Multiple Scattering of Polarized Light in Planetary Atmospheres. Part I. The Doubling Method

JAMES E. HANSEN

Goddard Institute for Space Studies, NASA, New York, N. Y.

(Manuscript received 15 September 1970)

ABSTRACT

The doubling method is described for multiple scattering of light in plane-parallel atmospheres. The polarization of the radiation and the azimuth dependence are correctly accounted for. The method is practical for application to realistic simulations of clear, hazy and cloudy planetary atmospheres.

1. Introduction

Van de Hulst (1963) showed that if the solution is known for multiple scattering from a plane-parallel atmosphere of thickness τ_0, then this solution may be used to obtain solutions for layers of thickness $2\tau_0$, $4\tau_0$, etc., by a "doubling" procedure. For the relatively simple case in which the polarization of the light is neglected, this doubling principle has been developed and used, in one form or another, by van de Hulst and Grossman (1968), Hansen (1969), Hansen and Cheyney (1968), Hansen and Pollack (1970), Twomey *et al.* (1966), Irvine (1968) and Hunt and Grant (1969).

In this paper we extend the doubling method to a rather general case in which the polarization of the light is correctly accounted for. As in our previous paper (Hansen, 1969), we begin the numerical computations with a layer of such small optical thickness that the scattering and transmission matrices are given to a high accuracy by the phase matrix for single scattering. We have, however, made some changes in the scheme for writing down the doubling procedure in order to allow a closer correspondence to the work of van de Hulst (1963).

In Part II of this series (Hansen, 1971) we present the results of computations for the reflection of sunlight by terrestrial water clouds.

2. The doubling method

Consider a plane-parallel atmosphere of optical thickness τ illuminated uniformly from above by a parallel beam of radiation. The incident beam may be described by its Stokes parameters (Chandrasekhar, 1960; van de Hulst, 1957) which define a column vector of four elements

$$\begin{bmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{bmatrix} \equiv \pi \mathbf{F}, \tag{1}$$

where I_0 is the net flux per unit area of the incident beam, and Q_0, U_0 and V_0 are simply related to the degree of polarization, plane of polarization, and ellipticity of the incident beam (Chandrasekhar, 1960).

Directions are specified by a zenith angle, $\theta = \cos^{-1}\mu$, and an azimuth angle ϕ, where θ is measured from the outward normal to the surface and we use only values in the range $0° \leqslant \theta < 90° (0 < \mu \leqslant 1)$. Looking downward, ϕ is measured counterclockwise from an arbitrary but fixed direction. The direction of the incident beam is (μ_0, ϕ_0).

It is desired to find the Stokes parameters for the light diffusely reflected and transmitted by the atmosphere, i.e., $\mathbf{I}_r(0,\mu,\phi)$ and $\mathbf{I}_t(\tau,\mu,\phi)$, respectively; here $\mathbf{I}$ is a column matrix of four elements. It is convenient to define reflection and transmission matrices, each composed of four rows and four columns, such that

$$\mathbf{I}_r(0,\mu,\phi) = \mu_0 \mathbf{R}(\tau; \mu, \mu_0, \phi - \phi_0)\mathbf{F}, \tag{2}$$

$$\mathbf{I}_t(\tau,\mu,\phi) = \mu_0 \mathbf{T}(\tau; \mu, \mu_0, \phi - \phi_0)\mathbf{F}. \tag{3}$$

If $\mathbf{R}$ and $\mathbf{T}$ are multiplied by $4\mu\mu_0$, they become identical to Chandrasekhar's $\mathbf{S}$ and $\mathbf{T}$; these matrices as well as Chandrasekhar's satisfy symmetry relationships (Hovenier, 1969).

