On Non-Geostrophic Baroclinic Stability: Part III. The Momentum and Heat Transports

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ABSTRACT

The results of Parts I and II are used to calculate the transports of heat and momentum that accompany growing baroclinic instabilities in Eady's model. The transports are calculated for both the conventional ("geostrophic") kind of baroclinic instability and for symmetric instability, without any restriction on the stratification, as measured by the Richardson number. The transports are calculated consistently to second order in the amplitude expansion of stability theory, so that the transports are the sum of an eddy transport term and a mean transport term.

The results show that both kinds of instability always transport heat upward and poleward, and always transport zonal momentum downward. Under geostrophic conditions the horizontal transport of zonal momentum depends on the horizontal shear of the basic flow. This shear is neglected in Eady's model so the horizontal momentum transports calculated here only contain the non-geostrophic contribution to the transport. The results show that this non-geostrophic transport is always equatorward for geostrophic instability, but for symmetric instability it may be either equatorward or poleward depending on the value of the Richardson number. It is suggested that the equatorward transport of zonal momentum by geostrophic instability is a more likely mechanism for Jupiter's equatorial acceleration than the transport by symmetric instability.

1. Introduction

In this paper we use the results of Parts I and II (Stone, 1966, 1970) to calculate the transports of heat and momentum that accompany the most rapidly growing perturbations in Eady's (1949) model of baroclinic stability. In effect, we will be extending Eady's original results for these transports under highly stable conditions (i.e., very large values of the Richardson number Ri) to conditions with arbitrary stability. Highly stable conditions are typical of the large-scale motions in the earth's atmosphere, but less stable conditions do occur on smaller scales, and may occur in other planetary atmospheres. Since the amplitudes of the perturbations are undetermined in Eady's linearized model, we can only calculate the directions of the transports and their spatial variations. We will calculate these properties consistently to second order in the amplitude expansion of the stability analysis; thus, strictly speaking, our results will only apply to perturbations of small amplitude. Nevertheless, such information has proved very useful in parameterizing the effect of atmospheric motions, as shown by studies such as those of Charney (1959) and Green (1970). By using the simple approximations to the eigenfunctions found in Part I we will be able to find especially simple expressions for the transports.

In Parts I and II three kinds of instability were discussed:

1) Baroclinic instability of the kind described by Eady in his original paper. Throughout this paper we will refer to this kind of instability as "geostrophic" instability, although, in fact, the most unstable perturbations are not geostrophic under conditions of moderate stability. In Eady's model this kind of instability only occurs for $\mathrm{Ri}>0.84$.

2) Symmetric instability, which is essentially an inertial instability. In Eady's model symmetric instability only occurs when $\mathrm{Ri}<1$.

3) Kelvin-Helmholtz instability, which, in general, occurs when $\mathrm{Ri}<\frac{1}{4}$.

Here we will only calculate the transports for the geostrophic and symmetric instabilities, since the eigenfunctions corresponding to Kelvin-Helmholtz instability in Eady's model contain discontinuities (see Eliassen *et al.*, 1953) and are therefore unrealistic.

Since Eady's model neglects horizontal shear in the unperturbed flow, some of the transport properties which we will calculate will be unrealistic. In particular, when the dominant instabilities are geostrophic, i.e., when $\mathrm{Ri}\gg 1$, the meridional transport of zonal momentum depends crucially on the horizontal shear (Pedlosky, 1964; Stone, 1969). Also the meridional variations of all the transports depend on the horizontal shear. Consequently, we will emphasize the other transport properties.

2. Mathematical model

Eady's (1949) model assumes a Boussinesq, adiabatic, inviscid and hydrostatic fluid, with density ρ and thermal expansion coefficient α, located on a plane rotating about a vertical axis with angular speed $f/2$ and with a gravitational acceleration g. If x, y, z are respectively, the zonal, meridional, and vertical rectangular coordinates, and t the time, then the conservation equations for the zonal velocity u, the meridional velocity v, the vertical velocity w, the hydrodynamic pressure P, and the potential temperature θ, are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.1}$$

$$\frac{du}{dt}=fv-\frac{1}{\rho}\frac{\partial P}{\partial x}, \tag{2.2}$$

$$\frac{dv}{dt}=-fu-\frac{1}{\rho}\frac{\partial P}{\partial y}, \tag{2.3}$$

$$\frac{\partial P}{\partial z}=\alpha\rho g\theta, \tag{2.4}$$

$$\frac{d\theta}{dt}=0. \tag{2.5}$$

The model assumes that the fluid is contained between two horizontal planes at $z=0$, H and is unbounded horizontally, so that the boundary conditions are

$$w=0 \quad \text{at} \quad z=0, H. \tag{2.6}$$

The basic flow state consists of a zonal wind of magnitude U with constant vertical shear and no horizontal shear, and a potential temperature field with constant stratification, $\partial\theta_0/\partial z$, and constant horizontal gradient, $\partial\theta_0/\partial y$, related to U by the thermal wind relation. We will put the equations in dimensionless form by using the following basic units: $|x|=|y|=U/f$; $|z|=H$; $|t|=1/f$; $|u|=|v|=U$; $|w|=fH$; $|\theta|=H(\partial\theta_0/\partial z)$; and $|P|=\alpha\rho g H^2(\partial\theta_0/\partial z)$. In terms of these dimensionless variables, Eqs. (2.1)–(2.6), become

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{2.7}$$

$$\frac{du}{dt}=v-\mathrm{Ri}\frac{\partial P}{\partial x}, \tag{2.8}$$

$$\frac{dv}{dt}=-u-\mathrm{Ri}\frac{\partial P}{\partial y}, \tag{2.9}$$

$$\frac{\partial P}{\partial z}=\theta, \tag{2.10}$$

$$\frac{d\theta}{dt}=0, \tag{2.11}$$

$$w=0 \quad \text{at} \quad z=0, 1. \tag{2.12}$$

The Richardson number is defined as

$$\mathrm{Ri}=\frac{\alpha g H^2}{U^2}\frac{\partial\theta_0}{\partial z}=\frac{f^2(\partial\theta_0/\partial z)}{\alpha g(\partial\theta_0/\partial y)^2}. \tag{2.13}$$

The second representation for Ri is derived by substituting the thermal wind relation into the first.

