NOTES AND CORRESPONDENCE

The Shape of Internal Waves of Finite Amplitude from High-Resolution Radar Sounding of the Lower Atmosphere

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ABSTRACT

A high-resolution, vertically pointing FM/CW radar is used to record internal gravity waves in the lower atmosphere. When the temperature inversion in the atmosphere is near the ground (measured in wavelengths of the gravity waves), the shape of the waves indicates that nonlinear effects become important. Examples of such waves are shown and their shape is discussed. Theoretical results of Hunt, based on the Stokes method of approximating the solutions for waves of finite amplitude, are used to compare observation with theory.

This note describes internal waves in the lower troposphere when the amplitude of the waves cannot be considered to be small compared with the depth of the layer between the ground and the density discontinuity. As in the case of water waves, the crests are sharpened relative to the troughs due to nonlinearity in the equations when the depth of the lower fluid is small. The observations were made with the Richter radar sounding system (Richter, 1969).

It has already been pointed out by Gossard et al. (1970) that atmospheric waves are occasionally observed which show evidence of strong nonlinear effects. The effect is especially pronounced in Fig. 2 of that paper. The Stokes method of approximating the wave solutions for surface waves of finite amplitude has been extended by Hunt (1961) to the theory of internal waves at a fluid interface. Hunt's results, for wave shape to third order for a model in which the upper fluid is infinite and the lower fluid is of depth $h$, are described by

$$\frac{\xi}{a} = \cos(kx - \omega t) + A_2 \cos(2kx - 2\omega t) + A_3 \cos(3kx - 3\omega t),$$

where

$$A_2 = \frac{ka}{1 + T} \left[ 1 + \frac{\Delta \rho}{\rho} \frac{3 - T^2}{4T^3} \right],$$

$$A_3 = \frac{k^2 a^2}{(1 + 4T^2)} \left[ \left( 1 + \frac{\Delta \rho}{\rho} \frac{3 - T^2}{4T^3} \right) \left( 1 + \frac{9 + 4T^2 + 3T^4}{4T^2(3 + T^2)} \right) \right] \left[ \frac{1}{1 + \frac{\Delta \rho}{\rho} \frac{2(3 - T^2)}{T^3(3 + T^2)}} \right].$$

$$T = \tanh kh,$$

and where $\rho$ is density, $\Delta \rho$ the difference in density between the fluids, $\xi$ displacement, and $a$ the amplitude of the wave for the infinitesimal amplitude approximation. It is assumed that $\Delta \rho \ll \rho$. In Eq. (1), notation similar to that of Thorpe (1968) has been substituted for that of Hunt, but certain misprints have been corrected. For the atmospheric case, the compressibility can be taken into account to first order by substituting potential temperature for density; however, $\Delta \rho / \rho$ enters very weakly in (1) as $kh$ becomes very small, and for realistic discontinuities the shape of the waves becomes independent of the magnitude of the potential temperature discontinuity.

The theory further predicts that the dispersion relation for phase velocity to third order is given by

$$\frac{\omega}{k} = \frac{g}{\rho} \frac{T}{1 + T} \left[ 1 + \frac{(ak)^2}{8T} \right],$$

$$\frac{T}{9 - 22T + 13T^2 + 4T^3}.$$
where \( \rho \) and \( u \) are the wave perturbations of pressure and wind, and \( \rho \) is the unperturbed density. The pressure sensor was at 31 m MSL and the wind measurements were made on a mast at an elevation of 55 m. The meteorological records that accompany the 13 November radar record are repeated here as Fig. 1. Although only rigorously correct for small-amplitude waves, Eq. (3) is used to estimate the phase velocity as 7.2 m sec\(^{-1}\) from the south-southwest. Since the ambient wind was northwesterly during the period, and therefore nearly parallel to the wave crests, the observed frequency is probably Doppler shifted by only a small amount. Therefore, the wavelength is calculated to be \( \sim 3.4 \) km.

The way a wave changes shape as the lower layer \( h \) proceeds from deep to shallow is shown in Fig. 2 where values of the wavelength \( L \) and amplitude \( a \) are chosen as 3 km and 80 m, respectively, to be representative of the 13 November record. Therefore, \( h = 400 \) m when \( a/h = 0.2 \) and 200 m when \( a/h = 0.4 \). Since \( h = 200 \) m on 13 November, the lower frame should be compared with the radar record. The crests for \( h = 200 \) m are significantly sharpened and the troughs broadened. The crest-to-trough amplitude is increased from 160 to 184 m. In Fig. 3, the theoretical curve has been superimposed (slightly displaced) on the radar record to facilitate comparison of observation with theory.

If wave velocity is to be calculated from (3), a value for \( \Delta \rho/\rho \) must be assumed since no local balloon sound-

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**Fig. 1.** Radar echo from clear atmosphere and simultaneous surface meteorological measurements showing internal waves in lower atmosphere on 13 November 1969.

The factor in brackets represents the modification of the infinitesimal amplitude phase velocity that must be applied when \( ak \) becomes significantly large.

Radar returns from the clear atmosphere commonly show prominent internal waves. The radar return is directly related to sharp gradients in refractive index (moisture), but the exact mechanism of reflection or backscatter remains in doubt. The effects of finite amplitude on wave velocity and shape are best illustrated by the 13 November 1969 case of Paper I. For this case, the wave shape shown by the radar provides clear evidence of important nonlinear effects. As pointed out in Paper I, the phase velocity and direction of small-amplitude waves can be deduced very simply from surface measurements of pressure perturbation and wind perturbation when the lower layer is shallow. The equation is

\[
\omega = \frac{1}{k} \rho \frac{u}{\rho} = \frac{1}{k} \rho \frac{u}{\rho} \tag{3}
\]

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*The importance of this relation in the interpretation of fluctuations in surface pressure and wind was pointed out by Gosard and Munk (1954), and the reader is referred to this reference for a more complete discussion.*

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**Fig. 2.** Shape of waves of finite amplitude calculated from Eq. (1) for three thicknesses of lower layer. Vertical scale is ratio of displacement to corresponding "small-amplitude" displacement.
Fig. 3. Expanded photograph of radar record in Fig. 1 with wave shape (dashed curve) calculated for $a/h = 0.4$.

Fig. 4. Selected examples of "finite-amplitude" waves in the very low atmosphere. Whereas the upper waves show no evidence of "breaking," the waves in the lower frames show evidence of vortices behind the crests. The differences in form are probably a result of differences in wind shear. The discrete echoes (dot angels) are probably insects. As they move with the air currents they display air movements within the wave structure.

*The captive balloon appearing as an echo in the lower right-hand corner of the figure was torn free of its tether and lost.

ing was available.\(^3\) A typical value of potential temperature change across the inversion under similar meteorological conditions is 10K; thus, this was assumed in making order of magnitude calculations. The calculated velocity and the modification factor in the brackets of Eq. (2) are $a/k = 12.73 [1.0114]$, 10.34 [1.0071], and 7.22 [1.014] m sec\(^{-1}\), corresponding to the conditions assumed in the three frames of Fig. 2. The modification factor shows that finite amplitude causes little modification of the infinitesimal amplitude phase velocity, and the estimate from (3) should be accurate. Furthermore, the factor decreases as $h$ is reduced until $h$ becomes very small; it then increases with further reduction of $h$.

In summary, the radar picture of internal waves of finite amplitude indicates that the Stokes form of the nonlinear theory to third order provides an accurate representation of internal wave shape even under fairly extreme conditions. Furthermore, a simple two-layer model provides an adequate approximation of a heterogeneous medium to describe the wave characteristics discussed in this paper.

REFERENCES


