Large-Scale Dispersion of Clusters of Particles in the Atmosphere
II. Stratosphere

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The characteristics of the large-scale relative particle displacement tensor, the correlation functions, and spectra of the relative particle velocities at 10-, 30-, 50- and 100-mb levels are investigated; pertinent results concerning relative turbulence and diffusion at various levels in both troposphere and stratosphere are discussed and summarized. It is found that a quasi-stationary process exists in the large-scale turbulence diffusion in both the troposphere and stratosphere, the rate of relative particle dispersion being greatest in the tropopause level and generally proportional to the variance of the relative velocity. In general, the autocorrelation functions for the relative zonal velocities in both the troposphere and stratosphere behave like an exponentially decreasing function, whereas those for the relative meridional velocities show a combination of an exponential function and a cosine function with a damping amplitude. The power spectra of the relative zonal velocities at all levels show the similar characteristics of increasing kinetic energy with decreasing frequency, whereas those of the relative meridional velocities show an energy peak near the frequency of 10^{-2} cycles hr^{-1}. The high frequency portion of the power spectra of both the zonal and meridional components of the relative velocities at all levels is found to be proportional to the minus third power of the frequency. The principal axis of the large-scale turbulent diffusion in the stratosphere is generally oriented ENE–WSW, whereas in the troposphere it is ESE–WNW.

1. Introduction

Studies of the characteristics of relative particle displacement are fundamental to the understanding and prediction of particle dispersion in instantaneous sources. In a recent paper (Kao, 1968) a theoretical analysis of the kinematics and dynamics of the relative dispersion of particles in a stratified rotating atmosphere was made, and a diffusion model for estimating the concentration distribution in a cluster of marked fluid particles was constructed. To gain an insight into the nature of the large-scale relative dispersion in the atmosphere an investigation was made of the characteristics of the relative particle displacement tensor, the correlation functions, and spectra of relative particle velocities in the troposphere at the 500- and 850-mb levels as well as near the tropopause at the 200-mb level (Kao and al-Gain, 1968). In the present paper, the study is extended to the stratosphere at the 10-, 30-, 50- and 100-mb levels.

In the troposphere, the stream functions for the 200-, 500- and 850-mb levels were computed and stored on tapes at the National Meteorological Center (NMC), which were used for the earlier study of relative particle diffusion in the troposphere. In the stratosphere, however, the stream functions have not been computed, and, therefore, are not available at NMC. For the investigation in this paper, it was necessary to use the height data for the 10-, 30-, 50- and 100-mb levels.[1] From these data the geostrophic velocities were computed and the geostrophic trajectories were constructed for the analysis. In addition to the characteristics of the relative diffusion in the stratosphere, pertinent results of the relative turbulence and diffusion at various levels in both troposphere and stratosphere are discussed and summarized in this paper.

2. Source of data and analysis

For this study the height data for the 10-, 30-, 50- and 100-mb levels for the period December 1963 through November 1964 were used. These data were extracted from the NMC tape. Nine particles initially forming a circular grid of radius 176.5 km (one particle in the center, eight particles at radius of one-half grid square from the center particle) were released simultaneously into geostrophic motion. These releases were equally divided at each of the four levels among three locations (46.7N, 141.4W; 46.7N, 18.6W; 46.7N, 100E). The choice of three positions eliminates the bias that could result from the use of only one position of release. Furthermore, trajectories of particles released at this latitude can be traced up to 8 days without going off the NMC map so that the analysis of relative dispersion of particles can be made.

The height maps were available at 0000 and 1200 GMT for every day of the annual period except for a few missing cases. Each cluster was initiated at 1200 GMT and allowed to run for 192 hr. A string of 17 consecutive maps was necessary for each such run. At each level 60 such releases, evenly spaced throughout

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[1] An extensive analysis of the relative particle velocity correlation functions and spectra computed from the height data at 500 mb indicates that they give a good approximation to those computed from the stream function at the same pressure level.

the year, were made. The trajectories were constructed over 2-hr time steps. The geostrophic velocity used to locate each new point on a trajectory was arrived at by averaging two velocities. For example, to compute the second position of a trajectory, the velocity at the initial position was computed from the height field in effect at the initial time. Then a preliminary second position was calculated by assuming constant velocity for the particle for a 2-hr period. A second velocity was computed at the end of the time step. Then the vector average of the two velocities was taken as the mean velocity of the particle throughout the 2-hr period, and the second position calculated accordingly. Ninety-six such cycles were required for constructing each trajectory.

