etc. Fortunately, some centers are always available for condensation when the air tends to become supersaturated with water vapor and, due to the effectiveness of these centers, any appreciable supersaturation is legitimately ruled out under usual conditions in the atmosphere. However, as Byers and Braham

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(1949) have observed, conditions in cumuli become quite unusual with the development of precipitation and are revealed by a change of the in-cloud motion field from one dominated by updrafts to one by downdrafts. In an attempt to find a mechanism for such a radical change, Das (1964), Takeda (1966) and Srivastava (1967) considered the effect of the kinematics of precipitation hydrometeors on the dynamics of one-dimensional cumulus motions. It should be noted, however, that these authors did not consider any thermodynamic consequence of precipitation formation.

In order to see how the thermodynamics of the cumulus may be affected by the formation of precipitation, we shall follow Byers and Braham and consider the updraft and the downdraft separately. In the case of the downdraft, which we take up first on account of its physical simplicity, we have compressional heating so that the downflowing air would tend to become unsaturated unless additional water vapor is supplied through evaporation from the centers. Assuming (i) that the centers consist of cloud droplets and precipitation drops and (ii) that, in the regions where downdrafts form, it is the relatively small number of precipitation drops that dominate the liquid water field, one would expect after a short while that the droplets will evaporate leaving only the precipitation drops in the downdraft. From this point onward we would thus have conditions similar to those considered by Kamburova and Ludlam (1966) and Das and Subba Rao (1968), involving an unsaturated downdraft. It is clear, therefore, that in the circumstances pictured above an unsaturated downdraft will result as a consequence of the liquid water field being dominated by precipitation hydrometeors.

In the case of the updraft the basic process is cooling due to expansion which in saturated air will lead to supersaturation unless there is sufficient condensation on centers. In the early life of the updraft and especially near the cloud base one expects to find an adequate number (and activity) of centers so that condensation can take place approximately at the moist adiabatic rate. If, however, the centers which have already been activated fail to remove the excess vapor, there will be supersaturation. This would activate fresh centers and one would assume this process to go on until even the Aitken nuclei are activated. In this physical sequence there is no failure of the moist adiabatic process except for the effect described by McDonald (1962), which one can possibly assume to be local and temporary. When condensation has proceeded long enough to produce droplets whose radii are about 10μ, the smaller nuclei are effectively removed by Brownian and turbulent coagulation with them (Greenfield, 1957). But the moist adiabatic process is still likely to prevail since there will be an adequate number of droplets acting as centers of condensation. However, when droplets grow to the extent where gravitational coagulation becomes possible, conditions are likely to change considerably. Thus, it has been demonstrated by Golovin (1963a) that under these circumstances the supersaturation in updrafts will have much greater magnitudes than if there were no such coagulation. Apparently this is due to the removal of smaller droplets by larger ones so that the total concentration of centers becomes inadequate for removing the supersaturation.

The picture of what can happen when precipitation forms in a cumulus updraft is easily conceived by an extrapolation of the physical sequence described in the last paragraph. When available in large enough numbers the precipitation elements are likely to scavenge the droplets to an appreciable extent and, with the nuclei already removed by coagulation with the droplets, to dominate the field of centers of condensation. If this situation does develop, the concentration of centers is likely to be inadequate for condensation to proceed at the moist adiabatic rate so that considerable supersaturation may result. At the present moment we do not have any observational evidence of such supersaturation but the author is not aware of any significant effort directed to gathering such evidence. On the other hand, the physical picture described above renders such supersaturation a distinct possibility.

Now, since the thermodynamics of moist air is dominated by the release of latent heat in condensation (and absorption of latent heat in evaporation), the physical process described above can obviously be of great importance in the dynamical life of the cumulus. Indeed, there appears to be a need for a new thermodynamic equation, applicable to cumulus dynamics, which takes cognizance of the above processes. Such a need seems even greater today on account of the advances made by Ogura (1963), Orville (1965) and Arnason *et al.* (1968) in the numerical modelling of moist convection, since a more realistic thermodynamic equation can give a truer picture of the cumulus life cycle.

The aim of the present paper is to indicate how the thermodynamic equation must be changed to incorporate the effect of the microphysics of condensation and evaporation. It is not intended to derive a complete form of the thermodynamic equation valid for all the spatial and temporal phases of the cumulus. Thus, neither the effects of the latent heat of freezing nor those of eddy mixing will be included in the equation. On the other hand, the centers of condensation-evaporation will be assumed to consist entirely of droplets (and drops), the dry nuclei being treated as droplets of no mass. Further, the derivation of the basic equation will assume the mass (and size) of the centers to be a function only of height and time so that at any time centers of only one size will be found at a given height. The limitation imposed by this assumption is rather severe but, as shown later, the results of the elementary analysis can be extended to and generalized for varying sizes of the centers.

