Growth, Motion and Concentration of Precipitation Particles in Convective Storms[1]

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ABSTRACT

Analytical solutions for the growth and vertical and horizontal motion of a precipitation particle growing by coalescence with cloud drops are derived under simplified steady-state assumptions. An equation is also developed for the concentration density of a continuous distribution of growing particles.

Assuming that the cloud water content varies linearly with height, and that the fall speed of a drop is proportional to the square root of its diameter, it is shown that the combination of a linearly increasing updraft surmounted by a sharply decreasing one sets a sharp upper limit to the particle size, and sorts the particles horizontally. Particles which spend their entire life in regions of horizontal convergence associated with increasing updraft are packed into a narrower shaft than that in which they originated. Initially smaller particles are carried above into the region of horizontal divergence associated with decreasing updraft and are displaced far to the sides of the cloud core. It is found that when the updraft increases sharply there is a very small range of initial sizes which can grow to fall-out size. These facts are used to suggest that a steady "balance level" (equal reflectivity in rising and falling particles) may be maintained at a height near and below an updraft maximum. Particle size spectra computed from the concentration density equation are continuous and well-behaved for rising, floating and falling particles alike, without necessarily even maximizing for the floating size.

1. Introduction

In the updraft of a convective cloud precipitation particles which have attained a certain minimum size grow predominantly by collisions and coalescence with cloud drops. The process of coalescence growth is of importance in understanding many features of precipitating cumuli, such as the behavior of the first radar echo, the vertical profile of the radar reflectivity factor, the development of a Doppler "balance level" (Atlas, 1966), the formation of zones of high precipitation content, the size distribution of precipitation particles, and dynamic effects of precipitation on the life cycle of cumulus. In this paper we shall mainly discuss phenomena which are of importance in the interpretation of the Doppler balance level, and the possibility of a high concentration of precipitation accumulating aloft in a steady-state convective storm.

The discussion will be facilitated by analytical solutions of the equations of coalescence growth under certain simple assumptions about the cloud properties. An analytical solution for the horizontal motion of the particles is also obtained. The solutions are used to derive two conditions which are believed to be of importance in the development of a balance level; namely, the condition under which precipitation particles do not grow fast enough to fall out of an increasing updraft, and an upper limit to the size of the precipitation particles that may be expected within a model updraft profile. An equation will also be formulated for the concentration density of a continuous distribution of growing particles and calculations will be made of the variations of precipitation particle content within a model convective cloud. The results have an important bearing on theories (Hitschfeld and Douglas, 1963; Sulakvelidze *et al.*, 1965) which postulate high concentration of rainwater aloft to explain the growth of large hail.

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2. Equations

a. Assumptions

We assume that the updraft W and the effective cloud water content (product of the cloud liquid water content M and the collection efficiency E) are linear functions of the height z such that

$$W=W_0+\alpha z, \tag{1}$$

$$ME=\gamma_0+\gamma_1 z, \tag{2}$$

where W_0, α, γ_0 and γ_1 are constants. Conditions within a real cloud may be represented by (1) and (2) by dividing it into layers in each of which the equations are obeyed approximately. The air density ρ is assumed to decrease exponentially with height as

$$\rho=\rho_0 \exp(-\lambda z), \tag{3}$$

where the constant λ is given by

$$\lambda=+\left(\frac{g}{R}-\bar{\gamma}\right)\Big/\bar{T}. \tag{4}$$

g being the acceleration due to gravity, R the specific gas constant of air, $\bar{\gamma}$ the mean lapse rate of virtual temperature, and $\bar{T}$ the mean virtual temperature of the atmospheric column. Eq. (3) is used to compute horizontal motion through the equation of continuity, and it is adequate for this purpose. The updraft, effective cloud water content, and air density are assumed to be constant in any horizontal section within the cylindrically symmetrical cloud. This corresponds to the assumption of "top-hat" profiles sometimes made in theoretical studies of convection. The fall velocity V of a precipitation particle is taken to be proportional to the square root of its diameter D, i.e.,

$$V^2=KD, \tag{5}$$

where K is a constant taken as 2×10^6 cm sec^{-2} in the calculations of Section 3. This equation is a good approximation for hailstones and for raindrops of diameter $\gtrsim 1$ mm. Steady-state conditions are assumed throughout.

We also make the simple assumption that precipitation growth occurs on certain relatively big drops whose size distribution may be prescribed at a certain level. It is assumed that the big drops do not collide among themselves.

b. Growth and vertical motion

Following Langmuir (1948) and Bowen (1950), we adopt the following equations for the growth and vertical motion of the particles:

$$\frac{dD}{dt}=\frac{ME}{2\rho_d}V, \tag{6}$$

$$\frac{dz}{dt}=W-V, \tag{7}$$

where t is the time and ρ_d the density of the growing particle. With the help of (5), Eq. (6) may be written

$$\frac{dV}{dt}=\beta_0+\beta_1 z, \tag{8}$$

where β_0 and β_1 are constants given by

$$\beta_0=\frac{K}{4\rho_d}\gamma_0, \tag{9a}$$

$$\beta_1=\frac{K}{4\rho_d}\gamma_1. \tag{9b}$$

Eq. (8) may be used in place of (6) to describe the particle growth. Differentiating (7) and (8) with respect to time and using (1), we find after some manipulation that

$$\frac{d^2V}{dt^2}-\alpha\frac{dV}{dt}+\beta_1 V=\beta_1 W_0-\alpha\beta_0, \tag{10}$$

$$\frac{d^2z}{dt^2}-\alpha\frac{dz}{dt}+\beta_1 z=-\beta_0. \tag{11}$$

These are ordinary second-order linear differential equations with constant coefficients. Their solutions may be obtained by standard methods and are given in the Appendix.