Now we could write two long equations giving the reflection and transmission matrices, $\mathbf{R}(2\tau)$ and $\mathbf{T}(2\tau)$, for a layer of thickness 2τ, in terms of the corresponding matrices $\mathbf{R}$ and $\mathbf{T}$, for a layer of thickness τ, as we did in our previous paper (Hansen, 1969). Here, however, we break up the equations following the scheme of van

de Hulst (1963) by letting

$$\mathbf{Q}_1=\mathbf{R}^*\mathbf{R}, \tag{4}$$

$$\mathbf{Q}_n=\mathbf{Q}_1\mathbf{Q}_{n-1}, \tag{5}$$

$$\mathbf{S}=\sum_{n=1}^{\infty}\mathbf{Q}_n, \tag{6}$$

$$\mathbf{D}=\mathbf{T}+e^{-\tau/\mu_0}\mathbf{S}+\mathbf{S}\mathbf{T}, \tag{7}$$

$$\mathbf{U}=e^{-\tau/\mu_0}\mathbf{R}+\mathbf{R}\mathbf{D}. \tag{8}$$

Then

$$\mathbf{R}(2\tau)=\mathbf{R}+e^{-\tau/\mu}\mathbf{U}+\mathbf{T}^*\mathbf{U}, \tag{9}$$

$$\mathbf{T}(2\tau)=e^{-\tau/\mu}\mathbf{D}+e^{-\tau/\mu_0}\mathbf{T}+\mathbf{T}\mathbf{D}. \tag{10}$$

In (4)–(10) all of the matrices have four rows and four columns; an arbitrary matrix of this type, say $\mathbf{X}$, stands for

$$\mathbf{X}=X^{ij}(\mu,\mu_0,\phi-\phi_0), \quad i,j=1,2,3,4, \tag{11}$$

where i and j are the indices for the rows and columns, respectively. The product of two matrices implies matrix multiplication and integration over the adjoining angles, an arbitrary $\mathbf{Z}=\mathbf{X}\mathbf{Y}$ being defined as

$$Z^{ij}(\mu,\mu_0,\phi-\phi_0)$$
$$=\frac{1}{\pi}\int_0^1\int_0^{2\pi}\left[\sum_{k=1}^{4}X^{ik}(\mu,\mu',\phi-\phi')Y^{kj}(\mu',\mu_0,\phi'-\phi_0)\right]\times\mu' d\mu' d\phi'. \tag{12}$$

$\mathbf{R}^*$ and $\mathbf{T}^*$ are the reflection and transmission matrices for layers of thickness τ when the layer is illuminated from below. In general $\mathbf{R}^*\neq\mathbf{R}$ and $\mathbf{T}^*\neq\mathbf{T}$, but, since we are considering homogeneous layers, the simple relations

$$\mathbf{R}^*(\mu,\mu_0,\phi-\phi_0)=\mathbf{R}(\mu,\mu_0,\phi_0-\phi), \tag{13}$$

$$\mathbf{T}^*(\mu,\mu_0,\phi-\phi_0)=\mathbf{T}(\mu,\mu_0,\phi_0-\phi), \tag{14}$$

are valid (Hovenier, 1969).

$\mathbf{D}$ and $\mathbf{U}$ correspond, respectively, to the diffuse radiation downward and upward at the mid-level of the combined layer of thickness 2τ. The sum in (6) is over the multiple reflections between the two layers of thickness τ, n indicating the number of times the radiation has crossed the middle boundary going up. In practice this sum is terminated after some finite number of terms, depending on the accuracy desired. Furthermore, the omitted terms may be approximated by the geometric formula because the ratio of successive terms approaches a constant value, as was found previously in the case in which polarization is neglected (van de Hulst, 1963; Hansen, 1969). Thus,

$$\mathbf{S}\approx\sum_{n=1}^{N-1}\mathbf{Q}_n+\frac{\mathbf{Q}_N}{1-\eta}, \tag{15}$$

where

$$\eta=Q_N^{ij}/Q_{N-1}^{ij}, \tag{16}$$

and N is a small integer (usually $\lesssim 5$) taken as large as accuracy requirements demand. As suggested by the notation in (16) the ratio η depends on the matrix element, but it does not depend significantly on angle; hence it may be computed for convenient angles, say, $\mu=\mu_0=0.5$.