2. The continuity equations for water substances

The most important trait of the thermodynamic equation for moist convection lies in the release (or absorption) of latent heat in phase changes of water substances, especially in the condensation of water vapor into liquid water (or vice versa). Thus, an equation of continuity for water substances forms an essential adjunct to the thermodynamic equation being sought.

Considering the cloud to be a mixture of dry air, water vapor and liquid water (dispersed in the form of moist nuclei, droplets and drops), we let ρ_d, ρ_v ρ_l be their respective masses per unit volume of cloud air. If $d\tau$ is an elementary volume of cloud air, we can then write the principle of conservation of mass as

$$\frac{\delta}{\delta t}\int_R(\rho_d+\rho_v+\rho_l)d\tau=0,$$

where the integral is taken over a region R of the cloud. Here the derivative $\delta/\delta t$ has only a formal sense since the boundaries of the (ρ_d,ρ_v) region[2] and the ρ_l region move with different velocities, $\mathbf{V}$ and $\mathbf{V}'$ say, where $\mathbf{V}'=\mathbf{V}+\mathbf{V}_l$ and $\mathbf{V}_l$ is the velocity of the ρ_l region relative to the (ρ_d,ρ_v) region. Assuming at a point of space and time that ρ_l is composed of droplets or drops of a single size so that $\mathbf{V}_l$ is a unique function of space and time, we can reduce the last equation to the differential form

$$\frac{\partial}{\partial t}(\rho_d+\rho_v+\rho_l)+\nabla\cdot[(\rho_d+\rho_v)\mathbf{V}+\rho_l(\mathbf{V}+\mathbf{V}_l)]=0. \qquad (1)$$

Now, since ρ_l is dispersed in the cloud in the form of centers of different sizes, one should really write a more general form of (1) as

$$\frac{\partial}{\partial t}(\rho_d+\rho_v+\rho_l)+\nabla\cdot\left[(\rho_d+\rho_v+\rho_l)\mathbf{V}+\int_0^\infty m_r n_r \mathbf{V}_r dr\right]=0, \qquad (1a)$$

where m_r and $n_r dr$ are, respectively, the mass and the number concentration of centers of radii lying between r and $r+dr$, and $\mathbf{V}_r$ is the velocity of separation of the centers of radius r from the gaseous phase. However, as a first step we shall concentrate on understanding the significance of (1).

Eq. (1) can be split into two parts since the mass of the dry air is conserved independently of the water substances. Thus, we can write

$$\frac{\partial\rho_d}{\partial t}+\nabla\cdot(\rho_d\mathbf{V})=0, \qquad (2)$$

[2] In a mixed fluid system with the possibility of the fluids separating from each other, the fluid region occupied by a given mass of one component, although coinciding with the region occupied by another component at a particular instant, will not generally continue to do so at a later instant.

for the dry-air component of the cloud. This equation has the usual significance of the equation of continuity, but in the present context it serves an additional purpose. In writing (2) in the customary form

$$\frac{d\rho_d}{dt}+\rho_d\nabla\cdot\mathbf{V}=0, \qquad (3)$$

we define the substantial time derivative

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{V}\cdot\nabla, \qquad (4)$$

which is referred to the gaseous phase. This is the basic substantial derivative of the system in which the gaseous and liquid components move with different velocities and have different expressions for the substantial derivative.

On account of the possibility of phase changes between water vapor and liquid water the two must always be treated together. The equation for the change of water substances is obtained by subtracting (2) from (1) so that we have

$$\frac{\partial}{\partial t}(\rho_v+\rho_l)+\nabla\cdot[(\rho_v+\rho_l)\mathbf{V}+\rho_l\mathbf{V}_l]=0.$$

Together with (4) this becomes

$$\frac{d}{dt}(\rho_v+\rho_l)+(\rho_v+\rho_l)\nabla\cdot\mathbf{V}+\nabla\cdot(\rho_l\mathbf{V}_l)=0. \qquad (5)$$

Using (3), (5) can then be written as

$$\frac{d}{dt}(\xi_v+\xi_l)=-\frac{1}{\rho_d}\nabla\cdot(\rho_l\mathbf{V}_l), \qquad (6)$$

where $\xi_v(\equiv\rho_v/\rho_d)$ and $\xi_l(\equiv\rho_l/\rho_d)$ are, respectively, the mixing ratios of water vapor and liquid water in the cloud. This form of the equation of continuity for water substances in a nondissipative atmosphere has been discussed by Das (1963).