To obtain velocities at times between the available maps, intermediate maps had to be constructed. By interpolation they were constructed for times of 2, 4, 6, 8 and 10 hr after each map on the tape. A program was designed by which the 1108 Univac computer read the tape, constructed the intermediate maps, and computed the resulting geostrophic velocities and trajectories.

3. The relative displacement of particles as a function of time

As a measure of the relative zonal and meridional particle separations as functions of time, the mean squares of these separations are computed at 2-hr intervals from 60 runs and 2160 particle pairs for each of the 10-, 30-, 50- and 100-mb levels, and are shown in Figs. 1 and 2. For all experiments, the initial mean distance of particle separation was 4.11×10^{10} m² for $\overline{x_r^2}$ and $\overline{y_r^2}$ at all levels.

The mean-square relative zonal displacements of particles for the 10-, 30-, 50- and 100-mb levels are shown in Fig. 1. It is seen from this figure that the relative zonal displacement at 100 mb is generally greater than that at 10, 30 and 50 mb, whereas at the latter three levels the relative displacements are about the same. Comparing with the relative zonal displacements of particles at 200, 500 and 850 mb (Kao and al-Gain, 1968), one finds that the mean-square relative zonal displacement generally increases with increasing mean zonal velocity, i.e., it has its maximum value at the jet stream level and decreases both upward and downward from the jet.

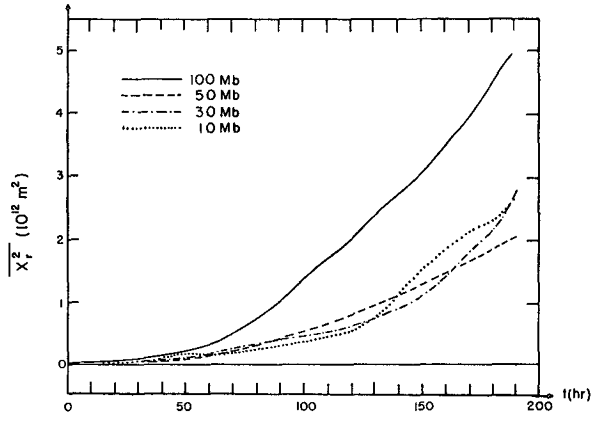

Fig. 1. Mean squares of the zonal component of the relative particle displacement.

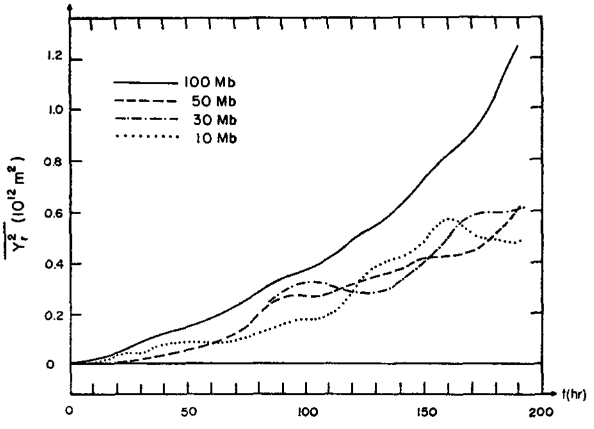

Fig. 2. Mean squares of the meridional component of the relative particle displacement.

The mean-square relative meridional displacements of particles for the 10-, 30-, 50- and 100-mb levels are shown in Fig. 2. When comparing these with the relative meridional displacement for the 200-, 500- and 850-mb levels, it is seen again that the maximum value is at the jet stream level. It may be noted that there exists a wavy type of oscillation in the relative meridional displacement, which is the result of the influence of the planetary waves in the atmosphere as shown theoretically in an earlier paper (Kao, 1962).