3. Initial layer for computations

The above equations provide the reflection and transmission matrices for layers of increasing thickness. As indicated in our previous paper (Hansen, 1969), one convenient way to begin the computations is with a layer of such small optical thickness that multiple scattering may be neglected in the starting layer. The reflection and transmission matrices for single scattering are given by:

$$\mathbf{R}_1(\tau;\mu,\mu_0,\phi-\phi_0)$$
$$=\frac{\tilde{\omega}_0}{4(\mu+\mu_0)}\left\{1-\exp\left[-\tau\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right]\right\}\times\mathbf{P}(\mu,\mu_0,\phi-\phi_0), \tag{17}$$

$$\mathbf{T}_1(\tau;\mu,\mu_0,\phi-\phi_0)$$
$$=\frac{\tilde{\omega}_0}{4(\mu-\mu_0)}\left[\exp\left(\frac{-\tau}{\mu}\right)-\exp\left(\frac{-\tau}{\mu_0}\right)\right]\times\mathbf{P}(\mu,\mu_0,\phi-\phi_0), \text{ if } \mu\neq\mu_0, \tag{18}$$

$$\mathbf{T}_1(\tau;\mu,\mu_0,\phi-\phi_0)$$
$$=\frac{\tilde{\omega}_0\tau}{4\mu_0^2}\exp\left[\frac{-\tau}{\mu_0}\right]\mathbf{P}(\mu,\mu_0,\phi-\phi_0), \text{ if } \mu=\mu_0. \tag{19}$$

Hence for a very thin starting layer, $\tau_0\ll 1$, we may take

$$\mathbf{R}(\tau_0;\mu,\mu_0,\phi-\phi_0)$$
$$=\frac{\tilde{\omega}_0\tau_0}{4\mu\mu_0}\left[1-\frac{\tau_0}{2}\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right]\mathbf{P}(\mu,\mu_0,\phi-\phi_0), \tag{20}$$

$$\mathbf{T}(\tau_0;\mu,\mu_0,\phi-\phi_0)$$
$$=\frac{\tilde{\omega}_0\tau_0}{4\mu\mu_0}\left[1-\frac{\tau_0}{2}\left(\frac{1}{\mu}+\frac{1}{\mu_0}\right)\right]\mathbf{P}(\mu,\mu_0,\phi-\phi_0), \tag{21}$$

where $\mathbf{P}(\mu,\mu_0,\phi-\phi_0)$ is the phase matrix referred to the meridian plane, which is the plane containing the direction of emergence (μ,ϕ) and the local normal. $\mathbf{P}$ is

normalized such that its integral over all directions is equal to 4π (Hansen, 1971).

The phase matrix is usually known with reference to the scattering plane as a function of the total scattering angle for single scattering, α, where

$$\cos\alpha=\pm\mu\mu_0+(1-\mu^2)^{\frac{1}{2}}(1-\mu_0^2)^{\frac{1}{2}}\cos(\phi-\phi_0), \quad (22)$$

where the plus sign applies for transmission and the minus for reflection. As shown by Chandrasekhar (1960), the phase matrix as a function of $(\mu, \mu_0, \phi-\phi_0)$ may be obtained by pre- and post-multiplying $\mathbf{P}(\alpha)$ by appropriate rotation matrices. Since we are working with the Stokes parameters given in (1), this transformation has the form

$$\mathbf{P}(\mu,\mu_0,\phi-\phi_0)=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos2i_2 & -\sin2i_2 & 0\\ 0 & \sin2i_2 & \cos2i_2 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

$$\times\mathbf{P}(\alpha)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & \cos2i_1 & -\sin2i_1 & 0\\ 0 & \sin2i_1 & \cos2i_1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \quad (23)$$

