Ageostrophic Stability of Divergent Jets

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ABSTRACT

The barotropic stability of divergent jets is studied under the condition of small but nonzero Rossby number. Small disturbances to the basic state consist of fast waves of inertia-gravity type and slow waves with speeds comparable to those of the basic current. It is shown that the fast waves are stable and do not amplify. It is also shown, for a velocity profile resembling that of the Gulf Stream, that the slow waves are destabilized by ageostrophic effects in the sense that waves of fixed wavenumber which are neutral for zero Rossby number amplify for nonzero Rossby number. For very shallow jets it is found that the complex wave velocity is small and that disturbances will be neutrally stable unless the jet has a countercurrent.

1. Introduction

This paper is devoted to a re-examination of the barotropic stability of divergent jets with special reference to the stability of the Gulf Stream. Despite the many recent studies of this problem there are a number of unresolved questions, even in situations for which the quasi-geostrophic theory would seem to apply. In addition, for the Gulf Stream problem, there is a legitimate doubt that the quasi-geostrophic theory is valid even as a low-order approximation. Since intuition alone is unlikely to suffice in resolving these matters, an analytical treatment is presented.

The first question to be discussed is that of the stability of inertia-gravity waves. The fact that these waves are unimportant is exploited in all of the models used for simplifying the primitive equations; the reason why they are unimportant is an open question, at least as far as formal mathematical proof is concerned. It will be shown below, when the Rossby number is small, that the class of perturbations corresponding to inertia-gravity waves is stable and will never amplify, at least under barotropic conditions. This result provides a formal justification for ignoring inertia-gravity waves.

Next, we will consider the effect of small but nonzero Rossby number in modifying the stability characteristics of quasi-geostrophic disturbances to a velocity profile resembling that of the Gulf Stream. There are a number of reasons for doubting the validity of quasi-geostrophic theory as applied to this problem; nevertheless, such treatments have been made (Stern, 1961; Lipps, 1963) and it is of interest to calculate the effects due to finite Rossby number. The result of the calculation is to prove wrong Lipps' speculation that the reason his computed growth rates are higher than observed (cf. Stommel, 1965, p. 196; also, Gulf Stream Summaries) is due to his neglect of higher order Rossby number effects. The Rossby number effect, for the profile studied, in fact, proves to be destabilizing, at least for waves with small growth rates.

Finally, we will consider a situation in which the Rossby number is small but the fluid is still divergent. This appears to be more applicable to the Gulf Stream problem than the usual quasi-geostrophic theory. The conclusion reached for this model is that the existence of a countercurrent is necessary for instability.

2. Formulation

We consider a two-layer fluid on the $\beta$ plane, the density $\rho$ of each layer being constant. Let $x$, $y$ and $z$ measure distance to the east, the north and the vertical, and let subscripts 1 and 2 refer to the upper and lower layers, respectively. The layers are of finite depth, with depths $D_{a}$ ($a=1,2$) in the absence of motion. We take the upper boundary of the fluid to be the free surface, $z=D_{2}+\eta$, where $\Delta \rho=\rho_{2}-\rho_{1}$, the interface between the layers to be $z=D_{2}+\eta=\rho_{1}/\rho_{2}$, and the lower boundary to be the rigid plane $z=0$. Also, we assume shallow water theory to be valid and denote the horizontal velocity by $u_{a}$ and the material derivative by $D/Dt_{a}$, i.e.,

$$\frac{D}{Dt_{a}} = \left( \frac{\partial}{\partial t} + u_{a} \cdot \nabla \right).$$

The equations of motion are then

$$\frac{Dq_{a}}{Dt_{a}} + f k \times q_{a} + g \nabla \eta_{a} = 0,$$

(1)

$^{3}$ No summation convention.
\[
\frac{Dh_\alpha}{Dt} + h_\alpha \nabla \cdot q_\alpha = 0, \tag{2}
\]

where the depths \(h_1\) and \(h_2\) are given by
\[
h_1 = D_1 + \eta_1 - \eta_2, \quad h_2 = D_2 + \eta_2 - (\rho_1 / \rho_2) \eta_1, \tag{3}
\]
and where \(f = f_0 + \beta y\) is the Coriolis parameter, and \(g' = g \Delta \rho / \rho_2\) the reduced gravity.

Our aim is to study the stability of small perturbations to the flow, i.e.,
\[
q_1 = [VU(y/L), 0], \quad q_2 = 0,
\]
\(V\) being a characteristic velocity and \(L\) a characteristic length. Two approximations will be made at the outset. First, we will neglect entire motions in the lower layer and thus eliminate potential energy conversion as a source of instability. This is justified in the quasi-geostrophic case, provided that a source of kinetic energy is present and that \(D_1 \ll D_2\) (Pedlosky, 1964, Section 4). We assume that it is true also for the ageostrophic case. Secondly, we restrict our attention to horizontal length scales so small that \(L^2 \ll V / \beta\). It is then permissible to neglect \(\beta y\) as compared to \(f_0\), provided that we simulate the trapping effect due to the earth's curvature (Jacobs, 1967). This will be achieved here by supposing the fluid between walls at \(y = LA, y = LB, B > A\).

