Relative Dispersion in the Enstrophy-Cascading Inertial Range of Homogeneous Two-Dimensional Turbulence

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1. Introduction

The well-known Richardson's law (1926) for the dependence of turbulence diffusivity on the relative particle separation \((\xi^2)\) in homogeneous three-dimensional turbulence can be obtained by using dimensional arguments (Obukhov, 1941) provided the relative particle separation is within the energy-cascading inertial range where \(\eta\), the total dissipation rate, is the essential characteristic parameter. Here, by dimensional arguments, a theory of the relative dispersion in the enstrophy-cascading inertial range of homogeneous two-dimensional turbulence will be derived.

2. Analysis

Through the studies of Kraichnan (1967), Leith (1968) and Batchelor (1969), the enstrophy-cascading range is characterized by an enstrophy (half-squared vorticity) cascade rate \(\eta\). In this range the characteristic time scale is \(\eta^{-1}\) and the energy spectrum \(E(k)\) has the form \(E(k) \propto \eta k^{-3}\), where \(k\) is the wavenumber. Now it is postulated that the turbulence diffusivity \(K\), defined by \(\frac{1}{2} \frac{d\langle \xi^2 \rangle}{dt}\), depends on \(\eta\) only. Hence, by dimensional arguments, we have

\[
\frac{1}{2} \frac{d\langle \xi^2 \rangle}{dt} = K = A \eta \langle \xi^2 \rangle,
\]

where \(A\) is a positive constant with the order of unity provided that turbulent mixing is the dominate process for dispersion, and \(t\) is the time. Integrating (1) yields

\[
\langle \xi^2 \rangle = \langle \xi_0^2 \rangle \exp(2A\eta t) \quad \text{or} \quad 2 \log(\frac{\langle \xi^2 \rangle}{\langle \xi_0^2 \rangle}) = 2A\eta t,
\]

where \(\langle \xi_0^2 \rangle\) is the integration constant referenced at \(t=0\). Eq. (2) then describes the relative dispersion in the enstrophy-cascading inertial range of homogeneous two-dimensional turbulence.

3. Discussion

Although it is known that the large-scale motions in the atmosphere are quasi-horizontal, a direct application of (2) to the large-scale dispersions in the atmosphere must be justified. Recently, Charney (1971) developed a geostrophic turbulence theory for three-dimensional quasi-geostrophic flow. According to him, the large-scale motions in the atmosphere are similar to geostrophic turbulence rather than to two-dimensional turbulence, i.e., the large-scale motions at high wave-
numbers are characterized by a pseudo-potential vorticity transfer function rather than an enstrophy cascade rate. By dimensional arguments, a result similar to (2) can be derived for the relative dispersion in geostrophic turbulence when the enstrophy cascade rate in (2) is replaced by the pseudo-potential vorticity transfer function.

Morel (1970) measured the large-scale dispersion by tracing ten independent pairs of simultaneous constant volume ballons at 200 mb level in the Southern Hemisphere general circulation, between 20 and 50S. In Fig. 1, his measurements of the relative dispersions $\langle X^2 \rangle^t$ and $\langle Y^2 \rangle^t$ in the respective longitudinal and meridional directions are presented in a semi-logarithmic plot against the time $t$ in days. In the early period of dispersion ($t<6$ days) both $\langle X^2 \rangle^t$ and $\langle Y^2 \rangle^t$ are fairly well correlated by two dashed lines, each of which have a slope of $(1/2.7)$ day$^{-1}$. However, in the later period ($t>5$ days), $\langle Y^2 \rangle^t$ approaches another straight-line relation with a slope $(1/19.4)$ day$^{-1}$, and $\langle X^2 \rangle^t$ a slope of $(1/11.5)$ day$^{-1}$. The reason why the slope changes suddenly after about 5 days is not clear. Nevertheless, a relevant time scale of the atmosphere will be calculated from Morel’s data by virtue of (2), i.e., in the early period when the clusters of particles are subject to an atmosphere which has a time scale $2.7 A/2.3$ day.

![Fig. 1. Root mean square of the relative dispersions as a function of time.](image)

![Fig. 2. Diffusivity $K$ as a function of the relative particle separation $l$.](image)

Since $A$ is a constant with the order of unity as expressed in (1), the time scale is then about 1 day.

In Fig. 2, Morel’s data on longitudinal dispersion was used to evaluate the turbulent diffusivity by

$$K = 2.3 \frac{d \log \langle X^2 \rangle^t}{dt} X^2,$$

where $\langle X^2 \rangle^t$ was converted to $l$ by using $1^\circ$ longitude = 100 km. In accordance with (2), Morel’s data are related by the expression $K = 9.86 \times 10^6 x^2 P$ for 50 km < $l$ < 2000 km. In comparison, Richardson’s data were reduced from a paper by Taylor (1959). It is seen that in the ranges of large-scale motions his data also indicate the relation $K \propto l^3$. It is interesting to note that Richardson discovered $K \propto l^3$ for three-dimensional turbulence, but he did not visualize $K \propto P$ for two-dimensional or geostrophic turbulence. Here it is noted that $K$ will be proportional to $P$ in the scale range where a characteristic frequency $f_p$ exists, i.e., $K \sim f_p P$ by dimensional arguments; the characteristic frequency could be the Brunt-Väisälä frequency, $(g/\rho_0)^{1/2}(d\tilde{T}/dz)$, in a stably stratified free atmosphere, or the internal frequency of turbulence, $(N/e)^{1/2}(g/\rho_0)$, in the so-called buoyancy subrange (Lin et al., 1969). Here $g$ is the gravitational acceleration, $\tilde{T}$ the mean temperature, $d\tilde{T}/dz$ the temperature gradient, and $N$ the total dissipation of temperature inhomogeneity by thermal conductivity. To clarify this speculation more measurements are urgently needed, and it is likely that a relative dispersion measurement can serve this purpose.
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REFERENCES