We will denote Eady's unperturbed flow state by zero subscripts. In terms of our dimensionless variables, this state is

$$v_0=w_0=0, \tag{2.14}$$

$$u_0=z, \tag{2.15}$$

$$\theta_0=z-\frac{y}{\mathrm{Ri}}, \tag{2.16}$$

$$P_0=\tfrac{1}{2}z^2-\frac{yz}{\mathrm{Ri}}. \tag{2.17}$$

We assume that the deviations from this state have an amplitude ϵ and that $\epsilon \ll 1$, so that we may expand the total solution in powers of ϵ, e.g.,

$$u=u_0+\epsilon u_1+\epsilon^2 u_2+\cdots. \tag{2.18}$$

Substituting similar expansions for all the dependent variables into Eqs. (2.7)–(2.12), and equating coefficients of the same order in ϵ, we obtain an expanded set of equations for the unknowns in the expansions.

The zero-order equations are, of course, automatically satisfied by the choices made for the basic state, Eqs. (2.14)–(2.17). The first-order equations are just the linearized equations in Eady's model of baroclinic stability. The coefficients of these latter equations depend only on z, so we can assume their solutions to be of the form $\exp[i(\sigma t+kx+ly)]$, and understand that the real part of the complex solution is the physically meaningful part. The resulting equations for the z dependence of the complex first-order solutions are

$$iku_1+ilv_1+\frac{dw_1}{dz}=0, \tag{2.19}$$

$$i(\sigma+kz)u_1+w_1=v_1-ik\,\mathrm{Ri}P_1, \tag{2.20}$$

$$i(\sigma+kz)v_1=-u_1-il\,\mathrm{Ri}P_1, \tag{2.21}$$

$$\frac{dP_1}{dz}=\theta_1, \tag{2.22}$$

$$i(\sigma+kz)\theta_1-\frac{v_1}{\mathrm{Ri}}+w_1=0, \tag{2.23}$$

$$w_1=0 \quad \text{at} \quad z=0, 1. \tag{2.24}$$

With a little algebra these equations can be reduced to a single equation for w_1, i.e.,

$$[1-(\sigma+kz)^2]\frac{d^2w_1}{dz^2}-2\left[\frac{k}{\sigma+kz}-il\right]\frac{dw}{dz}-\left[(k^2+l^2)\,\mathrm{Ri}+\frac{2ilk}{\sigma+kz}\right]w=0. \tag{2.25}$$

This equation and the boundary conditions (2.24) constitute the eigenvalue problem for the complex frequency σ, as originally formulated by Eady. Eq. (2.25) is identical to Eady's Eq. II, 11, taking into account the different way in which we have defined the parameters. Since we have used U/f as the unit of horizontal length, $k/(2\pi)$ and $l/(2\pi)$ are simply the Rossby numbers of the perturbation, defined in terms of the perturbation's zonal and meridional wavelengths, respectively.

The transports we wish to calculate are the total advective transports averaged over longitude, i.e., averaged over one wavelength in x. If we let a bar over a quantity denote this average, then the mean meridional and vertical transports of zonal momentum and heat for a Boussinesq fluid are proportional to $\overline{uv}$, $\overline{uw}$, $\overline{\theta v}$ and $\overline{\theta w}$, respecitvely. Since θ is the potential temperature the latter two transports include both the transport of sensible heat and the transport of potential energy. If we substitute the perturbation expansions (2.18) into $\overline{\theta v}$ and make use of the zero order solutions, (2.14)–(2.17), we obtain

$$\overline{\theta v}=\epsilon\theta_0\bar{v}_1+\epsilon^2(\overline{\theta_1 v_1}+\theta_0\bar{v}_2)+O(\epsilon^3). \tag{2.26}$$

Substituting the real part of the assumed complex forms of the first-order solutions and evaluating the x averages, we obtain

$$\overline{\theta v}=\epsilon^2\exp(-2\sigma_i t)\tfrac{1}{2}\,\mathrm{Re}(\theta_1 v_1^*)+\epsilon^2\theta_0 v_2+O(\epsilon^3). \tag{2.27}$$