It is convenient at this point to scale the variables. Omitting subscripts, which are now superfluous since \(q_2 = \eta_2 = 0\) by assumption, we define nondimensional variables by
\[
(x, y) = (x^* / \sqrt{y^*}), \quad t = (L / V) t^*, \quad q = V q^*, \quad \eta = f_0 V L / (g') \eta^*, \quad h = Dh^*, \tag{4}
\]
The nondimensional equations obtained through use of this scaling, with asterisks omitted, are
\[
\frac{Dq}{Dt} + k \times q + \nabla \eta = 0, \tag{5}
\]
\[
\frac{Dh}{Dt} + h \nabla \cdot q = 0, \tag{6}
\]
where
\[
\hat{h} = 1 + \epsilon \gamma^2 \eta. \tag{7}
\]
The nondimensional parameters are the Rossby number,
\[
\epsilon = V / (f_0 L), \tag{8a}
\]
and a nondimensional inverse radius of deformation \(\gamma\), given by
\[
\gamma = f_0 L / (g' D)^{1/2}. \tag{8b}
\]
If we now take
\[
q = [U(y) + u, v], \quad \eta = \Phi + \varphi, \tag{9}
\]
where \(\Phi' = -U\), and neglect products of the perturbation quantities, we obtain the linear equations
\[
\epsilon \left( \frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right) (u, v) + (-\zeta, u) + \left( \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y} \right) = 0, \tag{10}
\]
\[
\epsilon \gamma \left( \frac{\partial}{\partial t} - U \frac{\partial}{\partial x} \right) \varphi + \frac{\partial (H u)}{\partial x} + \frac{\partial (H v)}{\partial y} = 0, \tag{11}
\]
where \(H\), the unperturbed depth of the layer, and \(\zeta\), the unperturbed total vorticity, are given by
\[
H = 1 + \epsilon \gamma^2 \Phi, \quad \zeta = 1 - \epsilon U'. \tag{12}
\]
As is usual in stability problems (see, however, Case, 1960), we bypass the initial value problem and instead separate variables according to
\[
F(x, y, t) = \tilde{F}(y) e^{i k(x - ct)}, \quad k \geq 0, \quad c = c_r + i c_i, \tag{13}
\]
where \(F\) is any of \(u, v\) or \(\varphi\). Eqs. (10) and (11) then become \(n\) homogeneous system of ordinary differential equations with the homogeneous boundary conditions \(\varphi(A) = \varphi(B) = 0\), and \(c\) plays the role of an eigenvalue. The values of \(c\) for which
\[
\int_A^B \left( |\tilde{u}|^2 + |\tilde{v}|^2 + |\tilde{\varphi}|^2 \right) dy \\
\]
will be called the spectrum for this system; if \(c_i > 0\) the motion is unstable.

Invoking (13) and eliminating \(\tilde{u}\), we obtain two equations for the two remaining unknowns. With the definitions
\[
w = U(y) - c, \quad \tilde{\psi} = H / (ik), \quad Q = (H - \epsilon \gamma^2 \omega)^{-1}, \quad \nabla \Phi / A = \nabla \varphi = 0, \tag{14}
\]
and with the tildes omitted, we have
\[
q(e w \varphi' - \varphi) - \psi = 0, \tag{15}
\]
\[
Q(e w \Phi' - \Phi) - \varphi = 0, \tag{16}
\]
to be solved subject to \(\psi(A) = \psi(B) = 0\). Alternate formulations in terms of a single equation for a single unknown are
\[
(Q \varphi')' - \left[ \frac{k^2}{H} + \frac{\gamma^2 Q}{H} \right] \frac{(Q \psi)'}{e w} \psi = 0, \tag{17}
\]
or
\[
(q \varphi')' - (k^2 q + q' / e w) \varphi = 0, \tag{18}
\]
the latter equation being subject to \(\varphi = e w \varphi'\) at \(y = A\) and \(y = B\).
We note that if $|c| = O(1)$ as $\epsilon \to 0$, then, in the limit, $\varphi = \psi$ and (17) becomes the quasi-geostrophic equation

$$w\psi'' - (k^2 w + U'' - \gamma c)\psi = 0,$$  \hspace{1cm} (19)$$

for which one can show that $c$ lies in a certain circle $\Gamma$ in the $\epsilon$ plane. If $U_1$ and $U_2$ are constants such that

$$U_1 \leq U(y) \leq U_2, \quad U_2 > 0,$$

then the center of $\Gamma$ is at $c = -\frac{1}{2}(U_2 + b)$ and its radius is $\frac{1}{2}(U_2 - b)$, where $b = U_1$ if $U_1 \leq 0$ and $b = 0$ otherwise. We will refer to $\Gamma$ a number of times in what follows.

