Phase Angle Considerations in the Modeling of Intermittent Turbulence

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ABSTRACT

It has been suggested that the study of phase angles associated with the Fourier transforms of time series may yield information about the intermittent behavior of turbulent records. It is shown with numerical experiments that the phase angles and the fine-structure of the spectrum are both associated with the intermittency. The phase angles of turbulence appear to be nearly independent and uniformly distributed in the same sense that the spectrum has an approximate $-5/3$ power dependence on the wavenumber. But neither of these approximate models account for the observed intermittency. The intermittency therefore bears some relation to a higher order structure in Fourier space. The nature of this structure has not been found explicitly, although a qualitative explanation is offered.

1. Introduction

The study of atmospheric turbulence is to a large extent descriptive and an exact mathematical model has yet to be developed. Among the many observed properties of atmospheric turbulence three are of major interest here. First, over a large range of wavelengths the spectrum depends on the $-5/3$ power of the wavenumber. Second, it has been observed that the one-point statistics of the turbulent velocity fluctuations in this range are not Gaussian (Dutton *et al.*, 1969). Third, turbulence is observed to be intermittent in this range. A recent suggestion that the intermittency may be studied by analyzing the phase angles associated with the Fourier transform of a time history (Dutton and Lane, 1969) is investigated here. Because little is known about the statistical structure of these angles, we are primarily concerned at this point with finding whether they have any simple or detectable interrelationships.

An intermittent series has been described as one in which a relatively large fraction of the variance comes from a relatively small fraction of the total record (Dutton and Lane). Such intermittency is reflected in the probabilistic behavior and the normalized fourth moment, or kurtosis, can be used as a measure of the intermittency. For a Gaussian series the kurtosis is equal to 3, while for turbulent velocities it is somewhat higher, with values typically ranging from 3.5 to 6.0.

Turbulence velocity records are often examined through transformation into phase spaces, and of these the Fourier space and the associated energy spectrum are the most widely known. The spectrum represents the amplitude information in Fourier space. But there are two variables in a Fourier series, amplitude and phase, and very little attention has been given to the latter. It is these phases for which we want to find a model, and if possible, a model that reflects the observed intermittency.

We use model here in the sense that the $-5/3$ spectral slope gives a model for the behavior of the magnitude of the Fourier coefficients; the question is whether the behavior of the Fourier phases can be represented in as simple a manner.

2. Effects of the intermittency on the Fourier space variables

Despite a variety of attempts, we have not been able to organize the information contained in the phase angles associated with actual time histories of turbulent motion into any quantitative model. In this paper, we take the approach that as a first step the significance of the phase angles of turbulence records can be investigated by altering them artificially and then studying what effects various kinds of alterations produce in the time series.

That the phase angles have some effect on intermittency is clear because it can be shown (e.g., Spark, 1971) that statistical moments of order $\geqslant 3$ are both amplitude and phase dependent. The behavior of such moments is an important characteristic of the intermittency, which thus depends upon amplitude and phase.

In this section we shall explain an algorithm (Smith, 1971) for changing a time series into a new series that differs from the original one by having its Fourier phases altered while the spectrum remains the same, or by having its spectrum altered while the phases are the same.

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The line spectrum or periodogram may be written as

$$\Phi_n = a_n^2 + b_n^2, \tag{1}$$

where a_n and b_n are the Fourier cosine and sine coefficients, respectively. We may write these coefficients as

$$a_n = (\Phi_n)^{\frac{1}{2}} \cos\theta_n, \tag{2}$$

$$b_n = (\Phi_n)^{\frac{1}{2}} \sin\theta_n, \tag{3}$$

in which

$$\theta_n = \arctan(b_n/a_n). \tag{4}$$

The quantity θ_n is the phase considered in this study and is defined by (4). Using this formulation it is possible to see how a series is modified if (i) its spectrum Φ is replaced by a different spectrum Φ', yielding

$$\left.\begin{aligned} a_n' &= (\Phi_n')^{\frac{1}{2}} \cos\theta_n \\ b_n' &= (\Phi_n')^{\frac{1}{2}} \sin\theta_n \end{aligned}\right\}, \tag{5}$$

or (ii) its phase angles are changed from θ to θ', so that

$$\left.\begin{aligned} a_n' &= (\Phi_n)^{\frac{1}{2}} \cos\theta_n' \\ b_n' &= (\Phi_n)^{\frac{1}{2}} \sin\theta_n' \end{aligned}\right\}. \tag{6}$$

The new series is then obtained by inverse transformation once all or some of the coefficients have been modified. Application of this technique to modify velocity records obtained in clear air turbulence with instrumented aircraft produced the following results:

1) The smoothing of the original line spectrum to the functional form proposed by von Kármán (1948) for the $-5/3$ spectrum decreased the kurosis from 3.8 to 3.4.

2) The random rearrangement of the phase angles of the turbulence reduced the kurtosis to 2.7.

3) The substitution of phase angles associated with Gaussian noise series produced by a numerical random number generator reduced the kurtosis to 2.9.