c. Horizontal motion

The radial coordinate r of the particle is given by

$$\frac{dr}{dt}=u, \tag{12}$$

u being radial velocity of the air, which, assuming cylindrical symmetry, may be found from the continuity equation,

$$\frac{1}{r}\frac{\partial}{\partial r}(\rho r u)+\frac{\partial}{\partial z}(\rho W)=0. \tag{13}$$

This may be integrated to give

$$u=-\frac{r}{2}\frac{d}{\rho dz}(\rho W), \tag{14}$$

taking u as zero along the axis of symmetry. The ordinary derivative has been used on the right to emphasize that ρW is a function of z only. Using (1) and (3), Eq. (14) gives

$$u=\frac{r}{2}[(W_0\lambda-\alpha)+\lambda\alpha z]. \tag{15}$$

We now obtain the horizontal position of the particle by integrating (12) using the above value of u; thus,

$$\ln\left(\frac{r}{r_0}\right)=\tfrac{1}{2}(W_0\lambda-\alpha)t+\frac{\alpha}{2}\lambda\int_0^t z dt, \tag{16}$$

where r_0 is the radial coordinate of the particle at $t=0$, $z=0$. The integral on the right is obtained from (8) in the form

$$\int_0^t z dt=\frac{1}{\beta_1}(V-V_0-\beta_0 t). \tag{17}$$

Substituting in (16), we have finally

$$\ln\left(\frac{r}{r_0}\right)=\frac{1}{2}\left(W_0\lambda-\alpha-\frac{\lambda\alpha\beta_0}{\beta_1}\right)t+\frac{\lambda\alpha}{2\beta_1}(V-V_0). \tag{18}$$

It is seen that the horizontal displacement of a particle is proportional to its initial distance from the cloud axis; in particular, particles on the cloud axis do not undergo any horizontal displacement. Eq. (18) is not applicable when $\beta_1=0$. For this case the solution may be shown to be

$$\ln\left(\frac{r}{r_0}\right)=\tfrac{1}{2}(W_0\lambda-\alpha)t. \qquad (19)$$

For comparison, the displacement of an air particle is given by

$$\left.\begin{aligned} z_a&=\frac{W_0}{\alpha}(e^{\alpha t}-1)\\ \ln\left(\frac{r_a}{r_{a0}}\right)&=\frac{\lambda z}{2}-\tfrac{1}{2}\alpha t \end{aligned}\right\}, \qquad (20)$$

where the subscript a denotes an air particle, and we have taken $z_a=0$, $r_a=r_{a0}$ at $t=0$.

d. Concentration of precipitation particles

The changes in concentration of the precipitation particles during growth are of some importance in considerations of vertical distribution of the radar reflectivity factor in thunderstorms (Donaldson, 1961), and the occurrence of zones of high rain water content which are thought to play an important role in the growth of large hail. The problem of particle concentration has been considered by Wexler (1961) and Marshall (1961). Wexler considered a linear updraft profile and assumed the particle fall speed to be a linear function of the height. His Eq. (4) shows that the particle concentration becomes infinite at the level at which the particle is supported by the updraft. Marshall considered particles of constant fall speed in a varying updraft (with no horizontal motion) and found an infinite concentration when the particle fall speed equalled the updraft.

Following Marshall and Wexler for the moment let us consider a monodisperse distribution of particles of fall speed V_0. Let the particles be distributed at height z_0 over a circle of radius r_0, centered on the cloud axis, with a concentration n_0 (per unit volume). Because of the assumption of steady state we then have

$$n|W-V|r^2=n_0|W_0-V_0|r_0^2, \qquad (21)$$

where the symbols without the subscript refer to any height z. The above equation gives

$$\frac{n}{n_0}=\left|\frac{W_0-V_0}{W-V}\right|\left(\frac{r_0}{r}\right)^2. \qquad (22)$$

Since, as will be seen below, r_0/r generally remains finite, the concentration becomes infinite at the level where the fall speed of the particle is equal to the updraft. Of course, the infinity in concentration is a mathematical artifact which occurs at the height where $W=V$. Nevertheless, this circumstance has prevented the calculation of realistic profiles of concentration and the associated parameters of physical interest such as precipitation content and radar reflectivity.

The infinity arises at a discrete level because we have (previously) considered no size range for the growing particles. In reality we have a continuous distribution of drop sizes, and we should consider the concentration density N, such that N represents the number of particles in unit volume of air per unit fall velocity range of the particles. Let the concentration density of particles of fall speed V_0 be N_0 at height z_0. This is to say that the concentration of particles having fall speeds in the range V_0 to $V_0+\Delta V_0$ is $N_0\Delta V_0$ at height z_0. Then the number of particles in this fall velocity range going up per unit time through a circle of radius r_0 is $N_0\Delta V_0 \times(W_0-V_0)\pi r_0^2$, where W_0 is the updraft at height z_0. On reaching height z, the particle of fall speed V_0 will have attained a fall speed V, while the particle of fall speed $V_0+\Delta V_0$ will have attained a fall speed $V+\Delta V$. Moreover, because of horizontal motion, the particles will now be distributed over a circle of radius r. Hence, the flux of particles at height z is given by $N\Delta V(W-V)\pi r^2$, where N represents the concentration density of particles of fall speed V, and W the updraft at height z. In the steady state the fluxes at heights z_0 and z must be equal, i.e.,

$$N\Delta V|W-V|r^2=N_0\Delta V_0|W_0-V_0|r_0^2, \qquad (23)$$

whence

$$\frac{N}{N_0}=\left|\frac{W_0-V_0}{W-V}\right|\frac{dV_0}{dV}\left(\frac{r_0}{r}\right)^2. \qquad (24)$$