3. Location of the spectrum

It would be desirable to proceed without making any further approximations. This is impractical, however, and we will be content instead to consider only the case $\epsilon < 1$. If we can show that for unstable waves $|c| = O(1)$ as $\epsilon \to 0$, then the quasi-geostrophic theory is a valid first approximation and may be improved upon by carrying out a perturbation expansion in powers of $\epsilon$. Accordingly, it is desirable to show that for unstable waves $c$ lies in the circle $\Gamma$, and this is the object of the present section.

It should be noted that it is untrue that all the eigenvalues lie in $\Gamma$. For example, if $U(y) = 0$, the spectrum consists of the points $c = 0$, the geostrophic mode, $e^{-\frac{1}{2}c^2}$, the Kelvin waves, and $e^{-\frac{1}{2}c^2} = 1 + k^2 \times \left[\frac{\gamma^2 + \mu^2}{(B-A)^2}\right]$, $n = 1, 2, \ldots$, the Poincaré waves. In general, even for $U(y) \neq 0$, we must anticipate the existence of eigenvalues $\epsilon$ such that $\epsilon|c| \neq 0$ in the limit $\epsilon \to 0$. This is because the solutions of the initial value problem defined by (10) and (11) must, during part of their temporal history, have a time scale of order $\epsilon^{-1}$ in order that all the initial conditions be satisfied, a fact which is reflected in the modal analysis. The question, then, is whether the fast waves have imaginary parts.

To deal with the fast waves we set $b = \epsilon c$ and regard $b$ and $\epsilon$ and $c$ as complex variables. We restrict $b$ and $\epsilon$ to lie in the domains $|b| > a_1$, $|\epsilon| \leq \epsilon_0 \leq a_2$, where $a_1$ and $a_2$ are positive constants picked so that (20) has coefficients analytic in $b$ and $\epsilon$. Then, if $\mathbf{z}$ is the column vector $(\varphi, \psi)$,

$$\mathbf{z}' = (b - \epsilon U)^{-1} \mathbf{D} \mathbf{z},$$  \hspace{1cm} (20)$$

where the matrix $\mathbf{D}$ has elements

$$D_{11} = -1, \quad D_{12} = H_{-1} \left[\gamma^2 - k^2 b - \epsilon U \right]^2 \right] \right],$$

$$D_{21} = [\gamma^2 (b - \epsilon U) + H], \quad D_{22} = \xi.$$  \hspace{1cm} (21)$$

Now suppose that $U(y)$ is continuous with a continuous first derivative. The eigenvalue problem consists of solving (20) with $\psi(A) = 0$ and then finding $b$ such that $\psi(B) = 0$. The choice of $\varphi(A)$ is arbitrary to a large extent, and we choose $\varphi(A) = \delta$. Then, since the coefficients occurring in (20) and the initial conditions are analytic in $\epsilon$ and $b$, so is the solution, and the eigenvalue relation for the determination of $b$ is

$$\psi(b, y = B) = F(\epsilon, b) = 0,$$  \hspace{1cm} (22)$$

where $F$ is an analytic function of $\epsilon$ and $b$.

Setting $\epsilon = 0$ in (20), and solving the resulting equation, we obtain

$$F(0, b) = \frac{\left(\gamma^2 b^2 - 1\right)}{\mu} \sin \left[\mu (B-A)\right],$$  \hspace{1cm} (23)$$

where

$$\mu = \left(\gamma^2 b^2 - (\gamma^2 + \xi^2)\right)^{1/2}.$$  \hspace{1cm} (24)$$

Let $b_0$ denote the solutions of $F(0, b) = 0$. These solutions represent the Kelvin and Poincaré waves mentioned above. Using (23) and (24), we see that $F(0, b_0) \neq 0$. Hence, by virtue of the implicit function theorem, we may state that there exist neighborhoods $N(\epsilon)$ of $\epsilon = 0$ and $M(b) = b = b_0$ for each $b_0$ such that (22) has a unique solution $b = b(\epsilon)$ for $b$ in $M$ and $\epsilon$ in $N$. Furthermore, this solution is a single-valued analytic function of $\epsilon$, with $b(0) = b_0$.