3. The thermodynamic equation

Following Das (1963) we write the thermodynamic equation as a conservation of entropy principle in the form

$$\frac{\delta}{\delta t}\int_R(\rho_d\varphi_d+\rho_v\varphi_v+\rho_l\varphi_l)d\tau=0, \qquad (7)$$

where φ_d, φ_v and φ_l are, respectively, the entropy of a unit mass of dry air, water vapor and liquid water. Remembering again that liquid water moves with a velocity $\mathbf{V}+\mathbf{V}_l$, while air moves with velocity $\mathbf{V}$, one can reduce (7) to the differential form

$$\frac{d}{dt}(\rho_d\varphi_d+\rho_v\varphi_v+\rho_l\varphi_l)+(\rho_d\varphi_d+\rho_v\varphi_v+\rho_l\varphi_l)\nabla\cdot\mathbf{V}+\nabla\cdot(\rho_l\varphi_l\mathbf{V}_l)=0. \qquad (8)$$

Referring the entropies to a suitable basic state and remembering that we restrict our consideration to vapor-liquid transition alone, we shall write the following expressions for the entropies:

$$\varphi_d(T)=c_p\ln\theta_d, \tag{9}$$

$$\varphi_v(T)=\varphi_l(T)+\frac{L}{T}, \tag{10}$$

$$\varphi_l(T)=c\ln T, \tag{11}$$

where T is the temperature of the mixture on the Kelvin scale, θ_d is the potential temperature of the dry air, again on the Kelvin scale, c and c_p are, respectively, the specific heat of water and the specific heat of dry air at constant pressure, and L is the latent heat of vaporization of water. Using (3), (6) and (10), (8) can be written as

$$\frac{d\varphi_d}{dt}+\xi_v\frac{d\varphi_v}{dt}+\xi_l\frac{d\varphi_l}{dt}+\frac{L}{T}\frac{d\xi_v}{dt}+\xi_l\mathbf{V}_l\cdot\nabla\varphi_l=0. \tag{12}$$

By order-of-magnitude arguments (see Appendix) this can be simplified to

$$\frac{d\varphi_d}{dt}+\frac{L}{T}\frac{d\xi_v}{dt}=0. \tag{12a}$$

This would be the same as the basic equation for the pseudo-adiabatic process if we replaced $d\xi_v/dt$ by $d\xi_{vs}/dt$ where ξ_{vs} is the saturation mixing ratio corresponding to the temperature and pressure of the parcel. However, the mixing ratio is determined by the microphysics of condensation on and evaporation from the centers; thus (12a) should give a better representation of the state of affairs in a cloud than the traditional pseudo-adiabatic equation, provided $d\xi_v/dt$ is properly related to the physics of phase change of the water substances.

From (6) we have

$$\frac{d\xi_v}{dt}=-\frac{d\xi_l}{dt}-\frac{1}{\rho_d}\nabla\cdot(\rho_l\mathbf{V}_l). \tag{13}$$

We may recall that this assumes the liquid water at any particular level to be composed of drops (or droplets) of the same mass. As already remarked, the drops and droplets are only special classes of condensation centers including the condensation nuclei. If we treat the dry condensation nuclei as droplets of zero liquid mass and vanishing $\mathbf{V}_l$, we can generalize the picture of the cloud as containing centers of liquid mass m with a concentration of n centers per unit volume. Then

$$\xi_l=nm/\rho_d,$$

or, if we want to treat the concentration of the centers as the number mixed in a unit mass of dry air,

$$\xi_l=Nm, \tag{14}$$

where $N(\equiv n/\rho_d)$ is the number of centers contained in a unit mass of dry air. Differentiating (14) we have

$$\frac{d\xi_l}{dt}=m\frac{dN}{dt}+N\frac{dm}{dt}, \tag{15}$$

which is to be supplemented by expressions for dm/dt and dN/dt.

The derivative dm/dt does not represent all the mass change of a center due to condensation or evaporation but only the part of the change which will be observed by an observer moving with the gaseous parcel. On the other hand, the actual change of mass due to condensation or evaporation is really the "substantial change" of the mass of the center, We denote this substantial derivative of m as Dm/Dt and relate this to dm/dt by

$$\frac{Dm}{Dt}=\frac{dm}{dt}+\mathbf{V}_l\cdot\nabla m. \tag{16}$$

It is to be noted that the physical formulae for mass changes due to evaporation and condensation apply directly to Dm/Dt so that in actual applications of (16) one should determine dm/dt as $Dm/Dt-\mathbf{V}_l\cdot\Delta m$.

