Reply[1]

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8 November 1971

1. Introduction

I am pleased to have this opportunity to clarify my earlier analysis of the droplet collection equation (Long, 1971). Prof. Scott's comments have led me to review that treatment, and I feel able to respond satisfactorily to his objections, in the order they were presented.

2. Stochastic incompleteness of the collection equation

The stochastic completeness or incompleteness of the droplet collection equation has been discussed in this JOURNAL by Scott (1967, 1968), as well as by myself and others (Warshaw, 1967, 1968; Berry, 1968). Scott claims to have resolved the issue by demonstrating, in his earlier article, that the collection equation predicts not only the mean number of droplets in an interval dr, given by $\bar{N}(r,t)dr$, but the probability of all possible values of $N(r,t)dr$, as well. He showed that the mean number of droplets in a given size range equals the variance in the number of these droplets for an infinitesimal size range and, more importantly, for a finite size range as well, provided there is, in the latter case, no correlation between numbers of droplets of different sizes. Since equality of the mean and variance of a random variable happens to be a feature of the Poisson distribution, Scott apparently concluded that $N(r,t)dr$ must obey this distribution. But this feature of $N(r,t)dr$ does not ensure its Poisson distribution (see Warshaw, 1968). In fact,

$$\lim_{\Delta r\to 0}\{\bar{N}(r,t)\Delta r-\mathrm{var}[N(r,t)\Delta r]\}=0$$

is a feature of *any* reasonable probability distribution for $N(r,t)\Delta r$. We must conclude then that the prediction by the collection equation of $\bar{N}(r,t)dr$ is not equivalent to a prediction of the probability of all possible values of $N(r,t)dr$; hence, the collection equation is stochastically *in*complete.

In his third paragraph, Scott asserts that the probability of all stochastically possible events is included in the solution of the collection equation. If we interpret "event" to mean "droplet collection" event, then this statement is clearly false, since solutions of the equation give the droplet size distribution at any given time and not the probability of a collection event. On the other hand, if we interpret "event" to mean some value for $N(r,t)dr$ at some time t, then this statement must still be false by reason of the arguments given above.

3. Motivation for the analysis in Long (1971)

Because most of the remainder of Scott's objections apparently stem from a basic misunderstanding of the reasons for the type of analysis in my earlier article, I will now briefly outline its general motivation. Hopefully, this will clarify both the method and results of that treatment.

I was dealing there with the *validity* of those numerical solutions to the collection equation that must be obtained whenever we wish to allow realistically for the physical interactions of colliding and coalescing droplets. (Perhaps a better title for my earlier paper might have been "Validity of Numerical Solutions of the Droplet Collection Equation.") These numerical solutions yield the mean number of droplets, $\bar{N}(r,t)\Delta r$, in some finite size range Δr. Although the statistical dispersion in $N(r,t)\Delta r$ mentioned above and discussed

[1] The research reported here was supported by the Office of Naval Research under Contract N00014-67-A-0209-0009.

by Scott (1967) will affect the validity of these solutions to some extent, there is an additional source of dispersion in $N(r,t)\Delta r$, with potentially a much greater effect. The source of this additional dispersion is the inherent statistical dispersion in that finite change in the size distribution $\Delta N(r,t)\Delta r$ whose mean value is computed at each individual forward-integration time step to yield $\bar{N}(r,t)\Delta r$. This dispersion in $\Delta N(r,t)\Delta r$ is not taken into account in numerical solutions of the collection equation and may accumulate sufficiently over a number of time steps to invalidate any further numerical integration. In evaluating this accumulated dispersion, we cannot use the results of Scott showing equal mean and variance of $N(r,t)\Delta r$, since the source of this latter dispersion is entirely different. Rather we must examine the magnitude of the dispersion of $\Delta N(r,t)\Delta r$ itself. My earlier analysis was devoted to just such an examination.