We can now show that $b$ is pure real. The key point in the proof is to note that since $b$ and $\epsilon$ are analytic functions of $\epsilon$, the expansions

$$b = \sum_{n=0}^{\infty} \epsilon^n b_n, \quad \epsilon = \sum_{n=0}^{\infty} \epsilon^n \epsilon_n,$$  \hspace{1cm} (25)$$

converge for $\epsilon$ sufficiently small. Inserting these expansions into (20) and equating the powers of $\epsilon$, we obtain for $n \geq 1$ a sequence of inhomogeneous equations for the determination of $\epsilon_n$. The homogeneous form of each equation has a nontrivial solution satisfying $\psi(A) = \psi(B) = 0$, as does its adjoint, and these solutions may be taken to be pure real, since $b_0$ is pure real. The $b_n$'s are determined by invoking a Fredholm theorem, which states that if an operator $R$ is such that the homogeneous equation $Rx = 0$ has nontrivial solutions, then the inhomogeneous equation $Rx = y$ has solutions if and only if $y$ is orthogonal to all solutions of the adjoint homogeneous equation. The $b_n$'s thus determined prove to be pure real. Then, since the expansion (25) is convergent, $b(\epsilon)$ is pure real for $\epsilon$. Recalling the definition of $\epsilon$, we state that if $c$ lies in the domain $|c| > a_1/\epsilon_0$ it is pure real. Hence, the fast waves are neutrally stable for $\epsilon$ sufficiently small.

In discussing the slow waves, defined to be such that $|c| < a_1/\epsilon_0$, it is convenient to base the analysis on Eq. (17). We note that the coefficients of (17) are analytic in $\epsilon$ and $c$ provided that the cut $U_1 \leq \epsilon \leq U_2$ is deleted from the $c$ plane. We define neutral waves to be the limit as $c_i \to 0^+$ of amplifying waves and
confining our attention to a class of velocity profiles having the following properties:

1) \( U(y) \) has zeros at \( y = A, y = B, \) and at no more than one other point, \( y = y_0 \), with \( y_0 \) in the open interval \((A, B)\), and those zeros are simple zeros.

2) \( U(y) \) has no more than two extrema, a maximum at \( y = y_2 \) at which \( U_2 = U(y_2) > 0 \), and, if it exists, a minimum at \( y = y_1 \) at which \( U_1 = U(y_1) < 0 \). (If \( U_i = \infty \) it does not occur at a zero of \( U \), it need not be negative.) The extrema lie in \((A, B)\).

3) \( U'' \neq 0 \) at \( y = A, B, y_0, y_1, y_2 \).

4) \( U \) is analytic in \( y \) on \([A, B]\) and in a neighborhood of \([A, B]\) in the complex \( y \) plane.

This class of profiles pertains either to a simple jet or to a jet with a countercurrent.

For the above class of profiles it is possible to construct a contour \( J \) running from \( y = A \) to \( y = B \) in the complex \( y \) plane such that \( w \neq 0 \) on \( J \) except for the cases \( c = 0, c = U(y_i), i = 1, 2 \). With these cases excluded, the solution of (17) satisfying \( \psi(A) = 0, \psi'(A) = c \) is analytic in \( c \) and \( c \) for \( c \) real and in the range of \( U \). If we can rule out the cases \( c = 0 \) or \( c = U(y_i) \), the eigenvalue equation can be put into the form

\[
\psi(c, y = B) = G(c, c), \quad 0 = \psi(c, y = B)
\]

where \( G \) is analytic in \( c \) and \( c \) for \( |c| < a_1 / b_0, \ |c| < a_2 \). Then, if \( c_0 \) is the solution of (26) for \( c = 0 \), i.e., the solution for the quasi-geostrophic problem, we may conclude that there exists a positive integer \( N \) such that

\[
G(0, c_0) = 0, \quad \frac{\partial G}{\partial c} = G(0, c_0) \neq 0,
\]

and since if no such \( N \) exists, \( G(0, c) \) is independent of \( c \), contradicting the fact that \( c_0 \) must lie in the interior of \( \Gamma \). From (26) and (27), we conclude that there are at most \( N \) solutions \( c = c_j (\varepsilon), j = 1, 2, \cdots, N \), all of which are continuous in \( c \) with \( c_j (0) = c_0 \). Since \( c_0 \) is an interior point of \( \Gamma \), \( c(\varepsilon) \) must also lie in \( \Gamma \) for \( \varepsilon \) sufficiently small.

To exclude the cases \( c = 0, c = U(y_i) \), we use the relation

\[
E' = [\psi(Qy' - R^2 S \psi)]' = Q(\varepsilon w R \psi)'^2 + R^2 (\psi' - S \psi)^2 + I \psi^2,
\]

where

\[
R = H^{-1}, \quad S = \frac{d}{dy} \log(H w),
\]

and

\[
I = (k^2 + \gamma Qy' + H R y') / H - \frac{[(Qy')^2 + c^2 R^2 y'']}{(ctv)}. \tag{30}
\]

Eq. (28) is implied by (17), as is easily verified, and is valid at all points such that \( \psi / w \) is bounded and differentiable. Also,

\[
I = k^2 - \gamma^2 c / w \tag{31}
\]

for \( \varepsilon = 0 \). We show first that \( c \neq U(y_i) = U_i \) if \( k \neq 0 \) and if \( \varepsilon \) is sufficiently small. If \( c = U_i \), (17) has a regular singularity at \( y = y_i \), and the exponents relative to this point are

\[
\rho_\pm = \frac{1}{2} \left[ 1 \pm \sqrt{9 - 8 \gamma^2 U_i (H U_i')^2} \right]. \tag{32}
\]

