NOTES AND CORRESPONDENCE

The Effect of the Phase Function at Forward Angles on Light Pulses Scattered Backward from a Thin Turbid Medium

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(Manuscript received 21 November 1969)

1. Introduction

Small angle forward scattered radiation of wavelength λ, scattered from particles of radius $r>\lambda$, is a sensitive function of the size distribution of drops comprising clouds and hazes (Twomey and Howell, 1967). The analysis to be described will show that a short light pulse, doubly scattered in the *backward* direction produces a return signal with a marked temporal dependence on the scattering phase function at *forward* angles. This procedure, therefore, renders it feasible to study the phase function at forward angles with a light source and receiver which are located at the same boundary of a turbid medium.

Although the measurement of scattered light pulses from lidars is feasible, the analysis of such data is extremely complex, most of the complexity being caused by the need to consider the finite width and divergence of the laser beam and the finite vignetted field of view of the receiving telescope system. However, the finite transmitted beam width and the finite field of view of the receiver render it feasible to identify the singly and doubly scattered radiances in the return obtained from the lidar system. Observation of the triply scattered radiances is relatively improbable in most cases.

Much of the underlying physics of the lidar problem, considered by Eloranta and Weinman[2], can be illustrated by a Neumann solution of the time-dependent radiative transfer equation in one dimension (see van de Hulst, 1948, and Irvine, 1965). This exercise will illustrate the dependence of radiances singly and doubly scattered by media characterized by several anisotropic phase function.

2. The equation of transfer

Consider a homogeneous cloud which extends infinitely in the horizontal directions but is of finite thickness in the vertical direction. The source of illumination is also assumed to be of infinite horizontal extent. The scattered radiance is a function of only the vertical coordinates measured in the upward direction. The zenith angle, $\cos^{-1}\mu$, is reckoned positive for upward direction scattered radiances, with ϕ being the azimuth angle. The pulsed light source below the cloud illuminates the cloud perpendicularly, i.e., $\mu_0=1$.

Particles which comprise clouds and hazes scatter light anistropically, a considerable fraction of which is diffracted into a narrow forward cone. In order to illustrate the effect of this anistropy on the singly and doubly scattered radiance, the phase functions for water clouds and hazes are approximated by a "peaked semi-isotropic" phase function, i.e.,

$$P(\Theta)=\eta+4\pi(1-\eta)\delta(\mu-\mu')\delta(\phi-\phi'), \qquad (1)$$

where

$$\cos\Theta=\mu\mu'+\sqrt{1-\mu^2}\sqrt{1-\mu'^2}\cos(\phi-\phi'),$$

δ is the Dirac delta function, and η is chosen so that

$$\langle\cos\Theta\rangle=\frac{1}{4\pi}\int_0^{2\pi}\int_{-1}^{1}P(\Theta)\mu d\mu d\phi$$

is equal to the asymmetry function derived from the phase function obtained by Deirmendjian (1964).

The strongly anistropic phase functions previously considered by Irvine (1965) are also utilized, i.e.,

$$P(\Theta)=b\Phi_{\rm H.G.}(g_1)+(1-b)\Phi_{\rm H.G.}(g_2), \qquad (2)$$

where the parameters b, g_1 and g_2 employed in the Henyey-Greenstein functions

$$\Phi_{\rm H.G.}(g_i)=\frac{1-g_i^2}{(1+g_i^2-2g_i\cos\Theta)^{\frac{3}{2}}}$$

are summarized in Table 1. The approximate phase functions thus defined are shown in Fig. 1.

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[2] Paper presented at Conference on Laser Probing, National Center for Atmospheric Research, Boulder, Colo., 1968.

TABLE 1. Phase functions considered in this analysis.