In this equation σ_i is the imaginary part of σ ($\sigma_i<0$ for the unstable perturbations), Re indicates the real part, θ_1 and v_1 now refer solely to the z-dependent part of the first-order solution, and the asterisk denotes the complex conjugate. The leading contributions to the transport are of order ϵ^2, and there are two terms of this order. The first, $\frac{1}{2}\mathrm{Re}(\theta_1 v_1^*)$, is proportional to the eddy or Reynolds flux, and the second, $\theta_0\overline{v_2}$, is proportional to the flux due to the mean flow. In a similar manner we can show that the other transports are also composed of an eddy transport plus a mean transport:

$$\overline{\theta w}=\epsilon^2\exp(-2\sigma_i t)\tfrac{1}{2}\,\mathrm{Re}(\theta_1 w_1^*)+\epsilon^2\theta_0\overline{w_2}+O(\epsilon^3), \tag{2.28}$$

$$\overline{uv}=\epsilon^2\exp(-2\sigma_i t)\tfrac{1}{2}\,\mathrm{Re}(u_1 v_1^*)+\epsilon^2 u_0\overline{v_2}+O(\epsilon^3), \tag{2.29}$$

$$\overline{uw}=\epsilon^2\exp(-2\sigma_i t)\tfrac{1}{2}\,\mathrm{Re}(u_1 w_1^*)+\epsilon^2\overline{u_0 w_2}+O(\epsilon^3). \tag{2.30}$$

In order to calculate these transports to order ϵ^2, we need both the first- and second-order solutions in the amplitude expansions. Substituting the expansions (2.18) into Eqs. (2.7)–(2.12), we obtain the second-order equations

$$\frac{\partial u_2}{\partial x}+\frac{\partial v_2}{\partial y}+\frac{\partial w_2}{\partial z}=0, \tag{2.31}$$

$$\frac{\partial u_2}{\partial t}+u_0\frac{\partial u_2}{\partial x}+u_1\frac{\partial u_1}{\partial x}+v_1\frac{\partial u_1}{\partial y}+w_1\frac{\partial u_1}{\partial z}+w_2=v_2-\mathrm{Ri}\frac{\partial P_2}{\partial x}, \tag{2.32}$$

$$\frac{\partial v_2}{\partial t}+u_0\frac{\partial v_2}{\partial x}+u_1\frac{\partial v_1}{\partial x}+v_1\frac{\partial v_1}{\partial y}+w_1\frac{\partial v_1}{\partial z}=-u_2-\mathrm{Ri}\frac{\partial P_2}{\partial y}, \tag{2.33}$$

$$\frac{\partial P_2}{\partial z}=\theta_2, \tag{2.34}$$

$$\frac{\partial \theta_2}{\partial t}+u_0\frac{\partial \theta_2}{\partial x}+u_1\frac{\partial \theta_1}{\partial x}+v_1\frac{\partial \theta_1}{\partial y}-\frac{v_2}{\mathrm{Ri}}+w_1\frac{\partial \theta_1}{\partial z}+w_2=0, \tag{2.35}$$

$$w_2=0 \text{ at } z=0,1. \tag{2.36}$$

Since the first-order solutions are periodic in x, this will also be the case for the second-order solutions. Since we only need mean values of the second-order solutions, we will average Eqs. (2.31)–(2.36) over x and simplify them by using the periodicity of the solutions. If at the same time we substitute the real part of the complex forms of the first-order solution into the equations, and write the advective terms in flux form, we obtain

$$\frac{\partial \bar{v}_2}{\partial y}+\frac{\partial \bar{w}_2}{\partial z}=0, \tag{2.37}$$

$$\frac{\partial \bar{u}_2}{\partial t}+\exp(-2\sigma_i t)\frac{\partial}{\partial z}[\tfrac{1}{2}\,\mathrm{Re}(w_1 u_1^*)]+\bar{w}_2=\bar{v}_2 \tag{2.38}$$

$$\frac{\partial \bar{v}_2}{\partial t}+\exp(-2\sigma_i t)\frac{\partial}{\partial z}[\tfrac{1}{2}\,\mathrm{Re}(w_1 v_1^*)]=-\bar{u}_2-\mathrm{Ri}\frac{\partial \bar{P}_2}{\partial y}, \tag{2.39}$$

$$\frac{\partial \bar{P}_2}{\partial z}=\bar{\theta}_2, \tag{2.40}$$

$$\frac{\partial \bar{\theta}_2}{\partial t}-\frac{\bar{v}_2}{\mathrm{Ri}}+\exp(-2\sigma_i t)\frac{\partial}{\partial z}[\tfrac{1}{2}\,\mathrm{Re}(w_1 \theta_1^*)]+\bar{w}_2=0, \tag{2.41}$$

$$\bar{w}_2=0 \text{ at } z=0,1. \tag{2.42}$$

In these equations u_1, v_1, w_1 and θ_1 now refer only to the z-dependent parts of the first-order solutions, i.e., the solution of Eqs. (2.19)–(2.24). Therefore, the drive terms are independent of y, and so will be the solution for the mean second-order fields. Thus, Eqs. (2.37) and (2.42) require that $\bar{w}_2=0$ and Eqs. (2.38) and (2.39) reduce to two equations for $\overline{u_2}$ and $\overline{v_2}$,

$$\frac{\partial \overline{u_2}}{\partial t} - \overline{v_2} = -\exp(-2\sigma_i t)\frac{\partial}{\partial z}[\tfrac{1}{2}\,\mathrm{Re}(w_1 u_1^*)], \quad (2.43)$$

$$\frac{\partial \overline{v_2}}{\partial t} + \overline{u_2} = -\exp(-2\sigma_i t)\frac{\partial}{\partial z}[\tfrac{1}{2}\,\mathrm{Re}(w_1 v_1^*)]. \quad (2.44)$$

The homogeneous solutions of these equations are oscillatory in time, with constant amplitude, and may be neglected compared to the particular solution which grows exponentially in time. The particular solution for $\overline{v_2}$ is

$$\overline{v_2} = \exp(-2\sigma_i t)\left[\left(\frac{\sigma_i}{1+4\sigma_i^2}\right)\frac{\partial}{\partial z}\mathrm{Re}(w_1 v_1^*) + \left(\frac{1/2}{1+4\sigma_i^2}\right)\frac{\partial}{\partial z}\mathrm{Re}(w_1 u_1^*)\right]. \quad (2.45)$$