Since conditions 2) and 3) above imply that \( U'' / U' < 0 \), only the solution with exponent \( \rho_+ \) should be retained for a bounded solution. Since \( y = y_i \) is the only singularity, this solution is valid throughout the interval \([A, B]\), and since \( \rho_+ > 2, \psi / w = 0 \) at \( y = y_i \). Hence, \( E \) vanishes at \( y = y_i \), and at the boundaries, so its derivative must vanish somewhere. However, \( I \) is positive for \( \varepsilon = 0 \) and hence, for \( \varepsilon \) sufficiently small, the right side of (28) is positive, and \( E \) is monotonic. Consequently, the assumption \( c = U_i \) leads to a contradiction.

If \( c = 0 \) and if there is no countercurrent, (17) has regular singular points at \( y = A \) and \( y = B \) with exponents 0 and 1. The boundary conditions imply that the logarithmic solution must be discarded, so \( \psi \) may be taken to be a real function with \( \psi / w \) bounded and differentiable. From (28), we again find a contradiction.

In case a countercurrent exists, with \( U = 0 \) at \( y = y_i \), we use (28) to show that \( \psi(y_i) \neq 0 \). For \( c = 0 \), multiply (17) by the complex conjugate of \( \psi \), the conjugate of (17) by \( \psi \), subtract, and integrate from \( y = A \) to \( y = B \). The result is

\[
\text{Im} \int_A^B \frac{(Qy')'}{c w} \frac{d\psi}{c w} = \text{Im} \int_A^B Q \left[ \frac{\psi'}{c w} + \frac{\gamma^2}{H} \frac{\psi}{c w} \right] d\psi. \tag{33}
\]

We now take the limit \( c \rightarrow 0^+ \) with \( c \) in the range of \( U \). Since, by a previous argument, \( c \neq U_i \), the limit equation is

\[
\frac{c^{-\varepsilon}}{w} \sum_{\mu} \frac{\varphi_{\mu}^2}{U_{\mu}'} (\xi / H)'_\varepsilon = 0. \tag{34}
\]

Here \( y_i \) solves \( U(y_i) = c \), and the subscript \( c \) denotes the value of a function evaluated at \( y = y_i \). Since \( \xi / H \) is the potential vorticity of the basic state, (34) is a natural generalization of the corresponding quasi-geostrophic result. If \( c = 0 \), (34) reduces to

\[
\psi(y_i) = 0, \tag{35}
\]

a necessary condition. Since (35) cannot be satisfied, \( c \neq 0 \) even in the case of a countercurrent. Hence the cases \( c = 0 \) and \( c = U(y_i) \) have been excluded.

In summary, we have shown that if \( \varepsilon \) is sufficiently small and if \( k \neq 0 \), all eigenvalues \( c \) with nonzero imaginary parts must lie in the circle \( \Gamma \). This is the main result of this section. A secondary result is the relation
(34), giving a necessary condition for the existence of neutral modes.

We conclude by pointing out that if \( c_0 = U(y_0) \), where \( y_0 \) solves \( U''(y_0) = \gamma^2 U(y_0) \), then the eigenvalue relation (26) has a unique analytic solution \( c = c(\epsilon) \), with \( c(0) = c_0 \). To show this, we note that \( \psi_\epsilon \), the derivative of \( \psi \) with respect to \( \epsilon \), satisfies

\[
\psi''_\epsilon - (k^2 w + U'' - \gamma^2)\psi_\epsilon = (U'' - \gamma^2 U)\psi_\epsilon \quad \text{for} \quad \epsilon = 0.
\]

If \( c = c_0 \), an eigenvalue for the homogeneous equation, (36) has a solution satisfying \( \psi_\epsilon(A) = \psi_\epsilon(B) = 0 \) only if the orthogonality condition

\[
\int_A^B (U'' - \gamma^2 U)\psi_\epsilon^2 \, dy = 0
\]

is satisfied. Considering \( c_0 \) to be \( U(y_0) + i0+ \), letting

\[
f(y) = -[U''(y) - \gamma^2 U(y)]/w,
\]

and taking the imaginary part of (37) in the limit \( c_\epsilon \to 0+ \), we obtain a necessary condition for \( \psi_\epsilon \) to satisfy the boundary conditions, i.e.,

\[
\sum_{y_n} \frac{f(y_n)}{|U'(y_n)|} \psi_\epsilon(y_n) = 0.
\]