Designation	$P(\Theta)$ parameters	Description
A	Eq. (1) $\eta = 1$	Isotropic scattering
B	Eq. (1) $\eta = 0.15$	$\langle\overline{\cos\Theta}\rangle = 0.85$, comparable to that of cumulus clouds at $\lambda = 0.7\ \mu$
C	Eq. (2) $g_1 = 0.9$, $g_2 = -0.75$, $b = 0.95$	Similar to cumulus cloud at $\lambda = 0.7\ \mu$ (Irvine, 1965)
D	Eq. (2) $g_1 = 0.824$, $g_2 = -0.55$, $b = 0.9724$	Similar to maritime haze at $\lambda = 0.7\ \mu$ (Irvine, 1965)
E	Eq. (2) $g = 0.75$, $b = 1.0$	$\langle\overline{\cos\Theta}\rangle = 0.75$ (Irvine, 1965)

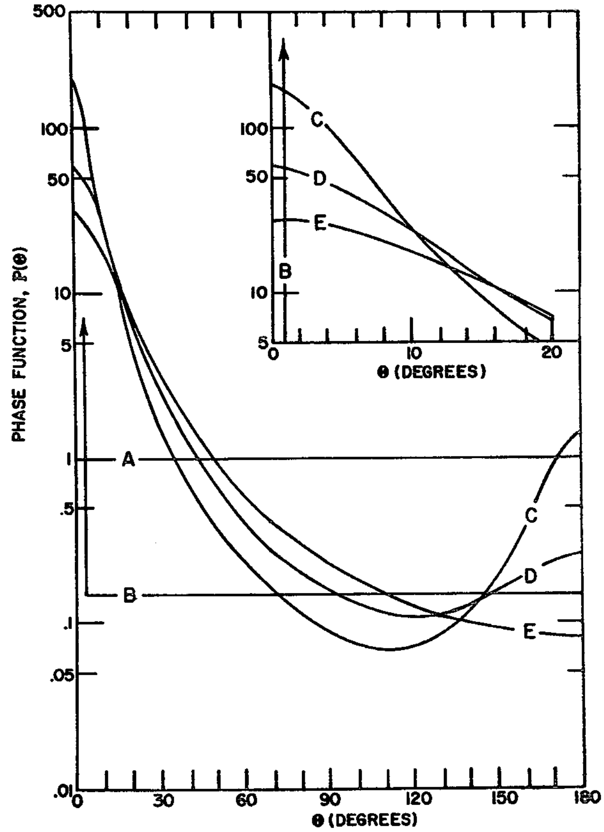

FIG. 1. Phase functions from Eqs. (1) and (2) utilizing parameters summarized in Table 1.

The equation of transfer appropriate to this problem (Bellman *et al.*, 1964) is

$$\mu\frac{\partial I}{\partial z}+\frac{1}{c}\frac{\partial I}{\partial t}+KI =\frac{K}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\phi;\mu',\phi')I(z,t,\mu',\phi')d\mu'd\phi', \quad (3)$$

where c and K are the velocity of light and the extinction coefficient, respectively. For simplicity, the turbid medium will be assumed homogeneous such that K and $P(\mu,\phi;\mu',\phi')$ are independent of z and the scattering will be regarded conservative.

The initial and boundary conditions determining $I(z,t,\mu,\phi)$ [W cm^{-2}] are

$$\left.\begin{aligned} I(z,0,\mu,\phi)=I(z,0,-|\mu|,\phi)=0\\ I(z_0,t,-|\mu|,\phi)=0\\ I(0,t,\mu,\phi)=\pi\delta(t)\delta(\mu-1)\delta(\phi)\end{aligned}\right\}, \quad (4)$$

where z_0 is the coordinate of the cloud boundary opposite to the source of light. Eq. (3) is transformed from a partial differential equation to an ordinary differential equation by the Laplace transformation

$$\mu\frac{\partial i}{\partial z}+(K+s/c)i =\frac{K}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\mu,\phi;\mu',\phi')i(z,s,\mu',\phi')d\mu'd\phi', \quad (5)$$