The rotation angles i_1 and i_2, may be derived from the spherical geometry in several different ways (Chandrasekhar, 1960; Hovenier, 1969); we use the relations

$$\cos i_1=\frac{\pm\mu(1-\mu_0^2)^{\frac{1}{2}}-\mu_0(1-\mu^2)^{\frac{1}{2}}\cos(\phi-\phi_0)}{(1-\cos^2\alpha)^{\frac{1}{2}}}, \quad (24)$$

$$\cos i_2=\frac{\mu_0(1-\mu^2)^{\frac{1}{2}}\mp\mu(1-\mu_0^2)^{\frac{1}{2}}\cos(\phi-\phi_0)}{(1-\cos^2\alpha)^{\frac{1}{2}}}. \quad (25)$$

Eqs. (24) and (25) are valid for $0<\phi-\phi_0<\pi$; the upper signs apply for transmission and the lower signs for reflection. For the special case $\mu=\mu_0=1$ we may use $\cos i_1=-\cos(\phi-\phi_0)$ and $\cos i_2=1$ for both transmission and reflection.

When only single scattering is included in the starting layer, τ_0 should be $\sim 2^{-20}$, depending on the accuracy demanded. The doubling may, however, be started at any τ_0 for which $\mathbf{R}(\tau_0)$ and $\mathbf{T}(\tau_0)$ are known. Since there are a number of computing methods which are satisfactory for optical thicknesses $\lesssim 1$, one of these may be employed to obtain $\mathbf{R}(\tau_0)$ and $\mathbf{T}(\tau_0)$; this may save many doublings at the expense of added programming complexity.

4. Fourier series expansion

It is advantageous to expand the azimuth-dependent functions in Fourier series in $\phi-\phi_0$. Each term in the Fourier series may then be treated independently, allowing large savings in computer storage requirements. Furthermore, the physical behavior of different Fourier terms differs markedly (van de Hulst, 1971; Hansen and Pollack, 1970), and this may be used to reduce the computing time (Hansen and Pollack).

Let $\mathbf{X}$ be any of the matrices in (4)–(10); we may expand it as[1]

$$X^{ij}(\mu,\mu_0,\phi-\phi_0)={}^0X_c{}^{ij}(\mu,\mu_0)$$

$$+2\sum_{m=1}^{\infty}\left[{}^mX_c{}^{ij}(\mu,\mu_0)\cos m(\phi-\phi_0)\right.$$

$$\left.+{}^mX_s{}^{ij}(\mu,\mu_0)\sin m(\phi-\phi_0)\right], \quad (26)$$

where

$${}^mX_c{}^{ij}(\mu,\mu_0)=\frac{1}{2\pi}\int_0^{2\pi}X^{ij}(\mu,\mu_0,\phi')\cos m\phi'd\phi', \quad m=0,1,2,\cdots, \quad (27)$$

$${}^mX_s{}^{ij}(\mu,\mu_0)=\frac{1}{2\pi}\int_0^{2\pi}X^{ij}(\mu,\mu_0,\phi')\sin m\phi'd\phi', \quad m=1,2,\cdots. \quad (28)$$

The rule for matrix multiplication, given by (12), becomes

$${}^0Z_c{}^{ij}(\mu,\mu_0)=2\int_0^1\sum_{k=1}^{4}{}^0X_c{}^{ik}(\mu,\mu'){}^0Y_c{}^{kj}(\mu',\mu_0)\mu'd\mu', \quad (29)$$

$${}^mZ_c{}^{ij}(\mu,\mu_0)=2\int_0^1\left\{\sum_{k=1}^{4}\left[{}^mX_c{}^{ik}(\mu,\mu'){}^mY_c{}^{kj}(\mu',\mu_0)\right.\right.$$

$$\left.\left.-{}^mX_s{}^{ik}(\mu,\mu'){}^mY_s{}^{kj}(\mu',\mu_0)\right]\right\}\mu'd\mu', \quad (30)$$

$${}^mZ_s{}^{ij}(\mu,\mu_0)=2\int_0^1\left\{\sum_{k=1}^{4}\left[{}^mX_c{}^{ik}(\mu,\mu'){}^mY_s{}^{kj}(\mu',\mu_0)\right.\right.$$

$$\left.\left.+{}^mX_s{}^{ik}(\mu,\mu'){}^mY_c{}^{kj}(\mu',\mu_0)\right]\right\}\mu'd\mu'. \quad (31)$$

Each of (4)–(10) hence represents three equations.