In deriving (39) we have used the fact that for \( c = U(y_0) \), \( \psi_\epsilon^2 \) is real and positive. Also, comparing (19) with

\[
\psi'' + \left[ \frac{\pi}{(B-A)} \right] \psi = 0,
\]

and using an oscillation theorem, we find that \( f(y) \) must satisfy the inequality

\[
f(y) \geq k^2 + \gamma^2 + \left[ \frac{\pi}{(B-A)} \right]^2 > 0,
\]

if (19) is to have a solution satisfying the boundary conditions. Hence (39) cannot be satisfied, \( \psi_\epsilon \) cannot vanish at \( y = B \),

\[
G_\epsilon(0,c_0) = \psi_\epsilon(0,c_0,y = B) \neq 0,
\]

and (27) is satisfied with \( N = 1 \). The implicit function theorem then guarantees the existence of a unique solution \( c = c(\epsilon) \), analytic in \( \epsilon \). The analyticity holds no matter which equation is used to specify the eigenvalue problem, and in the next section we will use (18) to examine the effect of increasing \( \epsilon \) from zero on the stability properties.

4. Effect of Rossby number on stability characteristics

As mentioned at the start of Section 3, a perturbation analysis can be carried out to improve the results of the quasi-geostrophic theory. Some of the labor involved in this procedure can be circumvented through use of the following device, which is due to Howard (1963). Consider the functional

\[
I[\psi] = \left\{ \frac{q\psi^2}{\epsilon w} \right\}_{A}^{B} - \int_{A}^{B} \left[ q(\varphi')^2 + k^2 \psi^2 \right] \, dy + q\varphi^2/\epsilon w \right\}_{A}^{B} \int_{A}^{B} \psi^2 \, dy.
\]

It is readily verified that \( I[\psi] = \gamma^2 \) and that \( I \) is stationary in the sense of the calculus of variations (though not an extremum); thus, it follows that the derivative of \( I \) with respect to any parameter does not involve contributions coming from the dependence of \( \varphi \) on that parameter. Letting \( n \) be any of \( \epsilon, k \) or \( \gamma \), we have

\[
\frac{\partial I}{\partial n} = \frac{\partial I}{\partial \epsilon} + \frac{\partial I}{\partial \varphi} \frac{\partial \varphi}{\partial n}
\]

from which \( \partial c/\partial n \) can be calculated. It should be noted that if \( c \) is real but not in the range of \( U \), \( \partial c/\partial n \) is real and the neutral wave in question is not a marginally stable wave.

Now, from (44), we obtain

\[
\left( \frac{\partial c}{\partial \epsilon} \right)_{\epsilon = 0} = \left\{ \int_{A}^{B} q\left( [\varphi']^2 + k^2 \varphi^2 \right) + \frac{q'}{2w_0} \varphi^2 \right\}_{A}^{B} \, dy
\]

\[
+ \omega_0 (\varphi')^2 \right\}_{A}^{B} \int_{A}^{B} \left[ -q' \right] \, dy,
\]

where \( \varphi_0 \) is the solution of the quasi-geostrophic equation (19), \( w_0 = U(y) - c_0, c_0 \) being the zero order wave speed, and \( q_0 = (\partial q/\partial \epsilon)_{\epsilon = 0} \). This provides the first-order term in a perturbation series for \( c \).

From the definition of \( q_1 \), we have

\[
q_1 = U' + \gamma^2 \Phi, \quad q_2 = 2\left( (U')^2 + \gamma^2 \Phi U' + k^2 (w_0)^2 \right).
\]

If the velocity profile \( U(y) \) is even in \( y \) and if \( A = -B \), \( \varphi_0(y) \) is either even or odd in \( y \), and \( (\varphi_0')^2 \) and \( \varphi_0^2 \) are even. Then \( q_1 \) is odd, \( q_2 \) is even, and the numerator of (45) integrates to zero while the denominator is nonzero. Hence, for a symmetric flow \( c = c_0 + O(\epsilon^2) \), i.e., there is no first-order Rossby number effect on \( c \).

The velocity profile which will be treated here is not symmetric and hence there will be a first-order Rossby number effect. The flow is

\[
U(y) = \cos \pi y + \frac{\sqrt{3}}{2}, \quad A = -7/6, \quad B = 5/6.
\]
and is plotted in Fig. 1. A countercurrent is included since (47) is meant to model the mean velocity of the Gulf Stream. The marginally stable waves corresponding to the above profile (with \( \epsilon = 0 \)) are readily calculated using standard methods, and these will be used in the formula for \( \partial \epsilon / \partial \epsilon \). In this way we find the perturbation off the neutral stability curve due to the Rossby number.