where

$$i(z,s,\mu)=\mathcal{L}[I(z,t,\mu)],$$

i.e., $\mathcal{L}$ represents the Laplace transformation operator. An approximate Neumann solution for the diffusely scattered radiance can be obtained for Eq. (5) in the form

$$i(z,s,\mu)\simeq i_1(z,s,\mu)+i_2(z,s,\mu), \quad (6)$$

where

$$i_n(z,s,\mu)=\int_0^{z_0} j_n(z',s,\mu)k(z-z',s,\mu)dz', \quad (7)$$

and

$$k(z,s,\mu)=\begin{cases}\dfrac{1}{|\mu|}e^{-(K+s/c)z/\mu}, & \dfrac{z}{\mu}>0\\[2ex] 0, & \dfrac{z}{\mu}<0\end{cases} \quad (8)$$

$$\left.\begin{aligned} j_1(z,s,\mu)&=\frac{K}{4}\mathcal{J}(\mu,1)e^{-(K+s/c)z}\\ j_2(z,s,\mu)&=\frac{K}{2}\int_{-1}^{1} i_1(z,s,\mu')\mathcal{J}(\mu,\mu')d\mu'\\ \mathcal{J}(\mu,\mu')&=\frac{1}{2\pi}\int_0^{2\pi}P(\Theta)d\phi\end{aligned}\right\}. \quad (9)$$

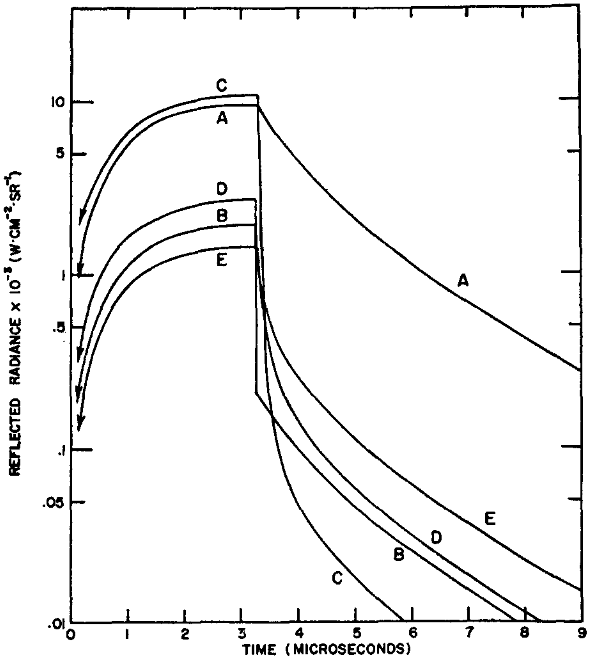

FIG. 2. Time dependence of doubly scattered radiances for a cloud with $K=1$ km^{-1} and $z_0=0.5$ km, consisting of particles which scatter by a phase function defined in Eqs. (1) and (2). The incident and reflected radiances both have $\mu_0=-\mu=1$.

We are primarily interested in radiances scattered backward toward the source of radiation, i.e., $\mu=-1$.

Applying the inverse Laplace transform to (7) yields

$$[I_1(0,t,-1)=\mathcal{L}^{-1}[i_1(0,s,-1)]$$

$$=\begin{cases}\dfrac{Kc\mathcal{G}(-1,1)e^{-cKt}}{8}, & t<2z_0/c\\ 0, & t>2z_0/c\end{cases} \tag{10}$$

and

$$I_2(0,t,-1)=\mathcal{L}^{-1}[i_2(0,s,-1)]$$

$$=\begin{cases}\dfrac{K^2c^2te^{-cKt}}{8}\displaystyle\int_0^1 \mathcal{G}(\mu,1)\mathcal{G}(-1,\mu)\frac{d\mu}{\mu+1}, & \text{for}\quad t<2z_0/c\\ \dfrac{K^2c^2e^{-cKt}}{4}\displaystyle\int_0^{z_0/(ct-z_0)} \frac{\mathcal{G}(\mu,1)\mathcal{G}(-1,\mu)}{(\mu^2-1)}\left(\mu t-\frac{z_0}{c}(\mu+1)\right)d\mu, & \text{for}\quad 2z_0/c<t<z_0(\mu+1)/c\mu\\ 0, & \text{for}\quad t>z_0(\mu+1)/c\mu\end{cases} \tag{11}$$