Computations of the zonal, meridional and the diagonal components of the turbulence diffusivity were made. It is found that most of the components generally vary with time, which reflects some of the effect of nonstationary motion in the atmosphere. For large diffusion times, the values of the zonal and meridional components of the turbulence diffusivity are estimated and listed as follows:

P(mb)	10	30	50	100	200	500	850
D_{xx}(10^{10} cm² sec^{-1})	4.4	5.3	2.5	5.8	23.6	8.3	7.0
D_{yy}(10^{10} cm² sec^{-1})	0.55	0.58	0.50	1.6	5.8	2.4	2.4

where $D_{xx}=\frac{1}{2}(\overline{dx_r^2}/dt)$ and $D_{yy}=\frac{1}{2}(\overline{dy_r^2}/dt)$. On the average, the zonal component of the turbulence diffusivity in the atmosphere is about four times as large as the meridional component. The distribution of the zonal and meridional components of the large-scale turbulence diffusivity in the troposphere and stratosphere is shown in Fig. 3.

The diagonal components of the relative displacement tensor were also computed for 10-, 30-, 50- and 100-mb

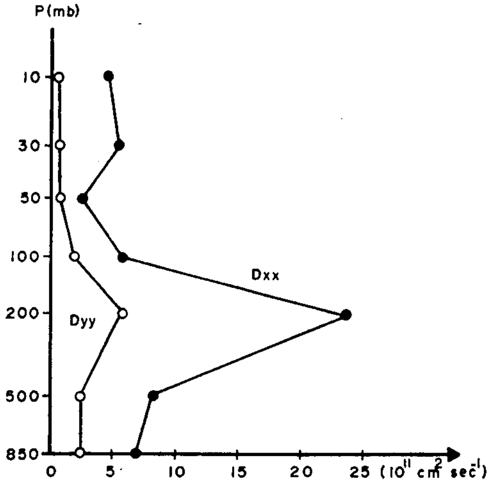

FIG. 3. Mean zonal and meridional components of the turbulence diffusivity.

levels, and are shown in Fig. 4. For large dispersion times, they are mostly positive at all levels, which indicates that the major axis of the large-scale particle dispersion is mostly oriented ENE–WSW for large dispersion times. Comparing with those for the 200-, 500- and 850-mb levels (Kao and al-Gain, 1968), one finds that the diagonal components of the relative displacement tensor below the jet stream level, which are mostly negative, are just opposite to those above the jet stream level.

4. The relative particle velocity as a function of relative distance

It is known that relative velocity between particles generally increases with increasing distance between particles (Richardson, 1926). However, in large-scale turbulence in the troposphere it is found that the relative particle velocity remains almost constant, indicating that the mechanism of relative turbulence in the large-scale atmospheric motion is more or less quasi-stationary (Kao and al-Gain, 1968). This is important to the fact that under such a condition the relative displacement tensor and, therefore, the relative particle dispersion, can be predicted with the use of correlation functions and spectra of the relative velocity components (Kao, 1968).

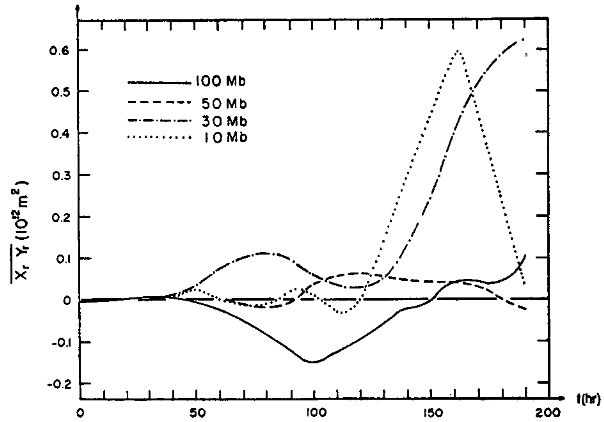

FIG. 4. Mean correlation of the zonal and meridional components of the relative particle displacement.

To show the degree of stationarity of the large-scale relative turbulence in the stratosphere, the root mean squares (rms) of the relative zonal and meridional velocities at the 10-, 30- and 50-mb levels were computed and plotted respectively vs those of the relative zonal and meridional distances between particles in Figs. 5 and 6. It may be noted that the rms of the relative zonal and meridional velocities increase slightly for a comparatively short particle distance; they then remain practically constant for large particle distance, indicating that a quasi-stationary state exists in the large-scale turbulence in the stratosphere. For practical purposes, one may take an average value of the rms of the relative velocity and consider it as a constant for the large-scale relative turbulence. The prediction formulas for the relative displacement tensor (Kao, 1968)

$$\overline{x_{ri}(t)x_{rj}(t)} = (\overline{v_{ri}^2}\,\overline{v_{rj}^2})^{\frac{1}{2}} \times \int_0^t \int_0^\eta [R_{v_{ri}v_{rj}}(\tau)+R_{v_{rj}v_{ri}}(\tau)]d\tau d\eta,$$

can, therefore, be applied to the stratosphere.