There are two modes solving (19) with \( \epsilon = 0 \), but one has wavenumber \( k = 0 \) and hence no time dependence. The other mode, which we will study, is given by

\[
\begin{align*}
\varphi_0 &= \cos \left[ \frac{\pi}{2} (y + \frac{1}{2}) \right], \\
k &= \frac{\sqrt{3}}{2}, \\
c_0 &= \frac{\sqrt{3}}{2} \left( \frac{\pi^2 + \gamma^2}{\pi^2 + \gamma^2 + i0+} \right). \tag{48}
\end{align*}
\]

In evaluating the integrals in (45) we will regard \( c_0 \) as being the limit of an unstable wave, as indicated by the last equation of (48). These integrals, with one exception, can be put into the form

\[
\int_{\alpha}^{\alpha + 2\pi} R(\cos \theta, \sin \theta) d\theta,
\]

where \( R \) is a rational function, and hence can be evaluated as contour integrals. The exception is due to a term linear in \( \gamma \) because of the presence of \( \Phi \). This is treated by expanding \( (\gamma + \frac{1}{2}) \) in a Fourier sine series in the interval \(-1 \leq (\gamma + \frac{1}{2}) \leq 1\) and evaluating each integral by converting it into a contour integral. The resulting infinite series can be summed exactly.

The details of this calculation will be omitted and we present only the final result. With

\[
F(\Gamma) = 2 \left( \frac{\sqrt{3}}{2} \left( \frac{1 - \Gamma_0^2}{1 - \Gamma_0^2} \right) \right) \times \log \left[ \frac{1 - \Gamma_0^2}{\sqrt{3} - 2\Gamma_0} \right], \tag{49}
\]

we obtain after a number of nontrivial integrations

\[
\text{Im} \left( \frac{\partial \epsilon}{\partial \epsilon} \right) = \left[ \frac{2\pi \Gamma (1 - \Gamma_0^2)}{7 - 4\sqrt{3} \Gamma} \right] \left[ \frac{\sqrt{3}}{2} \left( \frac{1 - \Gamma_0^2}{1 - \Gamma_0^2} \right) \right] \times \left( \frac{\sqrt{3}}{2} \right) \left( 1 - \Gamma_0^2 \right) \times [1 + \pi/2\sqrt{3} - F(\Gamma)]. \tag{50}
\]

We note that for \( \gamma = 0 \) (\( \Gamma = 0 \)) the right side of (50) vanishes. This is to be expected, since in this case there should be no Rossby number effect. The formula is inaccurate in the limit \( \gamma \to \infty \) (\( \Gamma \to \sqrt{3}/2 \)) because the product \( \gamma^2 \) has been treated as being small; accordingly the calculation is valid only when \( \gamma^2 \ll 1 \).

The result is given in Table 1. As can be seen, for this flow the Rossby number effect is destabilizing in the sense that a marginally stable configuration for zero Rossby number is unstable for finite Rossby number. This is a surprising result; it had previously been felt that Rossby number effects would be stabilizing. Further calculations based on different profiles have shown that the Rossby number effect may in some cases be stabilizing, in some cases destabilizing. We can therefore come to no definite conclusion regarding this matter except to say, in the present case for a profile resembling that of the Gulf Stream in many ways, that the first correction to quasi-geostrophic theory indicates a destabilizing Rossby number effect.

| Table 1. Values of \( \partial \epsilon / \partial \epsilon \) as a function of \( \Gamma \). |
|-------------------|-------------------|
| \( \Gamma \)      | \( \partial \epsilon / \partial \epsilon \) |
| 0.0               | 0.0               |
| 0.087             | 0.038             |
| 0.173             | 0.086             |
| 0.260             | 0.144             |
| 0.346             | 0.212             |
| 0.520             | 0.379             |
| 0.693             | 0.581             |
| 0.866             | 0.782             |
5. Small Rossby number divergent flow

The obvious difficulty in applying quasi-geostrophic theory to the Gulf Stream problem is that the Rossby number, based on a relative vorticity of 0.4 or 0.5 \times 10^{-4} \, \text{s}^{-1} (Stommel, 1965), is not particularly small. Not so obvious but equally important is the fact that the slope of the interface must also be small if the fluid is to be nondivergent in the lowest approximation. In the present paper this slope is the quantity \( \epsilon \gamma^2 \), and the conditions for quasi-geostrophic theory to be valid are \( \epsilon \ll 1, \epsilon \gamma^2 \ll 1 \). The first condition is satisfied marginally, the second not at all, since Stommel shows that \( \epsilon \gamma^2 = 1 \) if the 10C isotherm is to be the interface. This can be seen either by taking as the characteristic depth \( D \) the depth of the 10C isotherm at the midpoint of the stream or else by directly computing the slope of the isotherm.

It is pertinent to mention at this point that Stern (1961) discusses the case \( \epsilon \ll 1, \epsilon \gamma^2 \ll 1, \gamma^2 \gg 1 \). It is difficult to see how this could apply to the Gulf Stream, but one of Stern's conclusions, namely that an increase in \( \gamma^2 \) is stabilizing, turns out in the sequel to be true.