To find an expression for dN/dt we use a continuity equation for n. A simple form of the continuity equation results if we assume that there is no coagulation or splitting of the centers. This assumption is obviously unrealistic but is consistent with our earlier assumption of m being a function only of height and time. Under these assumptions the continuity equation can be written as

$$\frac{\partial n}{\partial t}+\nabla\cdot[n(\mathbf{V}+\mathbf{V}_l)]=0. \tag{17}$$

Substituting $n=\rho_dN$ in (17), we have

$$N\left[\frac{\partial\rho_d}{\partial t}+\nabla\cdot(\rho_d\mathbf{V})\right]+\rho_d\left(\frac{\partial N}{\partial t}+\mathbf{V}\cdot\nabla N\right)=-\nabla\cdot(n\mathbf{V}_l),$$

which with (2) becomes

$$\frac{dN}{dt}=-\frac{1}{\rho_d}\nabla\cdot(n\mathbf{V}_l). \tag{18}$$

Although (17) and (18) are physically identical, there is a difference of interpretation, which becomes apparent if we put $\mathbf{V}_l=0$ and note that in this case N is a conservative property of the cloud air while n is not. However, when $\mathbf{V}_l\neq0$, the interpretation of the conservatism of N becomes somewhat more involved. In order to see this let us rewrite (18) as

$$\frac{dN}{dt}+\nabla\cdot(N\mathbf{V}_l)+N\mathbf{V}_l\cdot\left(\frac{1}{\rho_d}\nabla\rho_d\right)=0. \tag{18a}$$

Now, the most important variation of ρ_d is in the vertical. Denoting the vertical component of $\mathbf{V}_l$ by $-V_T$ where V_T is the terminal velocity of the center, we have, approximately,

$$\mathbf{V}_l\cdot\nabla(\ln\rho_d)=-V_T\frac{\partial(\ln\rho_d)}{\partial z}=\sigma V_T,$$

where we have written $\sigma=-\partial(\ln\rho_d)/\partial z$. This last expression reduces (18a) to

$$\frac{dN}{dt}+\nabla\cdot(N\mathbf{V}_l)+\sigma N V_T=0, \tag{19}$$

in which the second term is the outflow of N through the boundary of the parcel to which d/dt refers. The third term appears to compensate for the fact that the outflow term for N includes a stretching, mostly in the vertical, of the gaseous phase entering into the definition of N.

It is noteworthy in (19) that V_T and $\mathbf{V}_l$ appear simultaneously. As a matter of fact $\mathbf{V}_l$ can be approximated by its vertical component in all cases except when the Brownian motion of the centers is comparable with the Stokes motion or when the horizontal air motions are fluctuating (on account of turbulence, for example) too fast for the falling center to attain the horizontal velocity of the air. Since the importance of these phenomena in cumulus dynamics is not yet well established, one can approximate (18) by

$$\frac{dN}{dt}=\frac{1}{\rho_d}\frac{\partial}{\partial z}(nV_T), \tag{19a}$$

and (19) by

$$\frac{dN}{dt}=\frac{\partial}{\partial z}(NV_T)-\sigma NV_T, \tag{19b}$$

without serious error. Since the quantity σ is of the order of 0.1 km^{-1}, the second term on the right-hand side of (19b) is likely to be small compared to the first when the clouds have a small vertical depth. The special forms (19a) and (19b) are not required for the following development.

If in (15) we substitute for dm/dt from (16) and for dN/dt from (18), we get

$$\frac{d\xi_l}{dt}=-\frac{m}{\rho_d}\nabla\cdot(n\nabla_l)+N\left(\frac{Dm}{Dt}-\mathbf{V}_l\cdot\nabla_m\right)$$
$$=-\frac{1}{\rho_d}\nabla\cdot(nm\mathbf{V}_l)+N\frac{Dm}{Dt},$$

or, since $\rho_l=nm$,

$$\frac{d\xi_l}{dt}=-\frac{1}{\rho_d}\nabla\cdot(\rho_l\mathbf{V}_l)+N\frac{Dm}{Dt}. \tag{20}$$

I we go back to (13) with (20) we obtain

$$\frac{d\xi_v}{dt}=-N\frac{Dm}{Dt}. \tag{21}$$

Substituting this in (12) we have

$$\frac{d\varphi_d}{dt}+\xi_v\frac{d\varphi_v}{dt}+\xi_l\frac{D\varphi_l}{Dt}=\frac{L}{T}N\frac{Dm}{Dt}, \tag{22}$$

which is the required thermodynamic equation. However, the simplified form corresponding to (12a), i.e.,

$$\frac{d\varphi_d}{dt}=\frac{L}{T}N\frac{Dm}{Dt}, \tag{22a}$$

will adequately replace the pseudo-adiabatic equation under circumstances envisaged by the assumptions made in the above derivation.