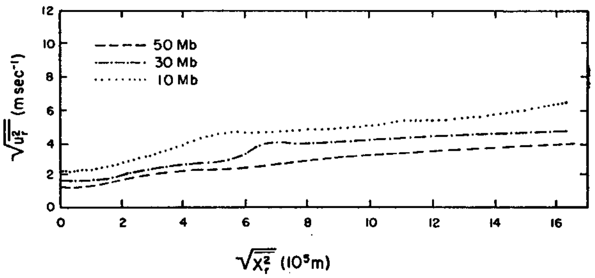

FIG. 5. Variation of relative zonal velocity with relative zonal displacement.

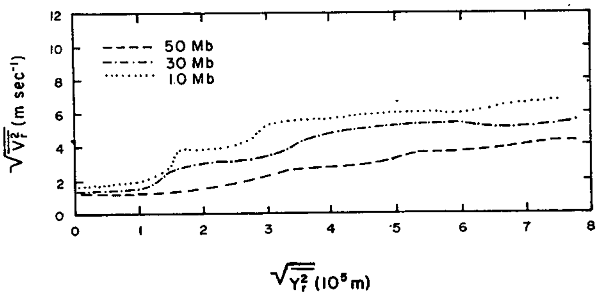

FIG. 6. Variation of relative meridional velocity with relative meridional displacement.

5. The autocorrelation functions and power spectra

The Lagrangian autocorrelation functions of the relative zonal and meridional velocities at the 10-, 30-, 50- and 100-mb levels were computed and are shown in Figs. 7 and 8. In general, the autocorrelation functions for the zonal component of the relative velocities in the stratosphere behave like an exponentially decreasing function, whereas those for the meridional component of the relative velocity show a combination of an exponential function and a cosine function with a damping amplitude. They generally agree with the theoretical model presented earlier (Kao, 1962), i.e.,

$$R_{u'u'}(\tau)=\exp(-\epsilon_x\tau),$$

$$R_{v'v'}(\tau)=\left(\frac{\bar{A}^2}{\bar{A}^2+2\overline{v_r^2}}\right)\frac{\sin\dfrac{\beta\Delta L}{2\pi}\tau}{\dfrac{\beta\Delta L}{2\pi}\tau}\cos\frac{\beta\bar{L}}{2\pi}\tau+\left(\frac{\bar{A}^2}{\bar{A}^2+2\overline{v_r^2}}\right),$$

$$\times\exp(-\epsilon_y\tau),$$

where $\bar{A}$ and $\bar{L}$ are the mean amplitude and wavelength of the planetary waves, ϵ_x and ϵ_y the resistance coefficient along the x and y axis, $\overline{v_r^2}$ the variance of the relative meridional velocity, and $\beta=2a^{-1}\Omega\cos\phi$, in which a is the radius of the earth, Ω the angular velocity of the earth, and ϕ the latitude.

Inspecting the aforementioned figures, it can be seen that the minimum correlation for the meridional component of the relative velocity is attained at a smaller lag time for the higher levels except for 100 mb, at which level the minimum is not distinct. At the 10-mb level, the minimum correlation is attained in about 25 hr, whereas at the 30- and 50-mb levels, the minimum correlation is attained at 35 and 45 hr, respectively. Similarly, the decrease in the correlation for the zonal component of the relative velocity is greater for the higher levels except, again, for the 100-mb level. Further discussion of this relation will be made in a later paragraph.