Using the above results for $\overline{w_2}$ and $\overline{v_2}$, we can now express the leading contributions to the transports entirely in terms of the correlations of the first-order solutions. Since the amplitude of these solutions is undetermined, we will for convenience choose

$$\epsilon^2 \exp(-2\sigma_i t) = 1. \quad (2.46)$$

Consequently, our expressions for the transports, correct to second order in the amplitude expansion, become

$$\overline{\theta v} = \tfrac{1}{2}\,\mathrm{Re}(\theta_1 v_1^*) + \left(z - \frac{y}{\mathrm{Ri}}\right)\left[\left(\frac{\sigma_i}{1+4\sigma_i^2}\right)\frac{\partial}{\partial z}\mathrm{Re}(w_1 v_1^*) + \left(\frac{1/2}{1+4\sigma_i^2}\right)\frac{\partial}{\partial z}\mathrm{Re}(w_1 u_1^*)\right], \quad (2.47)$$

$$\overline{\theta w} = \tfrac{1}{2}\,\mathrm{Re}(\theta_1 w_1^*), \quad (2.48)$$

$$\overline{uv} = \tfrac{1}{2}\,\mathrm{Re}(u_1 v_1^*) + z\left[\left(\frac{\sigma_i}{1+4\sigma_i^2}\right)\frac{\partial}{\partial z}\mathrm{Re}(w_1 v_1^*) + \left(\frac{1/2}{1+4\sigma_i^2}\right)\frac{\partial}{\partial z}\mathrm{Re}(w_1 u_1^*)\right], \quad (2.49)$$

$$\overline{uw} = \tfrac{1}{2}\,\mathrm{Re}(u_1 w_1^*). \quad (2.50)$$

Finally, we will examine the sources of the kinetic energy of the perturbations. The equation for the kinetic energy is found by multiplying the first-order equations obtained from (2.8)–(2.10) by u_1, v_1 and w_1, respectively, and adding them together. Integrating over all x, y, and z, we find

$$\frac{\partial}{\partial t}\int \tfrac{1}{2}\overline{(u_1^2+v_1^2)}\,dydz = \mathrm{Ri}\int \tfrac{1}{2}\,\mathrm{Re}(\theta_1 w_1^*)\,dydz - \int \tfrac{1}{2}\,\mathrm{Re}(w_1 u_1^*)\,dydz. \quad (2.51)$$

The left-hand side represents the change in kinetic energy of the perturbation. On the right-hand side the first term represents the release (or absorption) of potential energy from the unperturbed state, and the second term the release (or absorption) of kinetic energy from the unperturbed state.

3. Transports by geostrophic instabilities

We will now use the approximate first-order solutions found in Part I (Section 4) to compute all of the eddy correlations and transports which accompany geostrophic instability. The most rapidly growing perturbation for this kind of instability is the one with wavenumbers

$$k = \left(\frac{5/2}{1+\mathrm{Ri}}\right)^{\frac{1}{2}}, \quad (3.1)$$

$$l = 0. \quad (3.2)$$

If we let

$$c = -\frac{1}{2} - \frac{i}{2\sqrt{3}}, \quad (3.3)$$

the z dependence of the vertical velocity for this perturbation is given by

$$w_1 = [(c+z)^3 - c^3]\left[1 + (6+\mathrm{Ri})k^2\frac{(c+z)^2}{10}\right]. \quad (3.4)$$

Eq. (3.2) is exact, and the numerical results of Part II showed that Eqs. (3.1) and (3.4) have a maximum error of 7%, which occurs when $\mathrm{Ri} \approx 1$. Under geostrophic conditions, $\mathrm{Ri} \gg 1$, the error in these same expressions is <2%.

Since we are interested in the qualitative features of the transports, we will use an even simpler form for w_1 than Eq. (3.4). In particular, we will neglect the relatively small-amplitude second term in Eq. (3.4) and take

$$w_1 \approx (c+z)^3 - c^3. \quad (3.5)$$

Physically, this is equivalent to a "long-wave" approximation ($k \to 0$). Calculating the transports from this form for w_1 gives results that are exact for very long waves, but only approximate for the most unstable wave. However, the errors in the transports calculated for the most unstable wave by using Eq. (3.5) in place of (3.4) are only about 10% when $\mathrm{Ri} \to \infty$ and about 30% when $\mathrm{Ri} = O(1)$. Consequently, we can greatly simplify the algebra in the calculations without essentially changing the results by using Eq. (3.5) for w_1. To be consistent with this approximation for w_1, we will use the corresponding "long-wave" approximation for σ (see Part I, Section 4),

$$\sigma = kc, \quad (3.6)$$

with k given by Eq. (3.1), and retain only the dominant terms for small k in calculating the correlations.