As is apparent, the most unpleasant feature of (22) and (22a) is the mixing of two substantial derivatives. On closer study, however, one finds that this is unavoidable in a cloud where the products of condensation are moving with velocities different from that of the air. From the point of practical handling of the equations, however, the presence of Dm/Dt can really be viewed as a source of convenience since, through this, one can directly enter the thermodynamic equation with microphysical formulae for evaporation from (or condensation on) the centers, such as are given by Mason (1957, p. 109) or Kinzer and Gunn (1951).

4. The case of a steady one-dimensional draft: The quasi-adiabatic cloud lapse rate

The new thermodynamic equations (22) and (22a) eliminate the necessity that the air inside an adiabatic cloud should always be treated as just saturated. As already indicated in Section 1 the centers are unable, in general, to remove all the supersaturation vapor from the updraft and to supply all the vapor required for the saturation of the downdraft. Thus, the heat released (or absorbed) by condensation (or evaporation) in an adiabatic vertical draft (and, for that matter, in all adiabatic motions) will, in general, be smaller than in the saturation-adiabatic process. In order to see what this leads to it will be worthwhile to follow the customary procedure by which the pseudo-adiabatic lapse rate is derived from the pseudo-adiabatic equation, and obtain an analogous lapse rate from the thermodynamic equation (22a).

For a steady one-dimensional convective draft of vertical velocity w one can write $d\varphi_d/dt=w\,d\varphi_d/dz$. Substituting the expression for φ_d from (9) into (22a), using the familiar expression for potential temperature and assuming hydrostatic balance, we have

$$-\frac{dT}{dz}=\frac{g}{c_p}-\frac{NL}{wc_p}\frac{Dm}{Dt}, \tag{23}$$

where g is the acceleration due to gravity. The left-hand side of (23) is a lapse rate, say γ_c, and the first term of the right-hand side is the dry-adiabatic lapse rate γ_d, so that

$$\gamma_c=\gamma_d-(NL/wc_p)\,Dm/Dt. \tag{24}$$

TABLE 1. Pseudo-adiabatic cloud lapse rate γ_c in an updraft of 1 m sec^{-1} having a liquid water content of 10^{-8} gm cm^{-3} divided into droplets of radii r.

Radii of droplets r (μ)	Concentration of droplets (cm^{-3})	Pseudo-adiabatic cloud lapse rate γ_c (°C cm^{-1})	Remarks
1	2387	-1.9×10^{-4}	
2	298	$+2.6\times10^{-5}$	
—	—	3.9×10^{-5}	Moist adiabatic lapse rate
5	19	8.6×10^{-5}	
10	2.4	9.5×10^{-5}	
20	0.3	9.7×10^{-5}	
—	—	9.8×10^{-5}	Dry adiabatic lapse rate

One can give the name of *pseudoadiabatic cloud lapse rate* to γ_c. A more exact expression for the cloud lapse rate can be obtained from (22).

The most striking feature of the new lapse rate γ_c is its variability—it depends on the size, number and activity of the centers of condensation and evaporation, and also on the vertical speed of air (which, of course, should be large enough for the adiabatic assumption to be valid). On account of this variability it will be extremely difficult to predict cloud temperature distributions on the basis of (23) and (24), since for this purpose one needs information on elements which are not routinely measured; in addition the evolution of these elements from the condensation nuclei to the precipitation hydrometeors is not known. Systematic studies of such evolution have been taken up by a number of authors but a clear picture is yet to emerge. In the meantime we can make suitable model assumptions such as those of Das and Subba Rao (1968) or of Kamburova and Ludlam (1966) and study the meaning of the equations developed above.