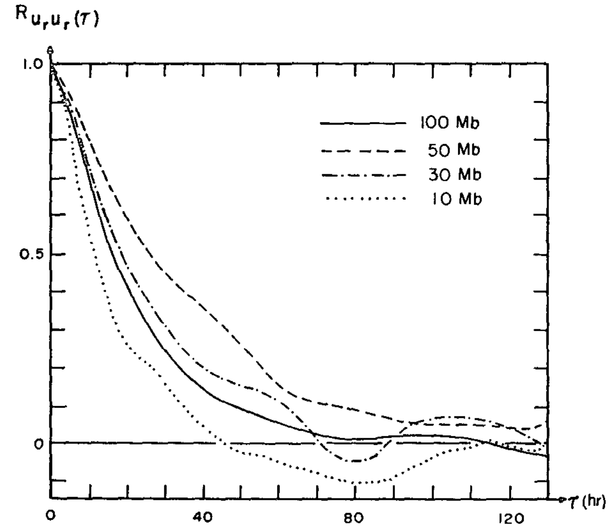

FIG. 7. Autocorrelation function of the relative zonal velocities.

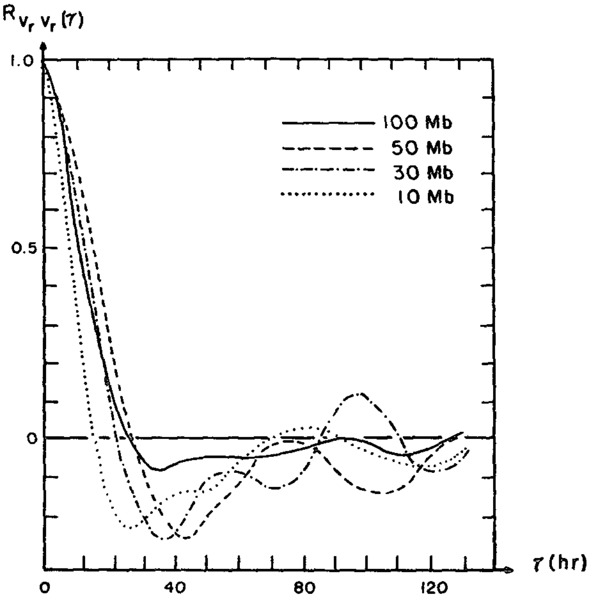

FIG. 8. Autocorrelation functions of the relative meridional velocities.

With the use of the above correlation functions the normalized power spectra of the relative zonal and meridional velocities at the 10-, 30-, 50- and 100-mb levels were computed and are shown in Figs. 9 and 10. The similarity in the normalized power spectra at various levels in the stratosphere as well as in the troposphere indicates again that a quasi-stationary process exists in the large-scale turbulence in the atmosphere. Figs. 9 and 10 show that the turbulent kinetic energy generally decreases with increasing frequency in the high frequency end, which indirectly indicates that the rate of relative dispersion of particles generally increases with the size of the particle cluster, except at the comparatively low frequency end at which energy peak of the power spectra of relative meridional velocity occurs. As was pointed out in the preceding paragraph, the minimum of the correlation function of the relative meridional velocity occurs at a smaller value for the higher levels (Fig. 8); thus, the energy peak of the relative meridional velocity occurs at a lower frequency for the lower levels in the stratosphere as shown in Fig. 10.

It may be noted in the comparatively high frequency range that the power spectra for both the relative zonal and meridional velocities are more or less proportional to the minus third power of the frequency, the same characteristic as in the troposphere (Kao and al-Gain, 1968). In these figures, the lower limit of the frequency is estimated by $(2T_{\max})^{-1}$, which, for a maximum interval $T_{\max}$ between observations, of 8 days, is about

0.003 cycle hr^{-1}; the upper limit of the frequency is estimated by $\bar{v}_r/\bar{d}$, which for a mean relative velocity $\bar{v}_r$ of 6 m sec^{-1} and a mean grid distance $\bar{d}$ of 350 km, is about 0.060 cycle hr^{-1}.

To obtain the value of power spectrum of the relative velocity at a frequency the normalized power spectrum needs to be multiplied by the mean-square value of the relative velocity. The mean-square values of the relative zonal and meridional velocities at the 10-, 30-, 50-, 100-mb levels, as well as at the 200-, 500- and 850-mb levels, are shown in Fig. 11. It may be noted that the mean squares of the relative velocities generally increase with increasing height from 850 to 200 mb, then decrease with height to about 50 mb, and then increase again with height, which indirectly indicates that the rate of relative dispersion is more or less proportional to the mean velocity in the atmosphere.