The approximate solutions for the z dependence of the other first-order fields are found by substituting Eqs. (3.2), (3.5) and (3.6) into Eqs. (2.19)—(2.23), We obtain

$$u_1=3i[\tfrac{2}{5}(1+\mathrm{Ri})]^{\frac{1}{2}}(c+z)^2, \tag{3.7}$$

$$v_1=-\frac{6}{5}(1+\mathrm{Ri})(c+z), \tag{3.8}$$

$$\theta_1=\frac{6i}{5}\left(\frac{2}{5}\right)^{\frac{1}{2}}\frac{(1+\mathrm{Ri})^{\frac{3}{2}}}{\mathrm{Ri}}. \tag{3.9}$$

From these first-order solutions we can calculate all six of the correlations between u_1, v_1, w_1 and θ_1 in a straightforward manner. It is convenient to write the results in terms of a vertical variable centered in the middle of the atmosphere, i.e.,

$$h=z-\tfrac{1}{2}, \tag{3.10}$$

so that $-\frac{1}{2}<h<+\frac{1}{2}$. The results are as follows:

$$\mathrm{Re}(u_1v_1^*)=-\frac{3}{5}\left(\frac{6}{5}\right)^{\frac{1}{2}}(1+\mathrm{Ri})^{\frac{3}{2}}(h^2+\tfrac{1}{12}), \tag{3.11}$$

$$\mathrm{Re}(u_1w_1^*)=-\left(\frac{3}{10}\right)^{\frac{1}{2}}(1+\mathrm{Ri})^{\frac{1}{2}}(h^2-\tfrac{1}{4})^2, \tag{3.12}$$

$$\mathrm{Re}(v_1w_1^*)=-\frac{6}{5}(1+\mathrm{Ri})(h^4-\tfrac{1}{16}), \tag{3.13}$$

$$\mathrm{Re}(\theta_1u_1^*)=\frac{36}{25}\frac{(1+\mathrm{Ri})^2}{\mathrm{Ri}}(h^2-\tfrac{1}{12}), \tag{3.14}$$

$$\mathrm{Re}(\theta_1v_1^*)=\frac{18}{25}\left(\frac{2}{15}\right)^{\frac{1}{2}}\frac{(1+\mathrm{Ri})^{\frac{5}{2}}}{\mathrm{Ri}}, \tag{3.15}$$

$$\mathrm{Re}(\theta_1w_1^*)=\frac{18(1+\mathrm{Ri})^{\frac{3}{2}}}{5\sqrt{30}\,\mathrm{Ri}}(\tfrac{1}{4}-h^2). \tag{3.16}$$

The vertical transports of momentum and heat depend only on the eddy transports [see Eqs. (2.48) and (2.50)] and therefore the above correlations immediately tell us the direction of these two transports. Eq. (3.12) shows that the vertical transport of zonal momentum is negative (downward) for all h and all Ri. This agrees with the direction in large-scale atmospheric perturbations (Palmén and Newton, 1969, Section 1.3.) Eq. (3.16) shows that the vertical transport of heat is positive (upward) for all h and all Ri. This agrees with the direction inferred for large-scale atmospheric perturbations (Palmén and Newton, 1969, Section 2.6).

If we substitute Eqs. (3.12) and (3.16) into (2.51), we find that the kinetic energy of the growing perturbation is drawn from both the potential and kinetic energy of the basic state. However, under geostrophic conditions ($\mathrm{Ri}\gg1$) the kinetic energy release is negligible compared to the potential energy release. Thus, the kinetic energy exchange between the basic flow and the perturbation would be dominated by the horizontal eddy correlations, $\mathrm{Re}(u_1v_1^*)$, as soon as any horizontal shear were added to the basic state. Only under non-geostrophic conditions ($\mathrm{Ri}\approx1$) is the kinetic energy release by the vertical eddy stresses a significant contribution to the perturbation's kinetic energy.

Substituting Eqs. (3.12), (2.46) and (3.13) into Eq. (2.45), we find that the mean meridional motion is

$$\bar{v}_2=\left(\frac{6}{5}\right)^{\frac{1}{2}}(1+\mathrm{Ri})^{\frac{3}{2}}\left[\frac{h(h^2+\frac{1}{4})}{\mathrm{Ri}+11/6}\right]. \tag{3.17}$$

In a model with horizontal shear in the basic flow the derivatives with respect to y in Eq. (2.33) would not be zero, and they would thus contribute another term to the solution for $\bar{v}_2$. Under geostrophic conditions ($\mathrm{Ri}\gg1$) this new term would be larger than the non-geostrophic term given by Eq. (3.17). Only under non-geostrophic conditions will the two terms be comparable, and then the solution (3.17) does represent an essential part of $\bar{v}_2$. This non-geostrophic contribution to v_2 [Eq. (3.17)] corresponds to a thermodynamically direct circulation, with $\bar{v}_2>0$ for $z>\frac{1}{2}$ and $\bar{v}_2<0$ for $z<\frac{1}{2}$.

Substituting Eqs. (3.1), (3.3), (3.6), (3.11), (3.12) and (3.13) into Eq. (2.49), we obtain the horizontal transport of zonal momentum

$$\overline{uv}=-\frac{3}{10}\left(\frac{6}{5}\right)^{\frac{1}{2}}(1+\mathrm{Ri})^{\frac{3}{2}}(h^2+\tfrac{1}{12})$$
$$+\left(\frac{6}{5}\right)^{\frac{1}{2}}\left(\frac{(1+\mathrm{Ri})^{\frac{3}{2}}}{11/6+\mathrm{Ri}}\right)h(h+\tfrac{1}{2})(h^2+\tfrac{1}{4}). \tag{3.18}$$

The first term represents the eddy transport and is negative (equatorward) for all h and Ri. The second term represents the mean transport and it changes sign with height. As in the calculation of $\bar{v}_2$, the neglect of horizontal shear eliminates terms from $\overline{uv}$ which would be larger than the terms in Eq. (3.18) when $\mathrm{Ri}\gg1$. Consequently, horizontal shear should be included to correctly calculate this transport under geostrophic conditions (see Pedlosky, 1964, and Stone, 1969). But under non-geostrophic conditions Eq. (3.18) does represent an important contribution to the transport. In order to find the net effect we integrate Eq. (3.18) over height, and obtain

$$\int_0^1 \overline{uv}dz=\frac{(1+\mathrm{Ri})^{\frac{3}{2}}}{5\sqrt{30}}\left[-\frac{3}{2}+\frac{1}{11/6+\mathrm{Ri}}\right]. \tag{3.19}$$

The net mean transport (the second term) is positive (poleward), but always numerically smaller than the net eddy transport (the first term). Consequently, the net non-geostrophic contribution to the meridional transport of zonal momentum is toward the equator for all Ri.

