# NOTES AND CORRESPONDENCE

## A Note on Predictability

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ABSTRACT

It is assumed that in making a forecast the initial state of the system is known subject to an initial error of measurement. Given an upper bound for the initial error, rigorous bounds are determined for the error of the forecast and for the time interval in which the error in the forecast is less than a pre-assigned value.

### 1. Introduction

The problem of predictability involves forecasting the future state of a physical system given imperfect knowledge of the initial state. For meteorological purposes this consists of finding a vector $z_i(t), i=1, 2, \ldots N$, satisfying the system

$$\dot{z}_i=\sum_j a_{ij}z_j+\sum_{j,k} b_{ijk}z_jz_k+q_i(t), \qquad (1)$$

and the initial condition

$$z_i(0)=c_i+h_i, \qquad (2)$$

where $c_i$ is a best estimate of the initial value and $h_i$ is an error term, in general unknown. The forecast $z_i^0(t)$ solves (1) with initial condition $z_i^0(0)=c_i$. Assuming that the reduction of the hydrodynamic equations to the form (1) and the numerical integration procedure for computing $z_i^0$ do not involve appreciable errors, the forecast is accurate if

$$x_i=z_i-z_i^0 \qquad (3)$$

is small in some sense.

One method of estimating the error is to integrate (1) for a variety of initial conditions. This is inherently inefficient, particularly for large systems. Another method, in principle very attractive, is to assume that the initial error $h_i$ is randomly distributed according to some probability law and then to forecast the probability distribution of $x_i$ by solving an appropriate Liouville equation (Epstein, 1969). Exact solution of the Liouville equation is beyond the capabilities of present day computers and approximations must be made. It is not known at present whether these approximations are valid.

The purpose of this note is to provide a deterministic upper bound for the error $x_i(t)$ given an upper bound for the initial error $h_i=x_i(0)$. Because the amount of computation necessary to provide this estimate is fairly small, the results contained herein provide a practical approach to the predictability problem and can be incorporated into forecast schemes.

### 2. Error estimates

Let $\mathbf{x}(t)$ and $\mathbf{h}$ be the column vectors whose components are $x_i(t)$ and $h_i$. Substitution of (3) into (1) leads to the vector equation

$$\dot{\mathbf{x}}=\mathbf{A}(t)x+\mathbf{f}(x), \quad \mathbf{x}(0)=\mathbf{h}, \qquad (4)$$

where the matrix $\mathbf{A}$ and vector $\mathbf{f}$ have components

$$A_{ij}=a_{ij}+\sum_k (b_{ijk}+b_{ikj})z_k^0(t), \qquad (5a)$$

$$f_i=\sum_{j,k} b_{ijk}x_jx_k. \qquad (5b)$$

Since $\dot{\mathbf{x}}$ is a polynomial in $x$, the solution $\mathbf{x}(t)$ is unique, and is a continuous function of $\mathbf{h}$. Therefore, $\mathbf{h}=0$ implies $\mathbf{x}=0$, and $|\mathbf{h}|\leqslant\epsilon$ implies $|\mathbf{x}(t)|\leqslant k$, for any constant $k>\epsilon$, at least in some time interval $0\leqslant t \leqslant T(k,\epsilon)$. We will seek an upper bound for $|\mathbf{x}(t)|$ and a lower bound for $T$, given the upper bound $\epsilon$ for $|\mathbf{h}|$.

The norm for a vector $\mathbf{y}$ is defined to be

$$|\mathbf{y}|=(\mathbf{y}^T\mathbf{F}\mathbf{y})^{\frac{1}{2}}, \qquad (6)$$

where $\mathbf{F}$ is a constant positive definite symmetric matrix and $\mathbf{y}^T$ is the transpose of $\mathbf{y}$. This is a convenient norm, since it allows us to weight the components of $\mathbf{x}$ differently. Let $r=|\mathbf{x}(t)|$. Then

$$\dot{r}=r^{-1}\{\mathbf{x}^T[\tfrac{1}{2}(\mathbf{FA}+\mathbf{A}^T\mathbf{F})]\mathbf{x}+\mathbf{x}^T\mathbf{F}\mathbf{f}(\mathbf{x})\}, \quad r(0)\leqslant\epsilon. \qquad (7)$$