It should be noted that the velocity differentials dV_0 and dV in the derivative on the right of this equation refer to the same group of particles at heights z_0 and z, respectively. We shall illustrate the evaluation of the derivative dV_0/dV in detail for the case of constant updraft and effective cloud liquid water content. Referring to (1), (2) and (9b), we see that in this case $\alpha=\gamma_1=\beta_1=0$; the resulting integration of Eqs. (7) and (8) gives

$$V=V_0+\beta_0 t, \qquad (25)$$

$$z=(W_0-V_0)t-\tfrac{1}{2}\beta_0 t^2, \qquad (26)$$

assuming $V=V_0$, $z=0$ at $t=0$. Eliminating t between the above two equations, we have

$$\beta_0 z=(W_0-V_0)(V-V_0)-\tfrac{1}{2}(V-V_0)^2. \qquad (27)$$

Differentiating this equation at constant z we obtain

$$\frac{dV_0}{dV}=\frac{W_0-V}{W_0-V_0}. \qquad (28)$$

Substituting in (24), and remembering that $W=W_0$, we have

$$\frac{N}{N_0}=\left(\frac{r_0}{r}\right)^2, \qquad (29)$$

which shows that in this case ($\alpha=\gamma_1=\beta_1=0$) all relative changes in concentration density can be ascribed to lateral effects only, the spectral shape remaining unaltered. In particular, the concentration does not become infinite at the size for which $V=W$, and the spectrum is continuous through this size for rising ($V<W$) and falling ($V>W$) drops. What happens is that the spectrum at a higher level is found from that at the initial level by modifying the concentration density by the $(W_0-V_0)/(W-V)$ and $(r_0/r)^2$ factors, and then spreading out the spectrum in accordance with the (dV_0/dV) factor [Eq. (24)]. While $(r_0/r)^2$ is well behaved, $(W_0-V_0)/(W-V)$ becomes very large at the same time that dV_0/dV becomes very small [Eq. (28)] near the level where $V\approx W$. Thus, a very narrow V_0 spectral region becomes very broad when $V\approx W$, and the net effect of all these factors is to produce a smooth and continuous N vs V spectrum without necessarily a pronounced peak at $V=W$.

In the general case V and z may be expressed as functions of V_0 and t by equations analogous to (25) and (26): $V=V(V_0,t)$ and $z=z(V_0,t)$. We then find by application of the rules of differentiation that

$$\frac{dV}{dV_0}=\frac{\partial V}{\partial V_0}-\left(\frac{\partial z}{\partial V_0}\right)\left(\frac{\partial V}{\partial t}\right)\Big/\frac{\partial z}{\partial t}. \tag{30}$$

For the case of linear variations of updraft and effective cloud water content, the functions $V(V_0,t)$ and $z(V_0,t)$ are given by the Appendix equations (A1), (A4) or (A6). Using (30) and (24), we then find that

$$\frac{N}{N_0}=e^{-\alpha t}\left(\frac{r_0}{r}\right)^2, \tag{31}$$

t being, of course, the time of growth from V_0 to V. Since $\alpha=0$ in the case of constant updraft, it is seen that (29) is valid even if the effective cloud water content varies linearly with height. It can, in fact, be shown that (29) is valid for arbitrary variations of the cloud water content provided the updraft is constant.

3. An example

Results of one set of calculations using the above equations are given in this section. The updraft is assumed to increase linearly from 2 m sec^{-1} at cloud base, $z=0$, to a peak value of 20 m sec^{-1} at $z=9$ km, and thereafter to fall sharply to zero at a height of 11 km. The effective cloud water content (ME) is assumed to increase linearly from 0 at cloud base to 3.3 gm m^{-3} at cloud top. The assumed profiles of updraft and cloud water content are first approximations to what is believed to occur inside thunderstorm during the middle phase of its life cycle. Values of other parameters are taken to be as follows: $\rho_d=1$ gm cm^{-3}; $K=2\times10^6$ cm

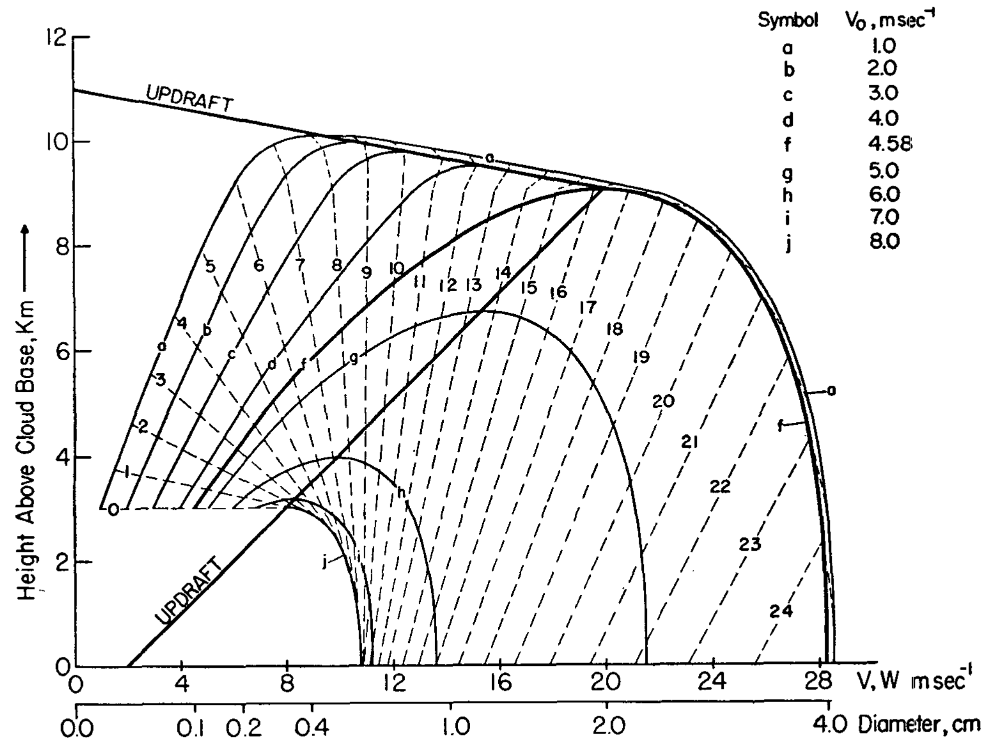

FIG. 1. Updraft W and particle fall speed and diameter vs height above cloud base for various initial particle fall speeds V_0. The precipitation particles are started at a height of 3 km above cloud base. Time in hundreds of seconds is indicated by the numbers on the isochrones shown as dashed lines. The growth curve (thick lines) is for the critical particle which is just balanced at the level where the updraft is maximum.