In (30) and (31) $m\gtrless 1$. While the case $m=0$ could be included in (30), with (29) omitted, it is worthwhile to program this azimuth-independent term separately. The azimuth-independent term behaves "conservatively;" i.e., η in (16) approaches unity as τ approaches infinity for the case $\tilde{\omega}_0=1$. Therefore, this term is sensitive to small errors, due, e.g., to the truncation in (15) or the replacement of integrals by finite summations, and it must be treated with greater care to obtain an accuracy consistent with that of the higher Fourier terms.

The matrices $\mathbf{R}^*$ and $\mathbf{T}^*$ do not need to be explicitly calculated because, according to (13), (14) and (26),

[1] A misprint occurs in Eq. (10) of our earlier paper (Hansen, 1969). There the factor $1/(2-\delta_{0,m})$ should read $(2-\delta_{0,m})/2\pi$. The correct equation was used in computations reported there and in other papers.

they differ from **R** and **T** only in the sign of the sine coefficients in the Fourier expansions; thus,

$$\left.\begin{aligned} {}^{m}R_c^{*ij}(\mu,\mu_0) &= {}^{m}R_c^{ij}(\mu,\mu_0) \\ {}^{m}T_c^{*ij}(\mu,\mu_0) &= {}^{m}T_c^{ij}(\mu,\mu_0) \end{aligned}\right\}, \quad m=0, 1, 2, \cdots, \quad (32)$$

$$\left.\begin{aligned} {}^{m}R_s^{*ij}(\mu,\mu_0) &= -{}^{m}R_s^{ij}(\mu,\mu_0) \\ {}^{m}T_s^{*ij}(\mu,\mu_0) &= -{}^{m}T_s^{ij}(\mu,\mu_0) \end{aligned}\right\}, \quad m=1, 2, \cdots. \quad (33)$$

We have made computations only for the special case in which the elements of the 4 by 4 phase matrix, $P^{ij}(\mu,\mu_0,\phi-\phi_0)$, are even functions of $\phi-\phi_0$ for the 2 by 2 submatrices (ij=11,12,21,22) and (ij=33,34, 43,44), and odd functions of $\phi-\phi_0$ for the submatrices (ij=13,14,23,24) and (ij=31,32,41,42); the even functions in $\phi-\phi_0$ contain cosine terms only and the odd functions contain sine terms only. This special case includes Rayleigh scattering and anisotropic Rayleigh scattering, as well as scattering by spherical particles and scattering by randomly oriented nonspherical particles which have a plane of symmetry (Hovenier, 1969). From (17)–(19) and the doubling equations, it is clear that if **P** has this special form, all of the 4 by 4 matrices above will have it. In this case (27) and (28) simplify to

$${}^{m}X_c^{ij}(\mu,\mu_0) = \frac{1}{\pi}\int_0^{\pi} X^{ij}(\mu,\mu_0,\phi') \cos m\phi' d\phi', \quad m=0, 1, 2 \cdots, \quad (34)$$

$${}^{m}X_s^{ij}(\mu,\mu_0) = \frac{1}{\pi}\int_0^{\pi} X^{ij}(\mu,\mu_0,\phi') \sin m\phi' d\phi', \quad m=1, 2 \cdots. \quad (35)$$

The integrations in (29), (30), (31), (34) and (35) may be performed numerically; we have used Gauss quadrature. This integration method is the most efficient of the numerical schemes which we have tested for these integrations and it has the additional modest advantage that it does not employ the endpoints of the integration interval; thus, the special cases $\phi-\phi_0=0$ and π need not be programmed for (24) and (25) because the phase matrix is not needed for those angles.