It may be noted from Figs. 8 and 11 that the larger the value of the mean square of the relative meridional velocity, the smaller is the value of the time lag for the occurrence of the minimum value of the autocorrelation of the relative meridional velocity. This indicates that the larger the value of the relative velocity, the faster is the particle separation; therefore, the shorter time is the lag needed for the particles to become uncorrelated.

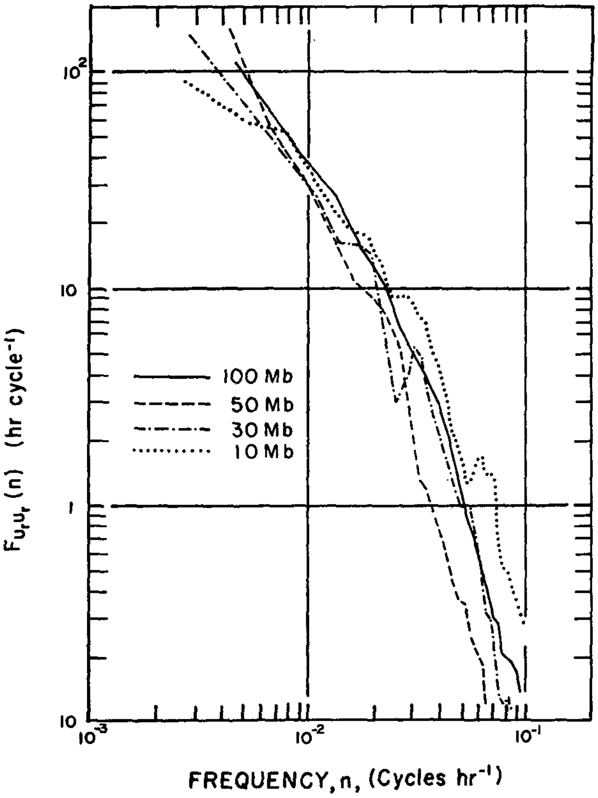

FIG. 9. Normalized power spectra of the relative zonal velocities.

6. The cross-correlation functions and cross spectra

To examine the effect of turbulent eddies of various frequencies on the degree of anisotropy of the field of relative turbulence and turbulent diffusion, the cross correlations and normalized cross spectra of the zonal

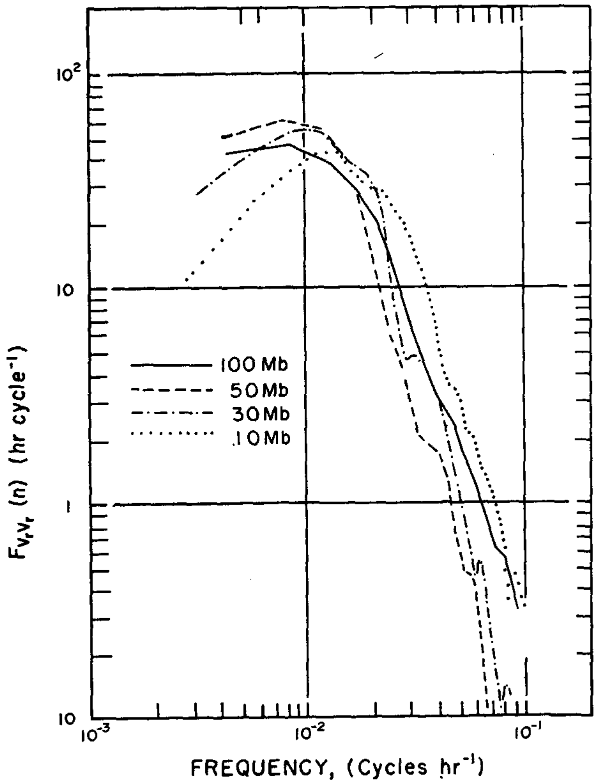

FIG. 10. Normalized power spectra of the relative meridional velocities.