sec^{-2}, constant with height; and $\lambda = 10^{-6}$ cm^{-1}, corresponding to $\bar{\gamma} = 6C$ km^{-1} and $\bar{T} = 273A$. While in actual fact K varies with height, such a variation may be accounted for in a first approximation, by choosing a suitable value for β_1 [see Eqs. (2) and (9b)].

a. Growth and vertical motion

Fig. 1 shows the growth of precipitation particles on a plot of height vs particle fall speed or diameter. The particles are started at $z=3$ km at $t=0$ sec. It is seen that particles having an initial fall speed >4.58 m sec^{-1} (curves g, h, i and j) grow fast enough to attain a fall speed equal to the updraft before reaching the level of peak updraft (9 km). Particles having initial fall speeds <4.58 m sec^{-1} (curves a, b, c and d) do not grow fast enough and are carried up into the region of decreasing updraft. The particle of initial fall speed equal to 4.58 m sec^{-1} (curve f, drawn thick) has the "critical" fall speed such that it is just supported at the level of the peak updraft. Isochrones are shown by dashed lines which are straight inside the region of increasing updraft, bounded on the right by the limb of the critical curve f corresponding to the phase when the particle is falling. This may be proved with the help of the solution given in the Appendix.

It is seen that the particles which enter the region of decreasing updraft above the peak spend a rather long time there. The growth curves are closely packed in this region, and for the sake of clarity only the one for $V_0 = 1$ m sec^{-1} (curve a) is shown. This particle comes out at $z=9$ km with a fall speed of 21.37 m sec^{-1}. Particles with initial fall speeds lying between 1.0 and 4.58 m sec^{-1} come out of the region of decreasing updraft with fall speeds varying between 21.37 and 20.0 m sec^{-1}. Their fall speeds at $z=0$ km are again very close, and in fact vary between about 28.5 and 28.3 m sec^{-1}. Thus, the distribution of vertical velocity consisting of an increasing updraft surmounted by a region of sharply decreasing updraft imposes a rather sharp limit on the sizes of the particles which fall out of the cloud. We shall see, however, that only a small fraction of the particles larger than the critical size (curve f) fall within the storm core. These are the particles which are on the cloud axis or sufficiently near to it; the majority of the particles which are farther away from the cloud axis are displaced to the sides where they fall outside the cloud core and possibly into the clear air. These facts are of some consequence in Atlas' (1966) interpretation of the balance level phenomenon and are discussed further in the following section.

b. Horizontal motion

Fig. 2 shows the horizontal position of the particles plotted against their vertical position. The initial posi-

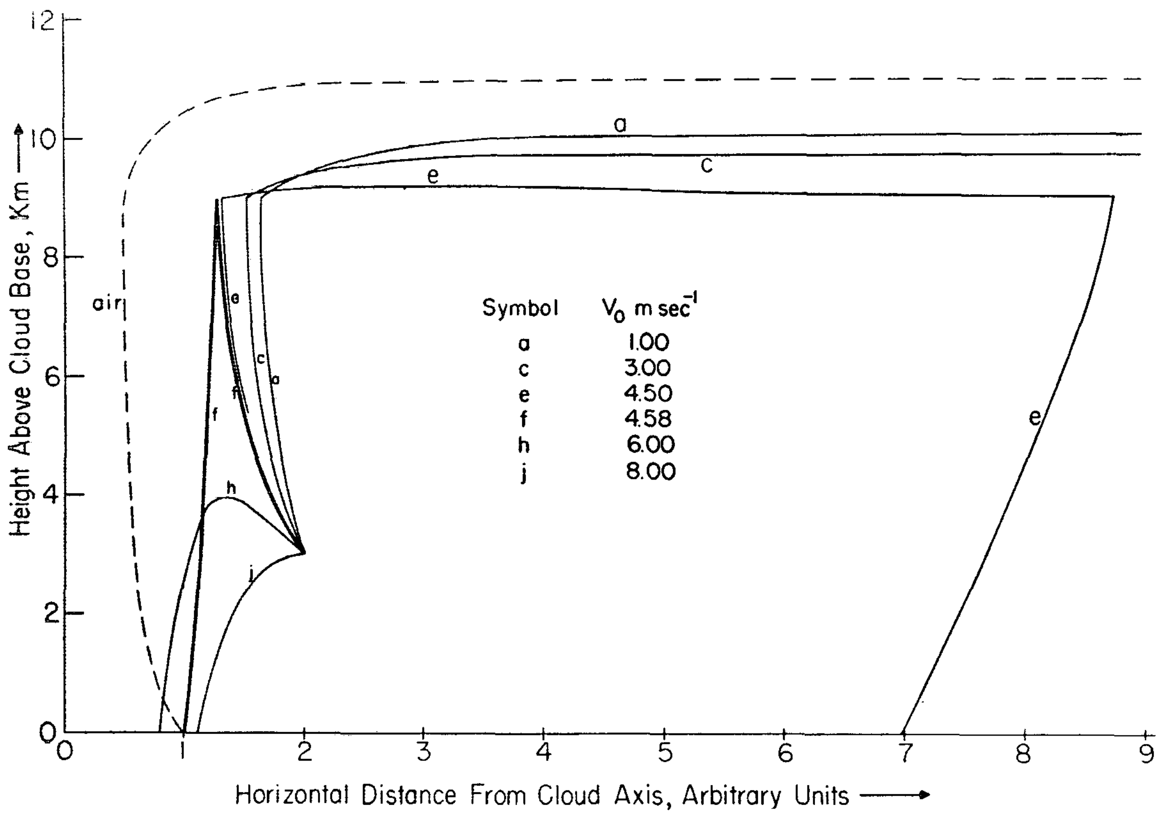

FIG. 2. Trajectory of air (dashed line) and precipitation particles (full lines). The scale of the abscissa is arbitrary. The precipitation particles are assumed to start at $z=3$ km at a distance of 2 units from the cloud axis.

tions of the particles were chosen arbitrarily. For comparison, the dashed curve shows the trajectory of an air particle. The updraft contracts a little near the cloud base, is essentially vertical over its middle portion, and spreads out in an anvil near the top in accordance with the pattern of convergence and divergence implied by the updraft profile.