4) The substitution of uniformly distributed angles obtained from a numerical random number generator led in one case to a kurtosis of 3.5 and in a second case to 2.8.

As another example, when the essentially flat spectrum derived from computer-generated Gaussian noise was changed to a smooth, $-5/3$ spectrum, the kurtosis increased from 3.0 to 3.1.

The above experiments are a limited group of case studies and the number of points involved, 4096 per series, is inadequate for statistical stability. With such a few data points, the observed change in the kurtosis may well be within the expected sampling error. Despite this limitation, we feel that these experiments indicate that the intermittency is reflected in the phase angles and that the interaction lies in some manner in the sequential arrangement by harmonics and not in the statistical distribution of the angles. It is also seen that the intermittency is related to the detailed structure of the line spectrum in the sense that if the spectrum is changed then both it and the original phase angles are associated with a new time series whose higher order moments are not the same as those of the original series. As will be shown in the next section, the intermittency need not *necessarily* be decreased when a line spectrum is smoothed, as might be expected.

3. Modeling the phase angles

A series, with zero mean, known at N points, may be expressed in Fourier form as

$$u_t = \sum_{n=0}^{(N/2)-1} (a_n^2 + b_n^2)^{\frac{1}{2}} \cos[(2\pi nt/N) - \theta_n]. \tag{7}$$

Here θ_n is the phase angle of (4) and it is evidently the phase of the nth harmonic when $t=0$. The phase is thus highly dependent on the initial conditions, and in this sense is a non-stationary statistic. Therefore, in studying the phase angles, the question that must be answered is whether there is any consistent property of the phases that can be analyzed despite their dependence on the starting point of the series.

Due to their definition with the arc tangent function in (4), the phase angles are only resolved over a range of 0 to 2π. Plots of the phase angles derived from turbulence records vs harmonic number showed that they appeared uniformly distributed and were apparently independent of each other. These phase angles were examined by such methods as linear regression, nonlinear regression, and least-squares approximation, but none of these methods led to any substantial reduction in the variance. The distributions of the angles obtained from turbulent velocity records were almost, but not quite, uniform. Deviations from the uniform distribution were not found to be correlated with kurtosis.

From the definition of intermittency it appears that the most intermittent series it is possible to construct would consist of a single peak like a delta function, because all the variance of such series would be concentrated at one point only. It is easily shown that if we consider a function defined on N points, and of value 1 at the point t_0 and 0 otherwise, then the phases as a function of harmonic number will be given by the linear relationship

$$\theta_n = 2\pi t_0 n/N, \tag{8}$$

in which the phase will only be determined within modulo 2π, however. The dependence on the starting point is explicitly clear. If the (discrete) delta function is placed at $t_0=0$, then all phase angles will be identically equal to zero. If the delta function is placed at $t_0 = N/m$, say, then

$$\theta_n = 2\pi n/m \text{ (modulo } 2\pi). \tag{9}$$

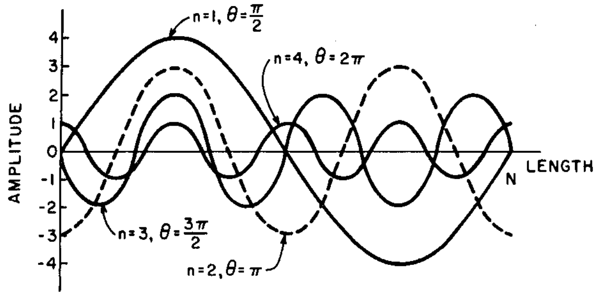

FIG. 1. Illustration of phase angles, $\theta_n = \pi n/2$, producing a peak at $N/4$.

If we plot the component harmonics of the series using the appropriate amplitudes and initial angles on a graph of wavelength vs amplitude, the harmonics will all be in phase at a distance $1/m$ of the fundamental wavelength. For example, in Fig. 1 the first four harmonics are drawn for a delta function placed at $t_0 = N/4$, so that the angles at the origin are given by $\theta_n = \pi n/2$. The amplitudes of each harmonic were chosen arbitrarily for this example. It is seen that the angles are in phase at a quarter of the fundamental wavelength, thus reinforcing each other to produce the peak of the delta function.

As all harmonics will be in phase only for a delta function, it seems attractive to speculate that the degree of intermittency of a series will depend on the extent to which the harmonics are in phase. But both the number of angles involved in the in-phase behavior and the amplitudes of the harmonics which are in-phase must be considered. Since we are studying turbulence which has a $-5/3$ power dependence of energy on the wavenumber, we would expect that in-phase behavior found at low wavenumber would correspond to greater observed intermittency than if the higher harmonics were in phase.