It is worthwhile to compare the above equations to those which result if polarization is neglected; by the phrase "polarization is neglected" we refer to the case in which an approximate value for the intensity is computed by treating it as a scalar independent of the other Stokes parameters. This approximation is often employed when only the intensity, and not the other Stokes parameters, is desired; it is very accurate for scattering by particles having a mean size at least on the same order as the wavelength (Hansen, 1971). In this approximation the matrices and vectors in the preceding equations are replaced by scalars; the phase matrix becomes the phase function which is equal to the matrix element in the first row and first column of the phase matrix. No transformation of the phase function, such as that indicated by (23) for the phase matrix, is necessary because the intensity is invariant for a rotation of the coordinate axis; i.e., $P(\mu, \mu_0, \phi-\phi_0)\equiv P(\alpha)$. For scalars the summations over k in (12), (29), (30) and (31) reduce to one term.

If, further, the phase function is an even function of $\phi-\phi_0$, as is true in the case for which we have made computations, then all of the sine coefficients (X_s, Y_s, etc.) are zero; thus, (30) may be simplified and (31) may be dropped. The difference between illumination from above and illumination from below then disappears, as indicated by (32), and the asterisks may be omitted from the doubling equations (4)–(10).

5. Discussion

In Part II (Hansen, 1971) we will illustrate that it is practical to use the doubling method on problems involving highly asymmetric phase matrices typical of planetary atmospheres; it is not difficult to obtain accuracies comparable to or better than those obtained observationally (two or three significant figures).

There are a number of other computational methods which have been successfully used for multiple scattering in planetary atmospheres, some including polarization, and a comparison of some of these to the doubling method is in order.

Chandrasekhar (1960) used an ad hoc method based on mathematical brilliance to force a solution to the problem of Rayleigh scattering. The solution depended on functions of only a single angle ("H functions," etc.); hence the method was tractable for small computers. It has not yet been shown that a *practical* extension of this approach can be made to problems involving a more general phase matrix such as that occurring for spherical particles.

The Monte Carlo method, used most prolifically by Kattawar and Plass (1968), is versatile and provides a tractable approach to a number of problems, including some problems involving unusual geometries. It is useful for more standard problems, such as that of a plane parallel homogeneous atmosphere, if a high accuracy is not demanded. A disadvantage of the Monte Carlo method is that it yields the intensity and polarization averaged over angular intervals, usually several degrees in width; the angular intervals may be reduced in size, but at the expense of a still lower accuracy or an increase in computer time. Hence it may be difficult for the Monte Carlo method to extract the full information available in precise polarization observations, particularly for optically thick atmospheres.

Dave (1970) has recently published polarization calculations made with the Gauss-Seidel iteration

scheme which was used previously by Herman (1965). Dave has succeeded in getting a reasonable accuracy (usually ~ 2 significant figures) for multiple scattering by monodisperse Mie particles; the method can also be used for polydispersions. The method can be reliable for a layer of moderate optical thickness ($\tau \lesssim 5$–10). By itself it cannot be practically used in calculations for scattering by thick clouds such as those typically occurring on Earth and on Venus.

Calculations in successive orders of scattering are very advantageous for some problems, particularly absorption line formation. In the classical method [that used, e.g., by Dave (1964)], integrations over optical depth, as well as integrations over angle, must be performed for each order of scattering; hence the method is limited to moderately thick layers ($\tau \lesssim 5$–10). Uesugi and Irvine (1969) have introduced a new and important method for computing successive orders of scattering in a semi-infinite homogeneous layer; their method involves integrations only over angle and it is capable of a high accuracy. However, if the method of Uesugi and Irvine is extended to include polarization, it takes a great amount of computer time and storage (machine memory) space when the single scattering albedo is close to unity.