The foregoing remarks imply that any low Rossby number theory must be modified by taking the quantity \( \Delta = \epsilon \gamma^2 \) to be of order unity even though \( \epsilon \) is small. Turning to Eq. (18), we find in the limit \( \epsilon \to 0, \gamma^2 \to \infty \), with \( \Delta \approx 1 \), that

\[
(H \varphi')' + \left[ \frac{\Delta c}{c(U - c)} - \frac{(HU')'}{U - c} - k^2 H \right] \varphi = 0,
\]

for \( \varphi = 0 \) at \( y = A \) and \( y = B \). To see the consequences of this limiting case, we multiply (51) by \( \bar{\varphi} \), the complex conjugate of \( \varphi \), and integrate over the interval \([A, B]\).

After an integration by parts, we obtain

\[
\epsilon \int_A^B f_z dy = \int_A^B \{ \bar{\varphi} (H(U')' - \Delta c) \} f_z dy,
\]

where

\[
f_1 = H(|\varphi'|^2 + |T|^2 |\varphi|^2), \quad f_2 = (|\varphi|/|w|)^2.
\]

The imaginary part of (52) is

\[
\Delta \int_A^B U f_z dy = \epsilon \int_A^B (H(U')' f_z dy,
\]

for \( c_i \neq 0 \), and addition of \( c_i \) times (54) to the real part yields

\[
\int_A^B [\Delta |c|^2 + \epsilon (H(U')' (U - 2c_i)) f_z dy = -\epsilon \int_A^B f_z dy.
\]

Now, unless \( U \) vanishes somewhere in \((A, B)\), the left side of (54) must be much greater than the right side, since \( \epsilon \) is small. Consequently, if \( c_i \neq 0 \), there must be a countercurrent. Turning to (55), we see that \( |c| \) must be small if the left side is to be negative. A crude estimate is \( |c| = O(\epsilon) \). A sharper estimate is obtained by noting that an asymptotic integration of (51), for \( c_i > 1 \), implies that \( \varphi' / \varphi_0 = (c_i / \epsilon) \), which in turn implies that \( f_1 / f_2 = c_i / \epsilon \). Then the ratio of the right side of (55) to the left side is roughly \( c_i / \epsilon \), and the assumption \( c_i > 1 \) leads to a contradiction and hence is incorrect. Consequently, if \( c_i \neq 0 \), it must be true that \( c_i = O(\epsilon) \).

Returning to (51), we set \( c = \lambda \), where \( \lambda = O(1) \). If \( U \) does not vanish anywhere in \((A, B)\), we may neglect \( \lambda \) next to \( U \), and (51) is then a regular self-adjoint Sturm-Liouville equation with real eigenvalues \( \lambda \). In this case the perturbations are non-amplifying, in agreement with the argument based on the use of (54).

If \( U = 0 \) at some point \( y_0 \) in \((A, B)\), a neglect of \( \lambda \) next to \( U \) is not uniformly valid. In this case, with \( \lambda = \lambda_i + \Delta \lambda \), \( \lambda_i > 0 \), we have

\[
\lim_{\epsilon \to 0} \frac{1}{U - \lambda} = \frac{1}{U} + \frac{i \pi}{|U'|} \delta(y - y_0),
\]

where \( P \) denotes the principal value, and (51) becomes

\[
(H \varphi')' + \left[ \frac{\Delta \lambda - (HU')'}{U} - k^2 H \right] \varphi = 0, \quad y \neq y_0,
\]

with \( \varphi(A) = \varphi(B) = 0 \), and

\[
\left. \frac{\varphi}{y_0^+} \right|_{y_0^+} = 0, \quad \left. \frac{\varphi}{y_0^-} \right|_{y_0^-} = \frac{i \pi \varphi_0}{H_0 |U_0'|} [(H_0 U_0)' - \Delta \lambda].
\]

We will not pursue the implications of this last set of equations except to remark that it is highly probable that complex eigenvalues result. The main points of interest are that in the model discussed in this section, which appears to be more relevant to the Gulf Stream than the usual quasi-geostrophic model, the complex wave speeds are small and the motion is stable unless the velocity profile has a countercurrent. The probable explanation for this deduction is that a critical layer is in general necessary for instability, and for very small wave speeds such a layer can exist only if the unperturbed velocity profile has one or more zeros.

6. Concluding remarks

When the theory of Section 4 is applied to a profile resembling that of the Gulf Stream, it is found that ageostrophic effects are destabilizing in the restricted sense used in the text. Since growth rates predicted by the quasi-geostrophic theory are already too high, one must conclude that the quasi-geostrophic theory is probably invalid for studying the stability of the
Gulf Stream. This conclusion is a tentative one, since the destabilization applies only for waves with very small growth rates. The theory of Section 5, which takes account of the divergence, is more realistic and predicts results in better qualitative agreement with observations. Nevertheless, the asymptotic analysis is rather delicate, and a numerical study is necessary to answer the question of whether the theory of linearized hydrodynamic stability is relevant at all to the problem of Gulf Stream meanders.

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REFERENCES


Gulf Stream Summaries, U. S. Naval Oceanographic Office.