This result for the nongeostrophic contribution to the eddy meridional transport of zonal momentum is independent of the special form assumed for the unperturbed zonal flow in Eady's model. If we let $\bar{u}(y,z)$ be the unperturbed zonal flow, then Eq. (2.21) is replaced by

$$i(\sigma+k\bar{u})v_1=-u_1-\mathrm{Ri}\frac{\partial P_1}{\partial y}. \tag{3.19a}$$

If we multiply this equation by v_1^*, reorder the terms, and take one-half the real part, we have

$$\tfrac{1}{2}\,\mathrm{Re}(u_1v_1^*)=\tfrac{1}{2}\sigma_i v_1v_1^*-\tfrac{1}{2}\,\mathrm{Ri}\,\mathrm{Re}\left(v_1^*\frac{\partial P_1}{\partial y}\right). \tag{3.19b}$$

The first term on the right-hand side represents the non-geostrophic contribution to the eddy transport and the second the usual geostrophic contribution. Since $\sigma_i<0$ for unstable modes, the non-geostrophic contribution to the transport is toward the equator for all $\bar{u}(y,z)$. Since $\sigma_i=O(\mathrm{Ri}^{-\frac{1}{2}})$ [cf. Eqs. (3.6), (3.3) and (3.1)], the two contributions will be comparable in magnitude if $\mathrm{Ri}=O(1)$.

Finally, we calculate the meridional heat transport by substituting Eqs. (3.1), (3.3), (3.6), (3.12), (3.13) and (3.15) into Eq. (2.47). We obtain

$$\overline{\theta v}=\frac{9}{25}\left(\frac{2}{15}\right)^{\frac{1}{2}}\frac{(1+\mathrm{Ri})^{\frac{5}{2}}}{\mathrm{Ri}}+\left(\frac{6}{5}\right)^{\frac{1}{2}}(1+\mathrm{Ri})^{\frac{3}{2}}\left(h+\frac{1}{2}-\frac{y}{\mathrm{Ri}}\right)\left[\frac{h(h^2+\frac{1}{4})}{11/6+\mathrm{Ri}}\right]. \tag{3.20}$$

The first term again represents the eddy transport and is always positive (poleward). The second term represents the mean transport and changes sign with h and y. To find the total net transport we integrate over height and obtain

$$\int_0^1\overline{\theta v}dz=\frac{(1+\mathrm{Ri})^{\frac{3}{2}}}{5\sqrt{30}}\left[\frac{18(1+\mathrm{Ri})}{5\,\mathrm{Ri}}+\frac{1}{11/6+\mathrm{Ri}}\right]. \tag{3.21}$$

The last term in (3.21) again represents the mean transport, and its net effect, like the eddy transport, is to transport heat poleward for all Ri. However, its contribution to the total transport is never more than 6%, so that for all practical purposes the mean transport could be neglected in calculating the total transport. Since only this mean transport is significantly affected by neglecting horizontal shear, we expect the total meridional heat transport given by Eady's model to be essentially correct for all Ri. Comparing this result with the transport observed for atmospheric perturbations (Lorenz, 1967, Fig. 47), we again find that the direction agrees.

4. Transports by symmetric instabilities

The most unstable perturbation corresponding to symmetric instability (see Part I, Sections 2 and 3) is the one with wavenumbers

$$k=0, \tag{4.1}$$

$$l\rightarrow\infty. \tag{4.2}$$

The solution for the height dependence of its vertical velocity, to first order in the amplitude expansion, is

$$w_1=\exp\left(\frac{-ilz}{1-\sigma^2}\right)\sin(\pi z), \tag{4.3}$$

with σ^2 given by

$$\sigma^2=1+\frac{\mathrm{Ri}l^2}{2\pi^2}\left[1-\left(1+\frac{4\pi^2}{\mathrm{Ri}^2l^2}\right)^{\frac{1}{2}}\right]. \tag{4.4}$$

Substituting Eqs. (4.1) and (4.3) into Eqs. (2.19), (2.20) and (2.23), we obtain the other first order fields:

$$u_1=\frac{1}{i\sigma}\exp\left(\frac{-ilz}{1-\sigma^2}\right)\left[\left(\frac{\sigma^2}{1-\sigma^2}\right)\sin(\pi z)-\frac{\pi}{il}\cos(\pi z)\right], \tag{4.5}$$

$$v_1=\exp\left(\frac{-ilz}{1-\sigma^2}\right)\left[\frac{\sin(\pi z)}{1-\sigma^2}-\frac{\pi}{il}\cos(\pi z)\right], \tag{4.6}$$

$$\theta_1=\frac{1}{i\sigma}\exp\left(\frac{-ilz}{1-\sigma^2}\right)\times\left[\left(\frac{\sigma^2+\frac{1}{\mathrm{Ri}}-1}{1-\sigma^2}\right)\sin(\pi z)-\frac{\pi}{il\mathrm{Ri}}\cos(\pi z)\right]. \tag{4.7}$$