In order to have a closer look at (24) let us remember that we generally expect condensation in an updraft ($w>0$, $Dm/Dt>0$) and evaporation in the downdraft ($w<0$, $Dm/Dt<0$) and write

$$\gamma_c=\gamma_d-(NL/|w|c_p)|Dm/Dt|, \qquad (24a)$$

so that γ_c is always less than the dry adiabatic lapse rate γ_d which is clearly the upper limit of γ_c. On the other hand one would be tempted to think that the lower limit of γ_c is the pseudo-adiabatic lapse rate. However, as shown by McDonald (1962), the lower limit can be as low as zero and even negative under certain circumstances. In order to see this we compute the values of γ_c under conditions close to those envisaged by McDonald. He considered an updraft of 1 m sec^{-1} containing 350–500 active nuclei cm^{-3} and a relative humidity of 101%. In our case we consider the nuclei to be water droplets of uniform size adding up to a liquid water content of 0.01 gm m^{-3}. Thus, to vary the concentration of the nuclei, we vary the size of the droplets while keeping the liquid water content constant. The results of the computation are shown in Table 1. It will be noticed from this table, with a sufficient number of active centers, that the moist adiabatic limit of the rate of latent heat release can be exceeded considerably as indicated by lapse rates less than the moist adiabatic. These results are in agreement with the conclusions arrived at by McDonald.

In order to see the full quantitative significance of (24a) one has to adopt a model of a vertical draft with different kinds of condensation centers and numerically integrate the equation. The computations of Das and Subba Rao (1968) have been performed with the liquid water content divided into drops of uniform size, the latter then being treated as the only centers of evaporation available in the cloud. It has been found for large drop sizes which correspond to a smaller number of centers that the downdraft has a lapse rate quite close to the dry adiabatic in spite of a large liquid-water content. The lapse rates become smaller for smaller drops, but in these computations the moist adiabatic condition has not been reached on account of restrictions on the lower limit of drop sizes imposed by computational facilities.

The computations of Das and Subba Rao also show that with the same amount of liquid water present the downdraft with the larger drops will be more unsaturated than the one with smaller drops. This was found on integrating the steady-state form of (21) side by side with (24). Since larger drops are responsible for heavier surface rain, an unsaturated condition accompanying a heavy thundershower is perfectly understandable in view of the present thermodynamic equation.

5. Generalization of the thermodynamic equation to a population of condensation centers of varying mass

The picture of a cloud having centers of one size at a given height and time is an oversimplification. The natural clouds have centers of varying sizes simultaneously present everywhere and have the added complication that the centers do not retain their identity as implied in the above development, but may collide and coalesce with one another. Thus, the mass change Dm/Dt of a center will be due to two processes, 1) vapor diffusion and 2) collision-coalescence (coagulation). It is only the former that is of direct thermodynamic relevance and the Dm/Dt of (22) and (22a) should refer to this process only. In other words, if we write $(Dm_r/Dt)_{\rm diff}$ for the diffusive mass changes of the center and $(Dm_r/Dt)_{\rm coag}$ for mass changes due to collision-coalescence, we should separate Dm_r/Dt as

$$\frac{Dm_r}{Dt}=\left(\frac{Dm_r}{Dt}\right)_{\rm diff}+\left(\frac{Dm_r}{Dt}\right)_{\rm coag},$$

and use an integral involving the term $(Dm_r/Dt)_{\rm diff}$ for the $N\ Dm/Dt$ in (22) and (22a). Clearly, this separation of the mass changes which go on simultaneously is only of a formal nature since the size distribution in the population of the centers changes continually with time (and space). Golovin (1963a) has formulated this problem and solved it analytically with special formulae for diffusion and coagulation. However, since his main interest was the change of a droplet spectrum from one centered about a small size to one centered about a larger size, the thermodynamic consequence of such population changes has not been brought out explicitly. The approach given below is directly related to the thermodynamic aspect of the change in the population of centers.

The basis of the following formulation is provided by the recent studies on the coalescence growth of larger droplets in a population (e.g., Twomey, 1966; Warshaw, 1967), which indicate that the changes in the size distribution in a population by collision-coalescence is quite slow at the smaller droplet sizes. On the other hand, it is at these sizes that the diffusion processes are likely to be the most important. Thus, for a short interval of time we may ignore the effect of changes in the size distribution by coalescence and consider only changes by diffusion. Consequently, we can treat the mass changes of the centers in two steps: 1) by diffusion alone (treating changes due to collision-coalescence as inoperative) and 2) by coalescence alone with the diffusion process inactive, and then superpose the two. When the mass changes of centers are by diffusion alone, one can generalize (21) to

$$\frac{d\xi_v}{dt}=-\int_0^\infty N_r\left(\frac{D}{Dt}\right)_r m_r dr, \tag{25}$$

where $N_r=n_r/\rho_d$, n_r is number of centers per unit volume and per unit interval in radii r, $(D/Dt)_r=d/dt+\mathbf{V}_r\cdot\nabla$, $\mathbf{V}_r$ is the velocity of the center of radius r relative to that of the gaseous phase, and m_r the mass of the center of radius r. In the future we shall write D_r for $(D/Dt)_r$ for the sake of conciseness.