The "invariant imbedding" method has been used by Bellman *et al.* (1966), Adams and Kattawar (1970), and others. The method involves numerically solving nonlinear integro-differential equations for the reflection and transmission matrices; physically this amounts to repeatedly adding thin layers to the atmosphere. The method is hence well suited for application to an inhomogeneous atmosphere; it is computationally stable unless both $\bar{\omega}_0 \approx 1$ and $\tau \to \infty$. However, the computer time is nearly proportional to the optical thickness, and hence with a general phase matrix the method would be practical only for a thin layer.

Abhyankar and Fymat (1970a,b,c) have developed a perturbation method for multiple scattering and they have successfully used it for isotropic and Rayleigh scattering. It remains to be demonstrated that this method can be practically used for problems involving general phase matrices.

Most of the computational methods presently used for multiple scattering by planetary atmospheres, as suggested by the above, encounter difficulties for the common case of a thick but finite planetary atmosphere. In the doubling method the optical thickness increases geometrically, and hence, as indicated by van de Hulst and Grossman (1968), the method is well suited for thick layers. Our calculations (Hansen, 1971) demonstrate that the doubling method is useful for many practical computations.

The doubling equations, as we have presented them, are valid for homogeneous atmospheres. With obvious modifications the same method may be used to add two dissimilar layers, and hence to build up an inhomogeneous atmosphere. However since this adversely affects a primary advantage of the doubling method, i.e., its computing speed, it will be difficult to effectively use the method for an atmosphere composed of many different layers.

With the doubling method the computer time required to obtain a given accuracy increases significantly as the phase matrix becomes more strongly dependent on the scattering angle; this happens, for example, when larger particles are considered. The same difficulty occurs, to a greater or lesser extent, in all of the multiple scattering methods discussed above. Therefore, in the case of scattering by large particles, it is important to take advantage of shortcuts in the numerical work; some suggestions are given by Hansen and Pollack (1970). Further discussions for the case of polarized light will be given in future publications.

Note added in proof. Howell and Jacobowitz (1970) have recently published computations for Rayleigh scattering made with their version of the doubling method. Their results, obtained for $\tau \leqslant \frac{1}{4}$, are no more accurate than those obtained by Collins (1968) with the Monte Carlo method; this represents an apparent contradiction to our statements above. Actually the doubling method, in the form described in this paper, can easily yield accuracies of 4–5 significant digits for Rayleigh scattering; this is true even in the most difficult case, ($\bar{\omega}_0 = 1$, $\tau \to \infty$), as indicated by comparisons to Chandrasekhar (1960) and Abhyankar and Fymat (1970c). It appears that Howell and Jacobowitz (1970) did not correctly handle the azimuth dependence. They do not mention the necessity for distinguishing between illumination from above and illumination from below [Eqs. (13), (14), (32) and (33)]; these equations are a statement of the fact that $\mathbf{R}$ and $\mathbf{T}$ for a homogeneous atmosphere are the same for illumination from below as for illumination from above only if, relative to a fixed coordinate system, the azimuth angle is reckoned in opposite senses in the two situations. That this distinction is necessary even for homogenous atmospheres was shown by Hovenier (1969). The notation for the azimuth dependence in the present standard text (Chandrasekhar, 1960) is, in the most generous interpretation, ambiguous; this may have increased the probability for errors.

Acknowledgments. I would like to thank J. W. Hovenier and H. C. van de Hulst for many valuable discussions. I was supported by a National Science Foundation Fellowship at Leiden Observatory, Netherlands, for six months in 1969, and am now supported by NASA Grant 33-008-012 through Columbia University.

REFERENCES

Abhyankar, K. D., and A. L. Fymat, 1970a: Theory of radiative transfer in inhomogeneous atmospheres. III. Extension of the perturbation method to azimuth-independent terms. *Astrophys. J.*, **159**, 1009–1018.