We now use these first-order solutions to calculate the same six correlations as in the preceding section. In the calculations we let $l\rightarrow\infty$, and retain only the leading terms. We find

$$\mathrm{Re}(u_1v_1^*)=-\mathrm{Ri}^{\frac{3}{2}}(1-\mathrm{Ri})^{\frac{1}{2}}\sin^2(\pi z)+O\left(\frac{1}{l^2}\right), \tag{4.8}$$

$$\mathrm{Re}(u_1w_1^*)=-[\mathrm{Ri}(1-\mathrm{Ri})]^{\frac{1}{2}}\sin^2(\pi z)+O\left(\frac{1}{l^2}\right), \tag{4.9}$$

$$\mathrm{Re}(v_1w_1^*)=\mathrm{Ri}\sin^2(\pi z)+O\left(\frac{1}{l^2}\right), \tag{4.10}$$

$$\mathrm{Re}(\theta_1u_1^*)=\frac{\pi^2}{(1-\mathrm{Ri})l^2}\left[1-\frac{\sin^2(\pi z)}{\mathrm{Ri}}\right]+O\left(\frac{1}{l^4}\right), \tag{4.11}$$

$$\mathrm{Re}(\theta_1v_1^*)=\frac{\pi^2}{l^2[\mathrm{Ri}(1-\mathrm{Ri})]^{\frac{1}{2}}}+O\left(\frac{1}{l^4}\right), \tag{4.12}$$

$$\mathrm{Re}(\theta_1w_1^*)=\frac{\pi^2\sin^2(\pi z)}{\mathrm{Ri}^{\frac{3}{2}}l^2(1-\mathrm{Ri})^{\frac{1}{2}}}+O\left(\frac{1}{l^4}\right). \tag{4.13}$$

We can again deduce the vertical transports directly from the correlations, since only the eddy transports contribute to the total mean vertical transports. Recalling that symmetric instability only occurs when $\mathrm{Ri}<1$ in Eady's model, we conclude from Eq. (4.9) that the vertical transport of zonal momentum is always negative (downward). Similarly from Eq. (4.13) we conclude that the vertical transport of heat is always positive (upward). Substituting these same two equations into the energy equation (2.51), we find that symmetric instability draws its kinetic energy from both the potential and kinetic energy of the basic flow. However, for the most rapidly growing symmetric instability, $l\rightarrow\infty$, the potential energy release is negligible.

Substituting Eqs. (4.4), (4.9), (4.10) and (2.46) into Eq. (2.45) and letting $l\rightarrow\infty$, we find for the mean meridional motion

$$\bar{v}_2=-\frac{3\pi}{2}\frac{\mathrm{Ri}^{\frac{3}{2}}(1-\mathrm{Ri})^{\frac{1}{2}}}{4-3\,\mathrm{Ri}}\sin(2\pi z). \qquad (4.14)$$

Therefore, $\bar{v}_2$ is positive in the upper atmosphere, $z>\frac{1}{2}$, negative in the lower atmosphere, $z<\frac{1}{2}$, and the circulation is thermodynamically direct.

Substituting Eqs. (4.4), (4.8), (4.9) and (4.10) into Eq. (2.49) and letting $l\rightarrow\infty$, we obtain the meridional transport of zonal momentum

$$\overline{uv}=-\tfrac{1}{2}\,\mathrm{Ri}^{\frac{3}{2}}(1-\mathrm{Ri})^{\frac{1}{2}}\sin^2(\pi z)$$
$$-\frac{3\pi}{2}\frac{\mathrm{Ri}^{\frac{3}{2}}(1-\mathrm{Ri})^{\frac{1}{2}}}{4-3\,\mathrm{Ri}}z\sin(2\pi z). \qquad (4.15)$$

The first term represents the eddy transport, and is negative (equatorward) for all $\mathrm{Ri}<1$ and for all z. This term has been calculated previously for Eady's model by Gierasch and Stone (1968). The second term represents the mean transport and is positive (poleward) in the upper atmosphere and negative (equatorward) in the lower atmosphere. To find the net transport we integrate over all z and obtain

$$\int_0^1\overline{uv}dz=\frac{3}{4}\frac{\mathrm{Ri}^{\frac{3}{2}}(1-\mathrm{Ri})^{\frac{1}{2}}}{4-3\,\mathrm{Ri}}[(\mathrm{Ri}-\tfrac{4}{3})+1]. \qquad (4.16)$$

The last term in the brackets represents the net mean transport, and is always poleward. Thus, the net mean transport is in the opposite direction from the eddy transport, and Eq. (4.16) shows that the sum is such that the total transport is poleward when $\mathrm{Ri}>\frac{1}{3}$ and equatorward[1] when $\mathrm{Ri}<\frac{1}{3}$.

Finally we calculate the meridional transport of heat by substituting Eqs. (4.4), (4.9), (4.10) and (4.12) into Eq. (2.47). Letting $l\rightarrow\infty$, we find

$$\overline{\theta v}=-\left(z-\frac{y}{\mathrm{Ri}}\right)\frac{3\pi}{2}\frac{\mathrm{Ri}^{\frac{3}{2}}(1-\mathrm{Ri})^{\frac{1}{2}}}{4-3\,\mathrm{Ri}}\sin(2\pi z). \qquad (4.17)$$

This transport arises entirely from the mean transport since the eddy transport is of order $(1/l^2)$. Its sign depends on z. To find the net transport we integrate Eq. (4.17) over all z and obtain

$$\int_0^1\overline{\theta v}dz=\frac{3\,\mathrm{Ri}^{\frac{3}{2}}(1-\mathrm{Ri})^{\frac{1}{2}}}{4(4-3\,\mathrm{Ri})}. \qquad (4.18)$$

Thus, the net meridional transport of heat by symmetric instability is always poleward.

