Refraction Correction of Rocket Tracking Radar Inputs in Near Real Time

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ABSTRACT

A technique has been developed to correct rocket tracking radar inputs in near real time (maximum computing time of 100 msec) for refractive effects, by iterating twice an approximate closed-form solution for an exponentially stratified model. Maximum computational discrepancies for a test atmosphere between the results using this technique and those using an accurate numerical integration of the ray-tracing equations were under 3 m for the range error and 0.2 mrad for both the bending and the elevation errors. The technique works well for any initial antenna elevation angle, including zero and negative values, and for all practical radar ranges.

1. Introduction

In rocket radar tracking it is often desirable to correct for refraction effects in real time. One way of achieving the necessary computing speed (corrections must be available in less than 100 msec) is to fit a reasonable model to the refractive profile in order to attempt to solve for the refraction corrections in closed form. In this report we:

1) Review briefly numerical ray tracing and a technique commonly used to achieve a closed-form solution for a linear atmosphere.

2) Introduce a short, iterative, refraction-computing technique which uses an exponential model.

3) Compare the results of using the iterative solution with those using numerical integration.

The iterative technique developed here works well at any elevation angle and improves on similar previously proposed closed-form solutions. The improvement is especially noticeable at low angles of elevation, where the effects of refraction are greatest.

2. Ray tracing—Linear atmosphere

All practical refraction-correction techniques are based on ray tracing. When the methods of geometrical optics can be used, a ray traveling through a refractive medium will bend in such a way that the curvature (the reciprocal of the radius of curvature \( R_c \)) at any point in the trajectory is (Kerr, 1951)

\[ \text{Curv} = 1/R_c = (1/n)(dn/dR_c), \]

where "\( n \)" is the local refractive index. The necessary conditions for (1) to be applicable are: (i) the index of refraction must not change greatly over one wavelength, and (ii) rays within a bundle must not converge (diverge) too rapidly; that is, rays may not cross each other.

In a spherically stratified medium, each individual trajectory lies in a plane which passes through the center of symmetry, and hence it is advantageous to use spherical coordinates. For this case it can be shown

\[ nr \cos \phi = c = \text{constant}, \]

where \( r \) is the magnitude of the radius vector \( r \) originating at the center of symmetry which follows the ray (Fig. 1), and \( \phi \) the emergence angle between the ray and a horizontal plane. Eq. (2) is known as Snell's law and expresses the fact that once the initial conditions are given, the emergence angle in a spherically stratified medium depends only on the local index of refraction.

From the geometry of an elementary part of the trajectory (Fig. 2), we have

\[ \tan \phi = (1/r)(dr/d\theta), \]

where \( \theta \) is the central angle swept by the radius vector \( r \). A transformation to dimensionless variables is convenient for numerical computations in the atmosphere. Let

\[ \begin{align*}
    r &= R + h_0 \\
    \xi &= \tan \phi \\
    \eta &= \ln(r/r_0) = \ln(1 + (r-r_0)/r_0) \\
    \nu &= \ln(n/n_0) = \ln(1 + (n-n_0)/n_0)
\end{align*} \]

where \( R \) is the radius of the earth and \( h_0 \) the radar elevation.

Differentiating Snell's law logarithmically and introducing (2) and (3), we obtain

\[ \begin{align*}
    d\eta/d\theta &= \xi \\
    d\xi/d\theta &= (1+\xi^2)(1+dn/\eta)
\end{align*} \]

which is easy to integrate numerically and specifies...
the dimensionless coordinates of the ray in terms of the independent parameter θ. We also note that
\[ P = 1 + \frac{dr}{dη} = 1 + (n/r)(dn/dr). \] (6)

In a linear atmospheric model, the refractive profile is divided into a number of sufficiently thin slices so that the vertical gradient of \( n \), \( (dn/dr) \), is sensibly constant within each slice. Since \( n \) is always very near unity and the percentage variation of \( r \) within each slice is very small, we see from (6) that \( P \) is also sensibly constant within each slice; thus, it can be substituted by its average value \( \bar{P} \) in (5). With these assumptions, (5) can be readily integrated in closed form. Techniques based on similar approaches have been well reported in the literature (Schulkin, 1952; Weisbrod and Anderson, 1959; Gardner, 1964). All these techniques differ only in the mathematical simplifications invoked, and the final answers often turn out to be the same. We can also point out that since
\[ (dn/dr_0) = (dn/dr) \cos φ, \] (7)
then, from (1) and (2)
\[ \text{Curv}=\frac{[n_0 r_0 \cos φ_0 (dn/dr)]}{(n^2 r)}. \] (8)
Thus, the assumptions needed to solve (5) in closed form when using a linear model also imply that the curvature remains sensibly constant within each slice. These assumptions are thus equivalent to assuming a circular trajectory within each slice.

3. Exponential model—Real time corrections

Most practical refraction reduction methods use the linear stratified model. When the reference layers are not too far spaced, the assumptions of this model are

Fig. 1. Geometry of ray tracing.