——, and ——, 1970b: Theory of radiative transfer in inhomogeneous atmospheres. IV. Application of the matrix perturbation method to a semi-infinite atmosphere. *Astrophys. J.*, **159**, 1019–1028.

——, and ——, 1970c: Imperfect Rayleigh scattering in a semi-infinite atmosphere. *Atron. Astrophys.*, **4**, 101–110.

Adams, C. N., and G. W. Kattawar, 1970: Solutions of the equations of radiative transfer by an invariant imbedding approach. *J. Quant. Spectry. Radiative Transfer*, **10**, 341–356.

Bellman, R., H. Kagiwada, R. Kalaba and S. Ueno, 1966: Numerical results for Chandrasekhar's X and Y functions of radiative transfer. *J. Quant. Spectry. Radiative Transfer*, **6**, 479–500.

Chandrasekhar, S., 1960: *Radiative Transfer*. New York, Dover, 393 pp.

Collins, D. G., 1968: Study of polarization of atmospheric scattered light using Monte Carlo methods. Radiation Research Associates, Rept. RRA-T86 (AFCRL-68-0310).

Dave, J. V., 1964: Meaning of successive iteration of the auxiliary equation in the theory of radiative transfer. *Astrophys. J.*, **140**, 1292–1303.

——, 1970: Intensity and polarization of the radiation emerging from a plane-parallel atmosphere containing monodispersed aerosols. *Appl. Opt.*, **9**, 2673–2684.

Hansen, J. E., 1969: Radiative transfer by doubling very thin layers. *Astrophys. J.*, **155**, 565–573.

——, 1971: Multiple scattering of polarized light in planetary atmospheres. Part II. Sunlight reflected by terrestrial water clouds (submitted to *J. Atmos. Sci.*).

——, and H. Cheyney, 1968: Near infrared reflectivity of Venus and ice clouds. *J. Atmos. Sci.*, **25**, 629–633.

——, and J. B. Pollack, 1970: Near infrared light scattering by terrestrial clouds. *J. Atmos. Sci.*, **27**, 265–281.

Herman, B. J., 1965: Multiple scatter effects on the radar return from large hail. *J. Geophys. Res.*, **70**, 1215–1225.

Hovenier, J. W., 1969: Symmetry relationships for scattering of polarized light in a slab of randomly oriented particles. *J. Atmos. Sci.*, **26**, 488–499.

Howell, H. B., and H. Jacobowitz, 1970: Matrix method applied to the multiple scattering of polarized light. *J. Atmos. Sci.*, **27**, 1195–1206.

Hunt, G. E., and I. P. Grant, 1969: Discrete space theory of radiative transfer and its application to problems in planetary atmospheres. *J. Atmos. Sci.*, **26**, 963–972.

Irvine, W. M., 1968: Multiple scattering by large particles. II. Optically thick layers. *Astrophys. J.*, **152**, 823–834.

Kattawar, G. W., and G. N. Plass, 1968: Radiance and polarization of multiple scattered light from haze and clouds. *Appl. Opt.*, **7**, 1519–1527.

Twomey, S. H. Jacobowitz and H. B. Howell, 1966: Matrix methods for multiple scattering problems. *J. Atmos. Sci.*, **23**, 289–296.

Uesugi, A., and W. M. Irvine, 1969: Multiple scattering in a plane-parallel atmosphere. I. Successive scattering in a semi-infinite layer. *Astrophys. J.*, **159**, 127–135.

van de Hulst, H. C., 1957: *Light Scattering by Small Particles*. New York, Wiley, 470 pp.

——, 1963: A new look at multiple scattering. Tech. Rept., Goddard Institute for Space Studies, NASA, New York, 81 pp.

——, 1971: Multiple scattering in planetary atmospheres. *J. Quant. Spectry. Radiative Transfer*, **11** (in press).

——, and K. Grossman, 1968: Multiple light scattering in planetary atmospheres. *The Atmospheres of Venus and Mars*, New York, Gordon and Breach, 288 pp.