As a consequence of (25) the generalization of (22) and (22a) is also straightforward; however, since the development of these formulae depends on the continuity equation for the concentration of centers, we need to know how (18) generalizes to this new situation.

In order to obtain an equation of continuity for N_r let us start from the conservation relation for a region R of the cloud, i.e.,

$$\frac{\delta}{\delta t}\int_R d\tau\int_0^r n_r dr=0,$$

where r is allowed to vary on account of vapor diffusion but no coagulation is envisaged. Since the centers of radius r move with velocity $\mathbf{V}_r$ relative to the air, we obtain, by usual procedures, the relation

$$\frac{d}{dt}\int_0^r N_r dr+\frac{1}{\rho_d}\int_0^r \nabla\cdot(n_r\mathbf{V}_r)dr=0,$$

where, as before, $n_r=N_r\rho_d$. Expanding the first term on the left by Leibnitz's rule, we rewrite the last equation as

$$N_r\frac{dr}{dt}+\int_0^r\frac{dN_r}{dt}dr+\frac{1}{\rho_d}\int_0^r\nabla\cdot(n_r\mathbf{V}_r)dr=0.$$

On differentiating this with respect to r one obtains

$$\frac{dN_r}{dt}=-\frac{1}{\rho_d}\nabla\cdot(n_r\mathbf{V}_r)-\frac{\partial}{\partial r}\left(N_r\frac{dr}{dt}\right) \tag{26}$$

for the equation of continuity for N_r.

In order to arrive at (25) we note that (6) can be generalized to

$$\frac{d\xi_v}{dt}=-\frac{d\xi_l}{dt}-\frac{1}{\rho_d}\int_0^\infty\nabla\cdot(n_r m_r\mathbf{V}_r)dr, \tag{27}$$

since $\rho_l=\int_0^\infty n_r m_r dr$. On the other hand

$$\frac{d\xi_l}{dt}=\frac{d}{dt}\int_0^\infty m_r N_r dr=\int_0^\infty m_r\frac{dN_r}{dt}dr, \tag{28}$$

since in the formulation envisaged by (26) the change in the liquid-water content is due to the expansion of the spectrum alone. On substituting for dN_r/dt from (26), (28) becomes

$$\frac{d\xi_l}{dt}=-\frac{1}{\rho_d}\int_0^\infty\left[m_r\nabla\cdot(n_r\mathbf{V}_r)+m_r\frac{\partial}{\partial r}\left(N_r\frac{dr}{dt}\right)\right]dr$$

$$=-\frac{1}{\rho_d}\int_0^\infty\nabla\cdot(n_r m_r\mathbf{V}_r)dr+\int_0^\infty N_r\mathbf{V}_r\cdot\nabla m_r dr$$

$$-\left[m_r N_r\frac{dr}{dt}\right]_{r=0}^{r=\infty}\quad\int_0^\infty N_r\frac{dr}{dt}\frac{dm_r}{dr}dr.$$

Obviously, $N_\infty=0$, $m_{r=0}=0$ and $(dm_r/dr)(dr/dt)=dm_r/dt$, so that we have

$$\frac{d\xi_l}{dt}=-\frac{1}{\rho_d}\int_0^\infty\nabla\cdot(n_r m_r\mathbf{V}_r)dr+\int_0^\infty N_r\left(\mathbf{V}_r\cdot\nabla m_r+\frac{dm_r}{dt}\right)dr,$$

or

$$\frac{d\xi_l}{dt}=-\frac{1}{\rho_d}\int_0^\infty\nabla\cdot(n_r m_r\mathbf{V}_r)dr+\int_0^\infty N_r D_r m_r dr. \tag{29}$$

Substituting (29) in (27), we have (25). Recalling that (25) is a generalization of (21), we can similarly generalize (22) and (22a) for a population of centers whose masses change by vapor diffusion. This completes our first step.

By way of introducing the second step let us note the interpretations of the terms on the right-hand side of (26). The first term, of course, represents the flux of the centers inward of the parcel on account of the relative motion of the centers and the gaseous phase. The second term arises out of the change of the sizes of centers by vapor diffusion. If we now introduce the effects of coagulation (or splitting) of centers, we shall have to write a third term on the right-hand side. Symbolically, then, (26) will be replaced by an equation of the type

$$\frac{dN_r}{dt}=\left(\frac{dN_r}{dt}\right)_{\text{flux}}+\left(\frac{dN_r}{dt}\right)_{\text{diff}}+\left(\frac{dN_r}{dt}\right)_{\text{coag}},$$

in which we already know the nature of the first two terms on the right-hand side. We want to dispose of the discussion on the third term by stating that its nature has been studied by Golovin (1963b), Twomey (1964, 1966), Berry (1967), Bartlett (1966) and Warshaw (1967), and that we do not intend to repeat their conclusions here. However, this should not be construed to mean that the coagulation term is not important in the thermodynamics of the clouds. In fact, as is clear from our discussion in Section 1, supersaturation in convective clouds is likely to be greater with coagulation of centers than without it, and inasmuch as supersaturation is a quantitative indicator of the failure of the classical moist adiabatic process, coagulation is expected to play an important role in the thermodynamics of the cumuli.