To check this hypothesis, an observed turbulence record consisting of 4096 points was Fourier analyzed and the corresponding line spectrum and phase angles determined. This series had an original kurtosis of 3.8. In order to induce stronger in-phase behavior than might have been present originally, alterations were made to the phase angles. If we let θ_n be the original phase angles and θ_n' the altered angles, the cases considered were as follows:

(a) $\theta_n' = 0$, $n = 1, 2, \ldots, 20$
$\theta_n' = \theta_n$, otherwise

(b) $\theta_n' = \pi n/4$, $n = 1, 2, \ldots, 20$
$\theta_n' = \theta_n$, otherwise

(c) $\theta_n' = \pi n/4$, $n = 1, 3, 5, \ldots, 39$
$\theta_n' = \theta_n$, otherwise

(d) $\theta_n' = 0$, $n = 2027, 2028, \ldots, 2407$
$\theta_n' = \theta_n$, otherwise

(e) $\theta_n' = \pi n/4$, $n = 2027, 2028, \ldots, 2047$
$\theta_n' = \theta_n$, otherwise

In cases (a), (b) and (c), the kurtoses were significantly increased to 4.7, 6.1 and 6.1, respectively. In case (a) a strong gust appeared at the start and at the end of the series, in cases (b) and (c) a gust appeared at one-eighth of the series length, and in case (c) an additional small negative gust appeared at five-eighths of the series length. For cases (d) and (e) the kurtosis remained unchanged and there was no noticeable change in the statistics of the series.

It seems clear that the intermittency was dependent on the position of the modified angles because the associated spectrum had a $-5/3$ power dependence on the wavenumber. Hence, the amplitudes at small wavenumber were much larger than at large wavenumber, so that modification of the angles had a greater effect in the alteration of the time series. In the case of a flat spectrum such a difference between large and small wavenumber is not found.

In a turbulent velocity series, then, the intermittency may be affected by any of three factors. First, the arrangement of the phase angles associated with a given spectrum at the small wavenumbers is crucial. Second, local variations in the spectrum give differing importance to the angles in local wavenumber regions. Third, structure in the phase angles, not in strict sequence, but in the sense of association at various harmonics, may produce intermittent behavior.

The second factor has an important implication. Suppose that a set of sequentially arranged phase angles in the original series is associated with local minima of the spectrum. Upon smoothing, the amplitudes will be increased and the increased energy now associated with this set of phase angles may then increase the intermittency.

Thus, we have the situation in which the general appearance of the spectrum may be a power law and in which the phase angles may appear to be independent, uniformly distributed variables. But the interaction between the local variations of the spectrum and the local behavior of the phase angles produces intermittency in the record. A qualitative model for the phase angles of intermittent turbulence, then, has some linear behavior of the phase angles sense of (9), perhaps occurring at widely separated harmonics, superimposed on an apparently random, uniform distribution.

4. Limitations and uses of the model

Although this quantitative model explains the observed results fairly well, it does not allow any exact prediction of the effect of inserting angles obeying (9), since the effect will depend on the relationship they bear to the angles left undisturbed as well as to the amplitude of the corresponding harmonics. Consideration of the model does, however, allow us to see why the analysis of the angles is so difficult, and why attempts to determine structure by regression techniques, for example, have failed.

A set of uniformly distributed and independent angles does not relate the angles to the intermittent phenomena and is characteristic of angles obtained from a random Gaussian series as well as of turbulence. It has been used by Syōno and Tanaka (1966) in the derivation of a theoretical model for the distribution of turbulent velocities. The difference between angles derived from a random Gaussian series and turbulence appears to lie in the modification to the uniform model described in the previous section.

Our qualitative model of the angles may be used to create a reasonably realistic simulation of turbulence using the following steps:

1) Generate a random Gaussian series.
2) Obtain its Fourier transform and the Fourier coefficients.
3) Form the phase angles from the coefficients.
4) Replace the original spectrum with a smoothed $-5/3$ spectrum (e.g., the von Kármán, 1948, spectrum).
5) Replace a suitable number of the angles with angles obeying (9).
6) Use (5) to find the new Fourier coefficients and back-transform these to obtain the simulated turbulence.

By manipulating the angles suitably, the required statistics and appearance may be obtained. This method is identical in spirit to one devised by Smith (1971) who inserted random gusts into a random Gaussian series at random intervals, then Fourier transformed, changed the spectrum to a $-5/3$ spectrum and back-transformed to obtain the simulated turbulence. A slightly different method for simulating turbulence is obtained by constructing a series of uniformly distributed angles between 0 and 2π on a mechanical random number generator, modifying some of these, then using a smoothed $-5/3$ spectrum as before.

5. Conclusions

A model of independent and uniformly distributed phase angles has been used in the literature (Edwards, 1964; Syōno and Tanaka, 1966) to model the Fourier phase angles of turbulence. Although this may be a reasonable approximation for some purposes, it does not serve to distinguish intermittent turbulence from Gaussian noise. The intermittency appears to be dependent on some higher order associations in Fourier space. The qualitative model proposed here explains how the phase angles of turbulence differ from those of Gaussian noise, although no mathematical formulation for it has been found. The higher order associations of the phase angles will not necessarily be found in a structure involving consecutive harmonics, and the effect of the phase structure on the intermittency will depend on the amplitude of the corresponding harmonics. It is therefore expected that any mathematical model in Fourier space for the intermittency will be extremely involved, and, moreover, that any model for the phase angles alone without reference to the line spectrum will be insufficient to explain the observed intermittency.

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