5. Discussion

The basic stability parameter in Eady's model is the Richardson number, and its value is determined by the vertical and horizontal temperature gradients in the basic flow state [see Eq. (2.13)]. The stability of the flow increases as Ri increases, i.e., the growth rate for the most unstable mode monotonically decreases as Ri increases (see Part I, Fig. 4). Thus, it is not surprising to find that the heat transports by the unstable motions are always in such a direction as to increase Ri and stabilize the flow. In particular, both geostrophic and symmetric instability always transport heat upward, thereby tending to increase $\partial\theta_0/\partial z$, and always transport heat poleward, thereby tending to decrease $\partial\theta_0/\partial y$. Both effects independently tend to increase Ri.

The effect of the instabilities on the vertical shear of the basic flow is also uniform. Both kinds of instability always transport zonal momentum downward and tend to decrease the vertical shear. The effect on the horizontal shear is more ambiguous. As we have already pointed out, the horizontal transport of zonal momentum under geostrophic conditions is dominated by effects left out of Eady's model; thus, our results only have meaning under non-geostrophic conditions. Here we find that geostrophic instabilities and symmetric instabilities when $\mathrm{Ri}<\frac{1}{3}$ transport momentum equatorward and tend to make the horizontal shear of the zonal flow negative. However, when $\mathrm{Ri}>\frac{1}{3}$, symmetric instabilities transport momentum poleward and tend to make the horizontal shear positive.

Many previous studies of baroclinic stability have already demonstrated that the direction of the net transports calculated using linearized theory agree with the observed transports in the earth's atmosphere (e.g., Eady, 1949, and Pedlosky, 1964). Since our results apply for all values of Ri, they can be used in modelling these transports without making the usual restriction that the atmosphere is very stably stratified ($\mathrm{Ri}\gg1$). In fact, we will make use of our results in this way in a subsequent paper.

[1] This result for the total meridional transport of zonal momentum by symmetric instability was first reported to me verbally by Michael McIntyre. Presumably the calculation of $\int_0^1\overline{uv}\,dz$ presented here duplicates his original (unpublished) calculation.

Since mixing length theory is frequently invoked to justify modelling eddy fluxes by eddy coefficients, it is worth examining our results to see how well such a representation would work for baroclinic instabilities in Eady's model. We calculated twelve of these eddy fluxes, Eqs. (3.11)–(3.16) and (4.8)–(4.13). Four of the eddy fluxes are down-gradient ($\overline{u_1 w_1}$ and $\overline{\theta_1 v_1}$ for both types of instability) and could be modelled by positive eddy coefficients. Two are up-gradient ($\overline{\theta_1 w_1}$ for both types) and could be modelled by negative eddy coefficients. The other six fluxes are all finite in spite of the fact that the corresponding gradients for the basic state are zero, and eddy coefficients could not be used to model these transports. These results illustrate the hazards of using mixing length theory and eddy coefficients to model dynamical transports of heat and momentum.

The question of the horizontal momentum transport by symmetric instabilities has been discussed in the literature in connection with its possible role in producing the strong equatorial current observed in the atmosphere of Jupiter. [A convenient summary of the observations of Jupiter is given in Peek (1958)]. The eddy component of this transport was first calculated by Gierasch and Stone (1968) and they found that it was equatorward and suggested that this could be the cause of the equatorial acceleration. This idea has been called "incorrect" by Hide (1970) on the grounds that the equatorial current is westerly relative to the underlying surface, and it is not possible for symmetric motions, which involve no longitudinal pressure gradients, to accelerate the atmosphere to a speed higher than the speed of the planet's surface at the equator. Hide's physical statement is correct, but as a criticism it misses the point of Gierasch's and Stone's suggestion. Since the transport in question represents solely horizontal advection, its effect is to accelerate the zonal motions in low latitudes relative to those in high latitudes. It is this acceleration in latitude that is actually observed and that Gierasch's and Stone's mechanism was proposed to explain. Any acceleration relative to the underlying surface would have to be explained by another portion of the momentum cycle.

A more relevant criticism of Gierasch's and Stone's mechanism has been made by McIntyre (1970) who pointed out that the mean motions produced by symmetric instabilities may make an important contribution to the horizontal momentum transport. Our calculation of the mean transport term vindicates McIntyre's caution, since we found that it tends to counteract the eddy transport, and, in fact, dominates it as long as $\mathrm{Ri}>\frac{1}{3}$. Thus, Gierasch's and Stone's mechanism for the equatorial acceleration will only work if $\mathrm{Ri}<\frac{1}{3}$. This restriction reduces the plausibility of the mechanism since in this range the stability theory only predicts symmetric motions, whereas the observations of Jupiter show both symmetric and nonsymmetric features and therefore imply larger values of Ri.

On the other hand, we found in Section 3 that the nongeostrophic component of the horizontal momentum transport by geostrophic instabilities is equatorward for all values of Ri. This transport is another possible cause for Jupiter's equatorial acceleration. It is more plausible than the symmetric instability mechanism since there is no analogous restriction on Ri. In addition, since the geostrophic instabilities are nonsymmetric, they could account for the acceleration relative to the underlying surface as well as that relative to high latitudes.

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