Particles having initial fall speeds equal to or greater than the critical value of 4.58 m sec^{-1} are seen to be carried inward toward the cloud axis. For example, the particle having an initial fall speed of 6 m sec^{-1} (curve h) at $z=3$ km, and located at a distance of 2 km from the cloud axis, emerges at cloud base at a distance of 0.8 km from the cloud axis. On the other hand, particles having initial fall speeds less than the critical fall speed, which enter the region of decreasing updraft, are carried away horizontally to large distances from the cloud axis. For example, curve e having an initial particle fall speed 4.5 m sec^{-1} (differing from the "critical" value by only 8 cm sec^{-1}), starting at a distance of 2 km from the cloud axis at $z=3$ km, enters the region of decreasing updraft at a distance of 1.3 km, comes out of this region at a distance of 8.8 km, and falls out at cloud base at a distance of 7 km. It may be concluded that most of the particles that are carried above the level of the peak updraft are carried away horizontally to large distances, and therefore may fall outside the core of the cloud, possibly even into clear air. In this event the particles will fail to grow to the sizes shown in Fig. 1 and their trajectories after growth ceases would differ from those shown in Fig. 2.

We see that a region of strong divergence overlying a region of convergence acts to sort the particles horizontally. If for the moment we ignore the initially small particles which enter the region of divergence and are carried away from the core of the cloud, then we see from Figs. 2 and 3 that the effect of the sorting is such that beyond a certain distance from the cloud axis the majority of the falling precipitation particles are above a certain size. In the present case if we assume that the cloud updraft has a radius of 2 km at $z=3$ km, then beyond a distance of about 1.1 km from the cloud axis only particles having fall speeds $\gtrsim 28.5$ m sec^{-1} will be found. Two consequences of the above may be noted: 1) the largest particles appear near or outside the cloud core or boundary, and 2) at a height just below the updraft maximum within the storm core, we would tend to find only the largest critical or near critical size particles falling while all others will be rising. Let us now consider the initially small particles which enter the region of divergence and are carried away from the cloud axis. The smaller of these particles may be carried to large distances in the anvil of the cloud. Somewhat larger particles may fall outside the cloud core into regions of low water content where their growth would be less than indicated in Fig. 1. Therefore, our conclusion 1) above is not strictly valid, and a large number of small particles may be found away from the cloud axis and outside the cloud core. In this connection, it should also be remembered that our conclusions above are based on the assumption of a "top-hat" profile of updraft. In actuality, the updraft profile may be "bell-shaped," and this would also allow smaller particles to occur away from the cloud axis.

c. Particle concentration and precipitation content

The concentration densities of the growing particles computed according to Eq. (31) are shown in Fig. 3. It

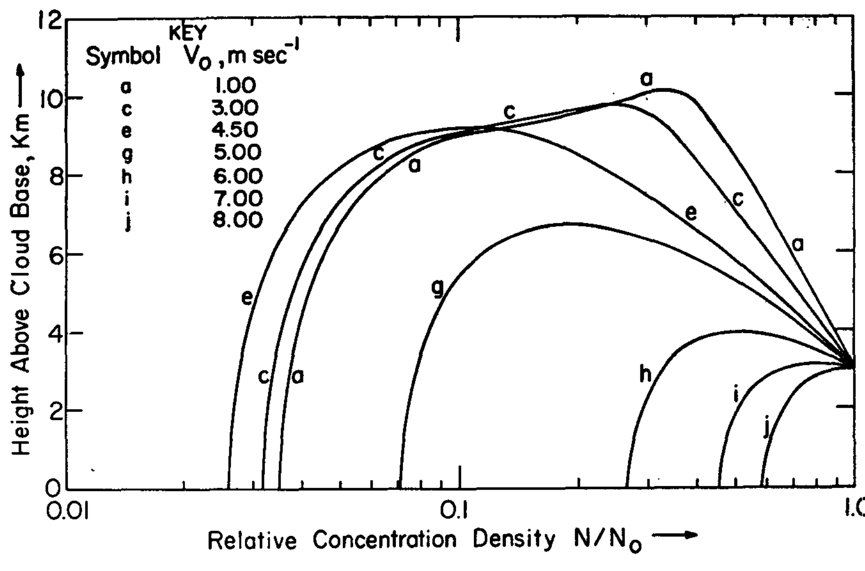

FIG. 3. Relative concentration density N/N_0 as a function of height. The concentration density N is the number of precipitation particles per unit volume per unit fall speed interval. N_0 is the initial concentration density at the starting height $z=3$ km.