Now, for any vector $\mathbf{y}$,

$$\mathbf{y}^T[\tfrac{1}{2}(\mathbf{FA}+\mathbf{A}^T\mathbf{F})]\mathbf{y}\leqslant\alpha(t)\mathbf{y}^T\mathbf{F}\mathbf{y}, \qquad (8)$$

where $\alpha(t)$ is the largest eigenvalue of $\frac{1}{2}(\mathbf{A}+\mathbf{F}^{-1}\mathbf{A}^T\mathbf{F})$. Hence,

$$\dot{r}\leqslant\alpha(t)r+[\mathbf{x}^T\mathbf{F}\mathbf{f}(\mathbf{x})]/(\mathbf{x}^T\mathbf{F}\mathbf{x})^{\frac{1}{2}}, \qquad (9)$$

and this implies that $r(t) \leqslant u(t)$, where

$$\dot{u} = \alpha(t)u + [\mathbf{x}^T\mathbf{F}\mathbf{f}(\mathbf{x})]/(\mathbf{x}^T\mathbf{F}\mathbf{x})^{\frac{1}{2}}, \quad u(0) = \epsilon. \tag{10}$$

Now let

$$\beta = \underset{y}{\mathrm{Max}}\,[\mathbf{y}^T\mathbf{F}\mathbf{f}(\mathbf{y})]/(\mathbf{y}^T\mathbf{F}\mathbf{y}), \quad 0 \leqslant |\mathbf{y}| \leqslant 1. \tag{11}$$

Since $f$ is a quadratic polynomial,

$$\beta p = \underset{y}{\mathrm{Max}}\,[\mathbf{y}^T\mathbf{F}\mathbf{f}(\mathbf{y})]/(\mathbf{y}^T\mathbf{F}\mathbf{y}), \quad 0 \leqslant |\mathbf{y}| \leqslant p, \tag{12}$$

and from $r \leqslant u$ there follows the inequality

$$[\mathbf{x}^T\mathbf{F}\mathbf{f}(\mathbf{x})]/(\mathbf{x}^T\mathbf{F}\mathbf{x})^{\frac{1}{2}} \leqslant \beta ur \leqslant \beta u^2, \tag{13}$$

whence

$$\dot{u} \leqslant \alpha(t)u + \beta u^2. \tag{14}$$

It follows that $u \leqslant v$, where

$$\dot{v} = \alpha(t)v + \beta v^2, \quad v(0) = \epsilon, \tag{15}$$

and $r(t) \leqslant v(t)$. Also, given $k > \epsilon$, it is seen that $t \leqslant \tau$ implies $r(t) \leqslant k$, where $\tau$ is the smallest solution of

$$v(\tau) = k. \tag{16}$$

Hence, we have an upper bound $v(t)$ for the error $|\mathbf{x}(t)|$ and a lower bound $\tau$ for the time interval in which the error is within specified limits.

## 3. Discussion

The argument given here is an adaptation of a well-known result in stability theory (Halanay, 1966, p. 71). It may be noted that if $\alpha(t) < 0$, the solution is stable in the sense of Liapunov. In general, of course, the solution is unstable in that a small value of $\mathbf{h}$ implies $\mathbf{x}(t)$ small only in a finite interval.

In carrying out the indicated calculation it is necessary to perform the following computations.

1) The actual forecast $z_i^0(t)$ must be computed.

2) At each time step the largest eigenvalue of $\frac{1}{2}(\mathbf{A} + \mathbf{F}^{-1}\mathbf{A}^T\mathbf{F})$ must be computed.

3) The constant $\beta$ given by Eq. (11) must be calculated.

4) Eq. (15) for $v(t)$ must be solved.

The only troublesome steps are 2) and 3). However, these calculations involve finding a constrained maximum, a problem for which efficient algorithms exist. We expect that very little computational time is involved.

The inequalities could be sharpened in a variety of ways. For example, if it is known *a priori* that no trajectories of $\mathbf{x}(t)$ can enter a certain region $D$ of phase space, additional constraints can be put on the maximization problems for determining $\alpha(t)$ and $\beta$. The resulting maxima, $\hat{\alpha}(t)$ and $\hat{\beta}$, will be less than or equal to their previous values, and the functions $\hat{v}(t)$ will thus be less than or equal to $v(t)$. It may also be possible to work with a Liapunov function $s = (\mathbf{x}^T\mathbf{C}\mathbf{x})^{\frac{1}{2}}$ in order to determine bounds for $r$. By suitable choice of the positive definite matrix $\mathbf{C}$ it might be possible to derive sharper bounds than those obtained here. We are particularly interested in the possibility that the magnitude of the error $\mathbf{x}(t)$ is determined by only a few of the components of $\mathbf{h}$. If this is so, an element of selectivity can be brought into the measurement problem.

We note finally that the bounds determined here are functionals of the forecast $z_i^0$, and hence differ from forecast to forecast. In order to determine how sharp the bounds are it is necessary to carry out some numerical experiments. Further results will be reported in due course.

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REFERENCES

Epstein, E. S., 1969: Stochastic dynamic prediction. *Tellus*, **21**, 739–759.

Halanay, A., 1966: *Differential Equations*. New York, Academic Press, 529 pp.