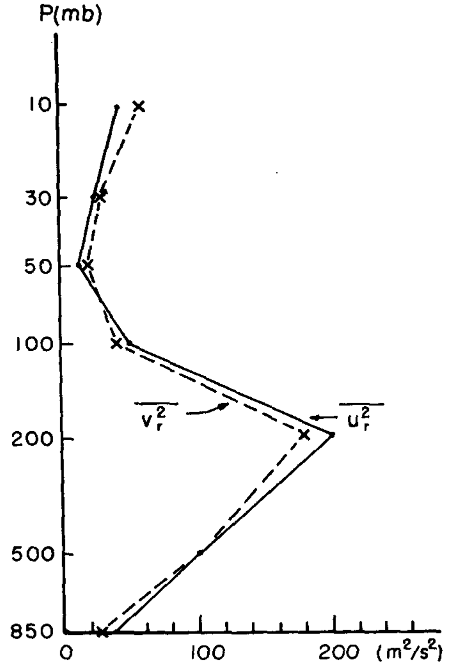

FIG. 11. Distribution of the mean squares of the relative zonal and meridional velocities.

and meridional components of the relative velocity at the 10-, 30-, 50- and 100-mb levels were computed and are shown in Figs. 12 and 13. For large time lags, the cross-correlation functions are generally positive at all levels in the stratosphere, reflecting in the positive value in the cross spectra at the low frequency end. The cross spectra in Fig. 13 also show that the minima at the lower stratosphere (100 and 50 mb) occur at more or less at the same frequency at which the upper level (10 mb) has its maximum. This indicates that the contribution of relative turbulence eddies in the relative turbulent motion to the diagonal components of the relative displacement tensor in the upper stratosphere has an opposite effect to that in the lower stratosphere.

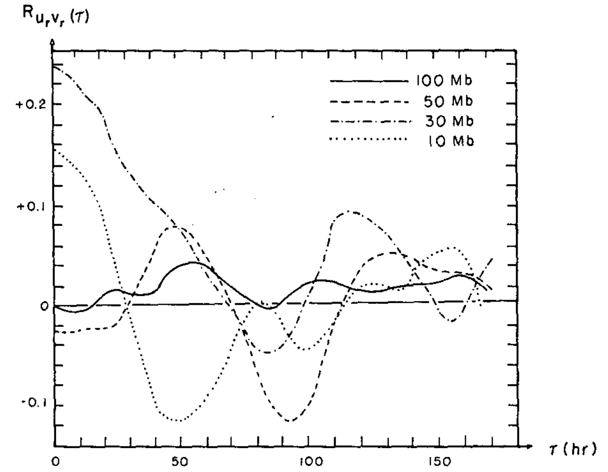

Fig. 12. Cross-correlation functions of the relative zonal and meridional velocities.

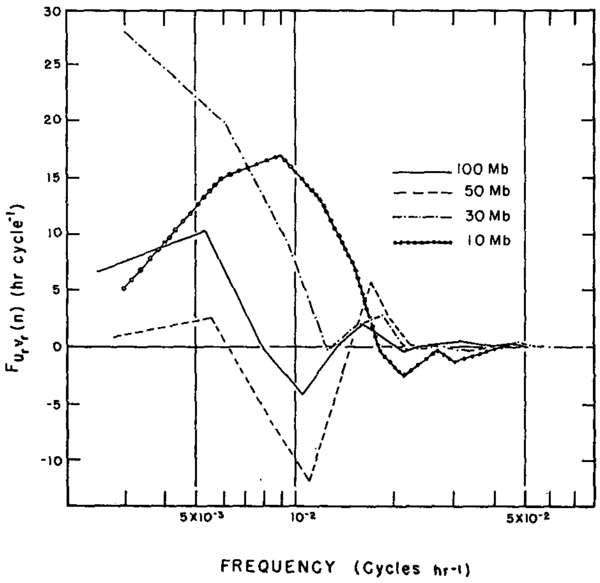

Fig. 13. Normalized co-spectra of the relative zonal and meridional velocities.