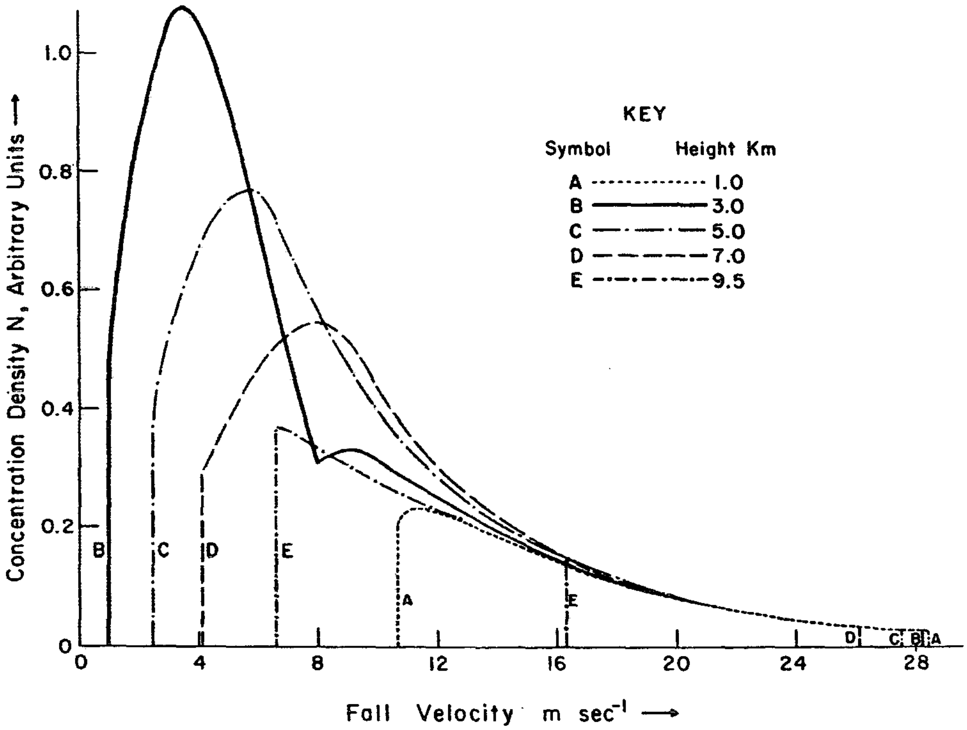

FIG. 4. Particle fall velocity spectra at selected heights. The scale of the ordinate, concentration density, is arbitrary. The thick portion of curve B represents the assumed spectrum at $z=3$ km.

is seen that the concentration densities of the particles decrease monotonically during their entire growth and, in particular, remain finite at the level where the particles are balanced by the updraft.

Fig. 4 shows the size distribution of the precipitation particles, assuming that the rising particles at $z=3$ km are distributed according to $N \approx V \exp(-cV^2)$, c being a constant; this corresponds to an exponential distribution of the Marshall-Palmer type. It is noteworthy that the form of the distribution is preserved at all levels, and the parts of the distributions corresponding to the rising and falling particles join continuously. Further, the distributions are very close together at the large size end; only the cutoff size increases with decreasing height. Fig. 5 gives the precipitation content as a function of height computed from the distribution curves of Fig. 4. It may be mentioned in computing the data on which Figs. 4 and 5 are based that many more trajectories were used than those shown in Fig. 1.

The preceding three figures are pertinent to certain problems in cloud physics. A high concentration of rainwater content aloft has sometimes been invoked to explain the growth of large hail (Hitschfeld and Douglas, 1963; Sulakvelidze *et al.*, 1965). But Fig. 5 shows that the precipitation content decreases monotonically with height, there being no maximum of the concentration aloft. Although this result is based on the restricted model of precipitation growth used in arriving at Fig. 5, it seems reasonable to predict that similar results will be found in more realistic models if a continuous distribution of precipitation particles is assumed for computing the concentration density of precipitation particles since this removes the infinite concentration which occurs when a discrete distribution is assumed. It is, therefore, suggested that in the *steady state*, a high concentration of rainwater aloft, at the level where the raindrops which constituted a spectral peak near cloud base are nearly balanced, may not be invoked to explain the rapid growth of large hail. However, "storage" of rainwater aloft is possible in time-dependent models

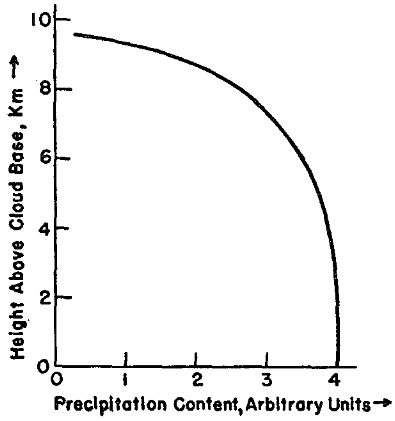

FIG. 5. Precipitation content as a function of height. The scale of the abscissa is arbitrary.

(Das, 1964; Srivastava, 1967). A temporary storage of precipitation aloft may also be inferred from the $z-V$ trajectories of Fig. 1 by referring to the range of particle sizes along any one isochrone. Only after some 2500 sec do the largest particles reach the cloud base. Prior to that time, therefore, the precipitation content will increase with time at cloud base. At heights above the base, the precipitation content will increase with time until the largest particles reach the height in question. For example, at $z=7$ km the precipitation content will keep increasing with time till about 2100 sec, and only thereafter will it settle to a constant value with the arrival from above of the critical particle.

Simple models of coalescence growth, somewhat similar to the one used here, have been employed by Roy and Srivastava (1958) and Murty and Roy (1962) to compute the raindrop size distribution that may be expected from a convective cloud. Murty and Roy pointed out that the final size of a precipitation particle at cloud base is larger the smaller the initial size of the particle (this may be seen from Fig. 1). Then, in line with observations that show that the concentration of giant cloud drops decreases with increasing size, Murty and Roy (1962) argued that at cloud base the concentration of raindrops should increase with increasing size, a conclusion at variance with experience. However, as Fig. 4 shows the difficulty does not arise if the particle concentration is computed according to Eq. (24).