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APPENDIX

Simplification of (12) by an Order-of-Magnitude Argument

Through the use of (10) one can rewrite (12) as

$$\frac{d\varphi_d}{dt}+(\xi_v+\xi_l)\frac{d\varphi_l}{dt}-\xi_v\frac{L}{T}\frac{d}{dt}(\ln T)+\frac{L}{T}\frac{d\xi_v}{dt}+\xi_l\mathbf{V}_l\cdot\nabla\varphi_l=0, \quad \text{(A1)}$$

in which we have assumed that L is a constant. Using the expression (9) for φ_d, rewriting the potential temperature in terms of temperature T and the partial pressure P_d for dry air, introducing (11) in (A1) and rearranging terms, one obtains

$$\left[1+\xi_v\left(\frac{c}{c_p}-\frac{L}{c_pT}\right)+\frac{c}{c_p}\xi_l\right]\frac{d}{dt}(\ln T)+\frac{c}{c_p}\xi_l\mathbf{V}_l\cdot\nabla(\ln T)+\frac{L}{c_pT}\frac{d\xi_v}{dt}=\frac{R}{c_p}\frac{d}{dt}(\ln P_d).$$

The most important implication of this expression arises in the case of vertical motion. If, therefore, one assumes a steady one-dimensional updraft w, writes $\mathbf{V}_l=-\mathbf{k}V_T$ where $\mathbf{k}$ is the unit vector in the vertical and V_T the terminal velocity of the centers, and regards the dry-air component as hydrostatically balanced, the last expression will yield

$$\left[1+\xi_v\left(\frac{c}{c_p}-\frac{L}{c_pT}\right)+\frac{c}{c_p}\xi_l\left(1-\frac{V_T}{w}\right)\right]\frac{dT}{dz}=-\left(\frac{g}{c_p}+\frac{L}{c_p}\frac{d\xi}{dz}\right). \quad \text{(A2)}$$

Clearly, the most elementary content of (A2) is

$$\frac{dT}{dz}=-\frac{g}{c_p},$$

which is the equation for the dry adiabatic lapse rate. The additional terms in (A2) perturb this elementary equation with varying magnitude. For $\xi_v\cong25\times10^{-3}$, $L=600$ cal gm^{-1}, $T\cong300$K, $\xi_l\cong10\times10^{-3}$, and for V_T having the same order of magnitude as w, we have

$$\left|\xi_v\left(\frac{c}{c_p}-\frac{L}{c_pT}\right)\right|\cong0.1, \quad \text{(A3)}$$

$$\left|\frac{c}{c_p}\xi_l\left(1-\frac{V_T}{w}\right)\right|\cong0.04, \quad \text{(A4)}$$

and there is very little chance of these values being exceeded in the atmosphere. Moreover, in most circumstances (A3) and (A4) might make contributions of opposite sign. Thus, the left-hand side of (A2) can hardly differ from dT/dz by more than 10%. On the other hand, in cumuli, especially near the base, ξ_v can easily change by a value of 3×10^{-3} through a height of 1.5 km; in which case we have

$$\left|\frac{L}{c_p}\frac{d\xi_v}{dz}\right|\cong4.8\times10^{-5},$$

as compared to $g/c_p\cong9.8\times10^{-5}$. Thus, the second term on the right-hand side of (A2) can have contributions of the order of 50% (or somewhat more, as is well known) of the dry adiabatic lapse rate. Thus, if we are satisfied with an accuracy of 10%, we can simplify (A2) to

$$\frac{dT}{dz}=-\left(\frac{g}{c_p}+\frac{L}{c_p}\frac{d\xi_v}{dz}\right), \tag{A5}$$

the terms of which can easily be seen to have originated from the first and the fourth term of (12). In other words, in the special case of a steady one-dimensional vertical draft these are the terms which predominate in (12) and there appears to be no reason why they should not do so in a more general dynamical situation. Since these are the terms of (12) which constitute the left-hand side of (12a), the latter can be treated as an order-of-magnitude simplification of (12).

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