Fig. 2. Geometry of an element of trajectory.
well justified and compatible with present-day accuracy of refractive index measurements. The need for many layers which require separate computations, however, greatly increases execution time; thus, this method is generally not feasible for use in real time. Furthermore, additional computations (in the form of interpolations) are needed in the last layer to accommodate the exact radar range, which in general does not coincide with the range of a reference level. The alternative of precomputing several key profiles and storing the results for various elevation angles is faster, but requires a large storage capacity.

Refractive profiles tend to follow an exponential rather than a linear law over very thick layers, so reference layers could be considerably spaced if an exponential model were used. The refractive index in an exponential model is given by

$$N = N_0 \exp[-a(r-r_0)], \quad (9)$$

where $N_0$ and $a$ are the parameters of the model, and $N$ is the usual modified index of refraction defined by

$$N = (n-1)10^6. \quad (10)$$

Freeman (1964), following the lead of Thayer (1961), showed how acceptable approximations could also be made in an exponential model to yield a closed-form solution for the refraction corrections. Rowlandson (1968) proposed using some empirical constants to improve the fit. Neither method works well below an elevation of 5°. Since corrections at low angles of elevation are of primary interest in this report, and since the approximations invoked by Freeman and Rowlandson are most critical in this region, we will, in our derivation, follow closely Freeman’s approach in order to investigate also the possibility of improving on his solution by keeping higher order terms.

$a. \ Target \ elevation$

In what follows, we will use dimensionless variables. This makes it easier to keep track of the order of magnitude of the various terms and also simplifies computations. Distances will be normalized by expressing them in terms of the radar radius vector $r_0$. The normalized slant range $S$ is then

$$S = SR/r_0, \quad (11)$$

where $SR$ is the measured radar range. Similarly, the normalized target elevation is

$$\rho = (r-r_0)/r_0 = \Delta h/r_0, \quad (12)$$

where $\Delta h$ is the actual (unknown) target elevation above the radar site. All distances are to be measured in the same units used for the earth’s radius $R$.

Let $\beta_0$ be the true target elevation, and $\phi_0$ be the radar-measured target elevation. From the geometry of Fig. 1, we have

$$\rho = (1+S^2+2S \sin \beta_0)^{1/2} - 1. \quad (13)$$

Expanding the radical by the binomial theorem to a second-order approximation gives us

$$\rho \approx \frac{1}{2}[(S \cos \beta_0 + \tan \beta_0)^2 - \tan^2 \beta_0]. \quad (14)$$

We note that

$$\rho/S = \sin \beta_0 + S/2 \cos \beta_0 \cdots, \quad (14a)$$

which indicates, for small angles of elevation ($\beta_0 < 5^\circ$), that $\rho$ is at least one order of magnitude smaller than $S$.

$b. \ Bending \ error$

Let

$$\mu = (n-n_0)/n_0 = K[\exp(-\lambda \rho) - 1], \quad (15)$$

where

$$\lambda = \alpha r_0 \quad \text{and} \quad K = 10^{-6}N_0/(1+10^{-6}N_0) \approx 10^{-6}N_0 \quad \text{[]}, \quad (16)$$

so $\lambda$ is the normalized refractive decay parameter. We note at all elevations that

$$|\mu/\rho| < \lambda K.$$  

Over the whole path, $\lambda$ rarely exceeds $10^3$. Hence, the ratio $|\mu/\rho|$ is at most $\frac{1}{3}$, and we can neglect second-order terms involving $\mu$ in comparison with those involving $\rho$.

By Snell’s law, the emergence angle is such that

$$\cos \phi = \cos \phi_0/(1+\rho)/(1+\mu). \quad (17)$$

The elementary bending rate, or curvature, as given by (1) and from the geometry of Fig. 2, is

$$(d\tau/dS) = r_0(1/n)(dn/dr) \cos \phi = (d\mu/d\rho) \cos \phi/(1+\mu)/(1+\mu). \quad (18)$$

Expanding to a second-order approximation,

$$(d\tau/dS) \approx -\lambda K \cos \phi_0 (1-\rho+\rho^2-2\mu) \exp(-\lambda \rho). \quad (19)$$

We introduce now a new variable to simplify the handling. Let

$$A = (\lambda/2)^{1/2} \quad \text{and} \quad \mu_0 = A \tan \beta_0 = A S \cos \beta_0 + w_0 \quad \text{[]}.$$ \quad (20)

Eq. (14) can then be rewritten as

$$\rho \approx (w^2-w_0^2)/(2A^2), \quad (21)$$

and also

$$dS = dw/(A \cos \beta_0). \quad (22)$$

Eq. (19) then becomes

$$d\tau \approx -2B_0(B_1 + B_2w^2 + B_3w^4 + B_4 \exp(\omega^2 - \omega^2) \exp(\omega^2 - \omega^2) dw, \quad (23)$$
where
\[
\begin{align*}
B_0 &= AK \cos \phi_0 \sec \beta_0 \\
B_1 &= 1 + \frac{\omega^2}{2 \omega} (1 + \omega^2/2 A^2) + 2 K \\
B_2 &= \frac{1}{2 A^4} (1 + \omega^2/2 A^2) \\
B_3 &= \frac{1}{4 A^4} \\
B_4 &= -2K
\end{align*}
\]
(24)