4. Balance level

From Doppler radar observations of convective storms, Atlas (1966) has identified a balance level where the mean Doppler velocity of the scatterers is zero with respect to the ground. In other words, the rising drops (having fall speeds less than the updraft), and the falling drops (having fall speeds greater than the updraft) have equal reflectivities at the balance level. Clearly, the balance level is related to the level at which the mean reflectivity-weighted drop is just supported by the updraft. Atlas has given an interpretation of the observed balance level phenomenon in terms of the coalescence growth of large cloud drops and their behavior at their turn-around point. He has also argued that a steady balance level is possible at the level of a secondary updraft maximum (below the level of the primary maximum which is generally believed to occur near the cloud top) provided 1) some particles attain fall speeds in excess of the maximum and turn downward, 2) other particles with smaller fall speeds continue to rise, and 3) the rising particles do not fall out from higher altitudes at places where they can fall back to the level of the secondary maximum with speeds greatly exceeding it. The validity of the latter condition in a steady-state storm has been questioned by Donaldson and Wexler (1968), and Atlas (1968) has provided a preliminary analysis in its justification. An illustration of these three features may be found in Figs. 1 and 2. Further, the solution given in the Appendix easily gives the condition under which a precipitation particle does not grow fast enough to fall out of a region of increasing updraft, and an upper limit to the size of the particles that may be expected from a region of decreasing updraft.

Let us suppose that the updraft increases fast enough so that the inequality $\alpha^2/4>\beta_1$ holds and Eqs. (A1), (A2) and (A3) are applicable. From (A1) we have

$$\begin{aligned} V-W &= V-W_0-\alpha z \\ &= (A_1-\alpha B_1)e^{p_1 t}+(A_2-\alpha B_2)e^{p_2 t}. \end{aligned} \tag{32}$$

At the level at which the growing particle is balanced, $V=W$; hence, the time at this level is given by

$$\begin{aligned} \exp[(p_1-p_2)t] &= -(A_2-\alpha B_2)/(A_1-\alpha B_1) \\ &= [p_2(W_0-V_0)-\beta_0]/[p_1(W_0-V_0)-\beta_0]. \end{aligned} \tag{33}$$

This equation would not be satisfied by any real value of t, i.e., the particle would not fall out of the updraft, if the right-hand side is negative or less than 1 (note $p_1>p_2$). The condition for this is

$$V_0 \leqslant W_0-\frac{\beta_0}{p_1}. \tag{34}$$

For constant liquid water content $\beta_1=0$, the condition is the same as given by Atlas (1968). Taking the numerical value of α as assumed by Atlas, i.e., $\alpha=4$ m sec^{-1} km^{-1}, and starting the particles at the height at which the effective cloud liquid water content is 1 gm m^{-3} ($\beta_0=0.5$ cm sec^{-2}), and taking the gradient of ME as 0.3 gm m^{-3} km^{-1} [corresponding to $\beta_1=1.5\times10^{-6}$ sec^{-2}, according to Eq. (9b)], we find that particles for which $V_0\leqslant W_0-1.396$ (m sec^{-1}) will continue to rise in the updraft. This condition may be compared with $V_0\leqslant W_0-1.25$ which holds when the effective cloud liquid water content is taken as constant and equal to 1 gm m^{-3}. We see, therefore, that when the updraft increases sharply with height there is a very restricted range of initial sizes which can grow to fallout size in the layer of increasing updraft. If particles having fall speed $V_0=W_0$ fall out at the base of such a layer, then no particles with fall speed smaller than W_0 by more than about 1.25–1.4 m sec^{-1} can fall out from higher levels, except at the very top of the updraft where W starts to decrease again. Accordingly, the third condition specified by Atlas (1966) for the existence of a steady balance level seems physically plausible.

Donaldson and Wexler (1968) have also argued that the updraft maximum does not set an upper limit to the size of the particles which may fall out at the height of the maximum. Let us therefore see if there exists an upper limit to the size of precipitation particles coming out of a region of decreasing updraft, i.e., where $W=W_0+\alpha z$ (α negative). Assume again that $\alpha^2/4>\beta_1$, so that Eq. (32) is applicable. Then it may easily be shown that

$$V-W\leqslant\beta^*/(p_1-p_2), \tag{35}$$

where $\beta^*=\beta_0+\beta_1 z^*$, and where z^* is the level at which the particle is balanced by the updraft. Consider as an example of the application of (35) the region between 9 and 11 km in Fig. 1. It is obvious that the largest particle that may be expected at $z=9$ km can result from the growth of a particle which is just balanced at 11 km. Hence, taking $\beta^*=1.65$ cm sec^{-2} (corresponding to $\gamma^*=3.3$ gm m^{-3}), $\beta_1=1.5\times10^{-6}$ sec^{-2} and $\alpha=-10^{-2}$ sec^{-1}, we have from (35) that $V\leqslant W+1.702$. If W is now taken to be the updraft maximum, V is the corresponding limiting fall speed. Hence, the maximum possible fall speed of a particle grown in the region is 21.7 m sec^{-1}. The minimum fall speed that a particle coming out of the region can have is obviously 20 m sec^{-1}. Thus, it is seen that a region of sharp decrease in the updraft very effectively limits the size of the descending particles.

At this point it is interesting to speculate whether or not a balance level is possible in a steady-state cumulus irrespective of the existence of a secondary updraft maximum and condition 3) assumed by Atlas (1966). Donaldson and Wexler (1968) have contended that a balance level is not possible in the steady state. Their argument is similar to that used by Murty and Roy (1962) to show a possible "inversion" of the computed raindrop size distribution. Donaldson and Wexler argued that since for every rising drop (positive Doppler velocity) there must correspond a larger and more reflective falling drop (negative Doppler velocity), the mean Doppler velocity will be negative at all levels except presumably at the cloud top. This argument presumes, as did that of Murty and Roy, that the concentrations of the rising and falling drops are equal. However, because of the horizontal sorting of particles this is not true. For example, curve a of Fig. 3 shows that at $z=6$ km the relative concentration N/N_0 of the rising drop of fall velocity 3.2 m sec^{-1} is 0.69 while that of the corresponding falling drop of fall velocity 27 m sec^{-1} is only 0.046. Hence, we may not conclude about the sign of the mean Doppler velocity, and therefore about the possibility or impossibility of a steady-state balance level, without further quantitative consideration of the backscattering cross sections of the rising and falling drops.