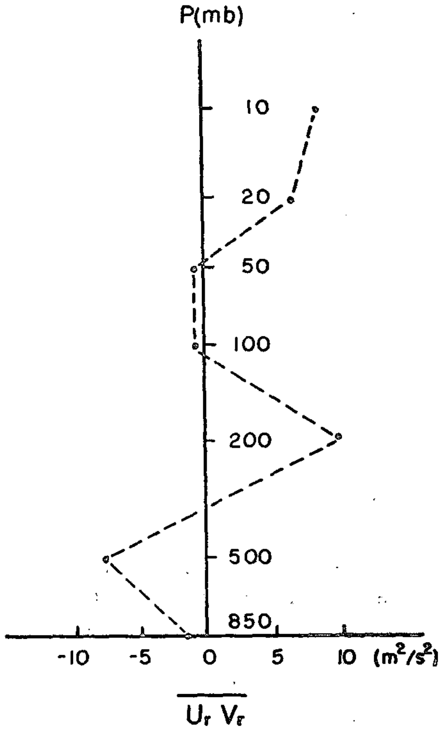

Fig. 14. Distribution of the mean correlation of the relative zonal and meridional velocities.

In accordance with the theory of relative dispersion of particles for large dispersion times, the degree of anisotropy of turbulent diffusion is primarily affected by the low frequency of the spectra (Kao, 1968). The cross spectra in Fig. 13 indicate that the contribution to the anisotropy of the relative diffusion for large diffusion times is primarily due to the positive low frequency spectra, which in turn contribute to the positive cross-correlation of the relative velocity for the large time lags (Fig. 4).

To obtain the value of the cross spectra at a frequency it is necessary to multiply the normalized cross spectra by the mean value of the covariance of the relative velocity components. The covariances of the relative velocities at various levels are plotted in Fig. 14, which shows that the covariance of the relative velocities is generally negative below about 300-mb level, but positive above 100 mb.

7. The orientation of the principal axis of diffusion

One of the quantities which are of importance in the prediction of the dispersion of clusters of particles is the orientation of the principal axis of diffusion. Using the formula

$$\alpha(t)=\tfrac{1}{2}\tan^{-1}\left(\frac{\overline{2x_r(t)y_r(t)}}{\overline{x_r^2(t)}-\overline{y_r^2(t)}}\right)$$

[presented in an earlier paper(Kao, 1968)], the angle

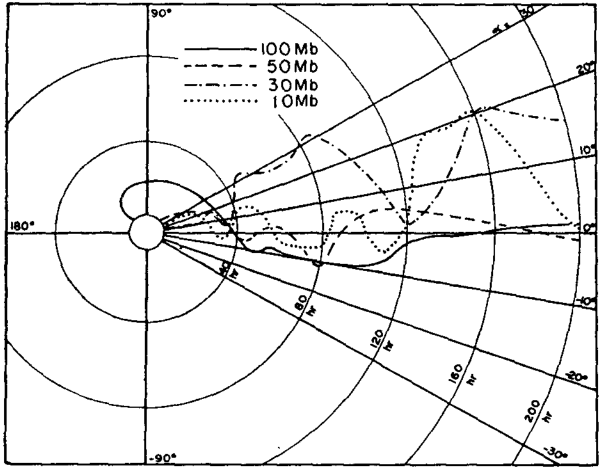

FIG. 15. Angle between the principal axis of diffusion and the x axis.

$\alpha(t)$ between the principal axis of diffusion and the x axis can be determined from the values of the components of the relative displacement tensor at various times after the release of clusters. They are computed at various levels and are plotted in Fig. 15. It may be noted that at the 10- and 30-mb levels, the angle α between the principal axis of diffusion and the x axis could be as large as 30°. With the use of the values of the mean square of the relative zonal and meridional particle displacement together with the orientation of the principal axis of diffusion, the concentration distribution of a cluster of particles at various times can be predicted.

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REFERENCES

Kao, S.-K., 1962: Large-scale turbulent diffusion in a rotating fluid with application to the atmosphere. *J. Geophys. Res.*, **67**, 2347–2359.

——, 1965: Some aspects of the large-scale turbulence and diffusion in the atmosphere. *Quart. J. Roy. Meteor. Soc.*, **91**, 10–17.

——, 1968: Relative dispersion of particles in a stratified, rotating atmosphere. *J. Atmos. Sci.*, **25**, 481–487.

—— and A. A. al-Gain, 1968: Large scale dispersion of clusters of particles in the atmosphere. *J. Atmos. Sci.*, **25**, 214–221.

Richardson, L. F., 1926: Atmospheric diffusion shown on a distance-neighbor graph. *Proc. Roy. Soc.* (*London*), **A110**, 709–737.