4. Magnitude of Δr

Scott is mistaken in suggesting that the magnitude of Δr can be freely chosen when computing the quantities displayed in Figs. 1–4; on the contrary, the value I used ($\Delta r = 1\ \mu$) is most appropriate. (Incidentally, this value for Δr appears explicitly along the ordinates in Figs. 1–3.) To show that this value for Δr is quite reasonable, we first note that each radial density presentation in Figs. 1–3 is a smooth fit to a *histogram* of the function displayed. This type of presentation is useful because it corresponds directly to the computational format of numerical solutions of the collection equation. In solving the collection equation numerically, we compute values of $\overline{\Delta N}(r,t)\Delta r$ for a number of different radius intervals $[r, r+\Delta r]$. The width Δr of these intervals will vary with r, but a number of investigators (e.g., Berry, 1967) have indicated that, for the smaller droplet sizes, Δr must be no more than a fraction of $1\ \mu$, although large droplet categories may be a few tens of microns wide. Ideally, the histogram interval Δr to use in obtaining Figs. 1–3 should have this same range of values. A reasonable compromise value of $1\ \mu$ was chosen for our analysis. Any much larger value, such as 1 cm as suggested by Scott, would have been both irrelevant and misleading.

5. An error

An entirely negligible error entered into my expression in Eq. (3) for the variance of $pr(r,\rho;\tau)$, the probability of a collection event involving two particular r and ρ droplets. It was my goal to neglect double-collection processes, and these are, in fact, assumed negligible in my formulation in Section 3 of this probability. But, as Scott correctly points out, consistency with this initial assumption required that I should have also neglected the factor $[1-pr(r,\rho;\tau)]$ in Eq. (3). Furthermore, this bracketed factor should have been deleted from Eqs. (9)–(16) in Section 5, leading to my expression for the variance of $\Delta N(r,t)$. But since this has a negligible effect on the graphical results in Figs. 1–4, the discussion in Sections 6, 7 and 8 remains valid.

Scott's next remarks on the inclusion of second-order terms indicate that apparently he has misunderstood my derivation in Section 5 of var$[\Delta N(r,t)]$. It should be clear that I do not focus there on the "mean number of collection events involving one particular r droplet and all the ρ droplets of range ρ to $\rho+d\rho$," but rather that I start with an expression for the variance in the number of collection events involving two particular r and ρ droplets and derive an expression for the variance in the change $\Delta N(r,t)$.

6. Appropriate measures of the validity of numerical solutions of the collection equation

Although I continue to feel that the statistical measures employed in my earlier article are the most appropriate to use in testing numerical solutions of the collection equation, Scott's comments on one of them, the relative dispersion of $\Delta N(r,t)$, do warrant some rebuttal.

Scott feels that this statistical measure is physically irrelevant to the problem at hand, but this impression is mistaken. In part b of Section 6, I examined the merits of the absolute and relative dispersion of $\Delta N(r,t)$ and combined the least ambiguous aspects of both these statistical measures into a test of the validity of the collection equation that was meaningful for all droplet sizes. I explicitly eliminated the relative dispersion from this test for all those r for which the relative dispersion becomes infinite. (See the first three paragraphs of part b.) Furthermore, all of my conclusions were drawn from *other* aspects of Figs. 3 and 4.

Scott believes that $\sigma[\Delta N(r,t)\Delta r]/\bar{N}(r,t)\Delta r$ would be a better test of the numerical solutions of the collection equation. From the standpoint of estimating the dispersion in $N(r,t)\Delta r$ that may accumulate over a number of time steps, this is indeed a useful quantity. But a clear idea of the importance of this accumulation relative to the numerically computed changes in $\bar{N}(r,t)\Delta r$ can be obtained only if we also know the ratio $\sigma[\Delta N(r,t)\Delta r]/\overline{\Delta N}(r,t)\Delta r$. But this ratio is just the relative dispersion of $\Delta N(r,t)\Delta r$.

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