5. Summary and concluding remarks

In this paper we have obtained analytical solutions for the growth and motion of precipitation particles under certain simplifying assumptions. It has been shown that the distribution of an increasing updraft surmounted by a sharply decreasing one sets a sharp upper limit to the size of particles emerging from the cloud, and sorts the particles horizontally so that most of the particles entering the region of decreasing updraft may be expected to fall outside the core of the cloud. It has also been shown that the growing particles are sharply sorted into rising and falling ones in the vicinity of an updraft maximum. It is felt that these three conclusions are of general validity and will continue to hold when the simple assumptions made here about the variations of updraft, cloud water content and relationship of particle fall speed to its diameter are replaced by more realistic ones. These three facts have been used to discuss certain problems relating to the balance level (Atlas, 1966, 1968).

An equation has also been developed for the concentration density of a continuous distribution of growing particles. Particle size spectra in terms of size or fall velocity are continuous and well-behaved for rising, floating and falling particles alike, without necessarily even maximizing for the floating size. Accordingly, serious doubt is placed on theories of hail growth requiring a large accumulation of precipitation aloft at least in the steady state.

It is hoped that by considering a continuous distribution of growing particles, it may prove possible to construct a steady-state model of convective cloud in which precipitation grows by the process of coalescence, and updraft and cloud water content are determined by dynamic and thermodynamic considerations. A previous attempt (Srivastava, 1964) in this direction was not very successful because of the occurrence of the artificial infinity in concentration of precipitation particles at the level at which they were supported by the updraft.

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APPENDIX

Solution to Eqs. (10) and (11)

Three cases arise, if $\beta_1\neq0$:

1) Case A: $\dfrac{\alpha^2}{4}>\beta_1,\ \beta_1\neq0.$

The solution is

$$V=A_1e^{p_1t}+A_2e^{p_2t}+W_0-\alpha\beta_0/\beta_1, \tag{A1a}$$

$$z=B_1e^{p_1t}+B_2e^{p_2t}-\beta_0/\beta_1, \tag{A1b}$$

where

$$p_1=\frac{\alpha}{2}+\left(\frac{\alpha^2}{4}-\beta_1\right)^{\frac{1}{2}}, \tag{A2a}$$

$$p_2=\frac{\alpha}{2}-\left(\frac{\alpha^2}{4}-\beta_1\right)^{\frac{1}{2}}, \tag{A2b}$$

and A_1, A_2, B_1 and B_2 are constants to be determined from the initial conditions. Let $z=0$ and $V=V_0$ at $t=0$.

Also, Eqs. (7) and (8) show that $dz/dt=W_0-V_0$ and $dV/dt=\beta_0$ at $t=0$. Making use of these conditions, we find

$$A_1=[(W_0-V_0-\alpha\beta_0/\beta_1)p_2+\beta_0]/(p_1-p_2), \quad \text{(A3a)}$$

$$A_2=-[(W_0-V_0-\alpha\beta_0/\beta_1)p_1+\beta_0]/(p_1-p_2), \quad \text{(A3b)}$$

$$B_1=(W_0-V_0-\beta_0 p_2/\beta_1)/(p_1-p_2), \quad \text{(A3c)}$$

$$B_2=-(W_0-V_0-\beta_0 p_1/\beta_1)/(p_1-p_2). \quad \text{(A3d)}$$

2) Case B: $\dfrac{\alpha^2}{4}=\beta_1,\ \beta_1\neq 0.$

The solution is

$$V=(A_1+A_2t)e^{\alpha t/2}+W_0-\alpha\beta_0/\beta_1, \quad \text{(A4a)}$$

$$z=(B_1+B_2t)e^{\alpha t/2}-\beta_0/\beta_1. \quad \text{(A4b)}$$

Again using the initial conditions, we have

$$A_1=-(W_0-V_0)+\alpha\beta_0/\beta_1, \quad \text{(A5a)}$$

$$A_2=\frac{\alpha}{2}(W_0-V_0)-\frac{\alpha}{2}\frac{\alpha\beta_0}{\beta_1}+\beta_0, \quad \text{(A5b)}$$

$$B_1=\beta_0/\beta_1, \quad \text{(A5c)}$$

$$B_2=W_0-V_0-\frac{\alpha\beta_0}{2\beta_1}. \quad \text{(A5d)}$$

3) Case C: $\dfrac{\alpha^2}{4}<\beta_1,\ \beta_1\neq 0.$

The solution is

$$V=(A_1\cos qt+A_2\sin qt)e^{\alpha t/2}+W_0-\alpha\beta_0/\beta_1, \quad \text{(A6a)}$$

$$z=(B_1\cos qt+B_2\sin qt)e^{\alpha t/2}-\beta_0/\beta_1, \quad \text{(A6b)}$$

where

$$q^2=\beta_1-\frac{\alpha^2}{4}. \quad \text{(A7)}$$

Once again using the initial conditions, we have

$$A_1=-(W_0-V_0)+\alpha\beta_0/\beta_1, \quad \text{(A8a)}$$

$$A_2=\left[\frac{\alpha}{2}(W_0-V_0)-\frac{\alpha}{2}\cdot\frac{\alpha\beta_0}{\beta_1}+\beta_0\right]\Big/q, \quad \text{(A8b)}$$

$$B_1=\beta_0/\beta_1, \quad \text{(A8c)}$$

$$B_2=\left[(W_0-V_0)-\frac{\alpha\beta_0}{2\beta_1}\right]\Big/q. \quad \text{(A8d)}$$

For $\beta_1=0$, the above solutions are not applicable. In this case we have the solution

$$V=V_0+\beta_0 t, \quad \text{(A9a)}$$

$$z=\frac{\beta_0 t}{\alpha}+\left(\frac{W_0-V_0}{\alpha}-\frac{\beta_0}{\alpha^2}\right)(e^{\alpha t}-1). \quad \text{(A9b)}$$

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