Eqs. (10) and (11) were solved numerically.

3. Results and discussion

The dependence of the backward directed radiances on a number of phase functions will be illustrated. Attention is confined to a medium with visibility $\simeq 4$ km, i.e., $K=1$ km^{-1} and an optical thickness 0.5 so that triply scattered radiances are relatively insignificant.

Eq. (10) indicates that the singly scattered radiance reflected 180° is proportional to the extinction coefficient and the phase function for $\Theta=180°$. The extinction coefficient determines the slope of the decay of the radiance. When $t>2z_0/c$, i.e., the time is sufficient for light to traverse the medium in both directions, the singly scattered radiance disappears.

Fig. 2 shows the doubly scattered radiance reflected by 180°, computed from (11), for the phase functions shown in Fig. 1.

The results for the isotropic phase function A agree with those obtained by Bellman *et al.* (1964) for a medium with optical thickness 0.5 and $t<1$ normalized time unit.

The effect of the δ function peak of the "peaked semi-isotropic" phase function B is evident near $t\sim 2z_0/c$. For $t<2z_c/c$, the following factors contribute to the scattered radiance:

1) Radiance scattered by a small forward angle $\Theta_1 \simeq 0°$ followed by large angle scattering $\Theta_2=180°-\Theta_1$ or *vice versa*, such that the resulting radiance is proportional to the product of the phase function for these angles, i.e., $(1-\eta)\eta$.

2) Radiance scattered by a large angle Θ_1 followed by scattering by another large angle $\Theta_2=180°-\Theta_1$, such that the doubly scattered radiance is proportional to the product of the phase functions for these two angles, i.e., η^2.

When $t>2z_0/c$ radiance scattered by step 1) no longer contributes to the doubly scattered radiance. A discontinuity, depending on the phase function for forward scattering, $1-\eta$, becomes evident for I_2 $(0, 2z_0/c, -1)$.

The phase functions C, D and E become less peaked in the forward direction in the order stated. The radiances, doubly scattered by media characterized by these phase functions, fall less abruptly at $t \gtrsim 3.3$ μsec as the phase functions become less peaked. This is caused by the upper limit of integration in (11) which truncates the integral at values of μ which change with t between $2z_0/c < t < (\mu+1)z_0/\mu c$. Because the range of μ is determined by the time, the temporal variation of $I_2(0, t, -1)$ responds to the product of the phase functions $\mathcal{I}(-1, \mu)$ and $\mathcal{I}(\mu,1)$. The term $\mathcal{I}(-1, \mu)$ is less sensitive to changes in μ than $\mathcal{I}(\mu,1)$. The final result, therefore, depends markedly on the shape of the small angle diffraction peak in $P(\Theta)$.

4. Conclusion

The present analysis shows that the singly scattered reflected radiance yields information on the phase function for backscattering, and the extinction coefficient of a cloud. The *backward* directed doubly scattered radiance provides a measure of the phase function at small *forward* angles. Analysis of the various orders of scattering may therefore provide a method to remotely investigate the phase functions of turbid media where conventional measurements are physically awkward.

Acknowledgments. The authors wish to thank Mr. E. Eloranta for numerous interesting discussions on his laser experiments in fogs which motivated this study. Computations were conducted at the University of Wisconsin computer center. This research was supported by funds from the National Science Foundation under Grant GA-1020.

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