This can be integrated to yield
\[
\tau \approx B_0 C_1 G(w) + C_2 (w E - w_0) + C_3 (w^2 E - w_0^2) + C_4 G(2w) \]  
(25)

where
\[
\begin{align*}
E &= \exp(w_0^2 - w^2) \\
F(x) &= 2 \exp(x^2) \int_x^\infty \exp(-y^2) d\alpha \\
G(x) &= EF(w) - F(w_0)
\end{align*}
\]
(26)

\[
\begin{align*}
C_1 &= 1 + \left[ \frac{\omega^2}{2 \omega} \right] [1 + \left[ \frac{\omega^2}{2 A^2} \right] ] \\
C_2 &= \frac{1}{2 A^4} (1 + \omega^2/2 A^2 - 3/4 A^4) \\
C_3 &= \frac{1}{4 A^4} \\
C_4 &= -2K
\end{align*}
\]
(27)

The function \( F(x) \) varies smoothly from \( \pi^4 \) to zero when the argument \( x \) varies from zero to an infinitely large value, and admits a simple and accurate rational approximation:
\[
\begin{align*}
F(x) &= 1.00229 \{ x + 0.039262 + 0.1856633 [ x - 9.02478 + 115.2736 (x + 12.29742)^{-1} ]^{-1} \}^{-1} \]  
(28)

The minimax rational approximation developed for \( F(x) \) is accurate to five decimals for an infinite range of \( x \), and can be computed very fast, even on a small computer.

An order-of-magnitude analysis was next made of the terms involved in (25). Over thick layers, typical values are: \( A \approx 20, \omega \approx 2 \) for elevation angles under 5°. The order of magnitude of the various constants \( C_i \) is then
\[
\begin{align*}
C_1 &\approx 1 \\
C_2 &\approx 10^{-3} \\
C_3 &\approx 10^{-4} \\
C_4 &\approx 10^{-4}
\end{align*}
\]  
(29)

The function
\[
f_1(w) = wE - w_0 = w \exp(w_0^2 - w^2) - w_0
\]
(30)

reaches a maximum when \( w = 2^{-1} \). From (20) it follows that \( f_1 \) is also bounded by \( 2^{-1} \approx 0.707 \). Similarly, the function
\[
f_2(w) = w^2 E - w_0^2 = w^4 \exp(w_0^2 - w^2) - w_0^2
\]
(31)

reaches a maximum when \( w = \frac{1}{\sqrt{2}} \); hence, from (20), \( f_2 \) is bounded by \( \frac{1}{\sqrt{2}} \approx 1.837 \). Since \( F(x) \) has an upper bound \( \pi^4 \), and \( E \) is bounded by unity, \( G(x) \) is also bounded by \( \pi^4 \). This means that the terms involving \( C_2, C_3 \) and \( C_4 \) in (25) are at least three orders of magnitude smaller than the term involving \( C_1 \) and can be neglected. Examining now in detail the term \( C_1 \) as given in (27) we can see that all of its various components are at least two orders of magnitude smaller than the first one, which is unity. Finally, note that
\[
\cos \phi_0 \sec \beta_0 \approx 1 - \epsilon \tan \phi_0 + \epsilon^2/2 \cdots,
\]
(32)

where
\[
\epsilon = \phi_0 - \beta_0
\]
(33)

is the elevation error. Since \( \epsilon \) never exceeds 20 mrad, the product \( \cos \phi_0 \sec \beta_0 \), occurring in (24) differs from unity at most by quantities of the second order of magnitude. In view of the many unavoidable measurement uncertainties, it was finally decided to make \( C_1 \) equal to unity, drop the terms involving \( C_2 \) through \( C_4 \), and make \( B_0 \approx AK \), with the final result:
\[
\tau \approx AK [EF(w) - F(w_0)]
\]
(34)

which is basically similar to Rowlandson's (1968) formulation for long ranges but which remains also accurate at short ranges. This analysis also shows that it is not practical to improve further on Freeman's (1964) approach by keeping higher order terms in the derivation.

\( c. A \) short iterative solution for the bending

Eq. (34) still suffers from a disadvantage also present in Freeman's formulation; it implies implicitly the unknown target elevation angle \( \beta_0 \) in the computation of \( \omega_0 \) and \( w \) in (20). Rowlandson (1968) proposed using an auxiliary angle \( \gamma \), which lies somewhere between \( \phi_0 \) and \( \beta_0 \) in place of \( \beta_0 \), that would be expressible in terms of known quantities by using an empirical fit. The true elevation \( \beta_0 \), however, can be obtained by an iterative procedure starting from the known radar elevation angle \( \phi_0 \). A complete iterative solution for \( \beta_0 \), however, is neither necessary nor desirable. As already pointed out by Freeman, using the radar elevation angle \( \phi_0 \) instead of \( \beta_0 \) in the computations tends to underestimate the bending because the line of sight lies above the real path (see Fig. 1). Using the true value of \( \beta_0 \), while improving matters, tends to overestimate the bending as the computation path then lies below the real path, and these differences are most marked at low elevations.

The elevation error \( \epsilon \) and the bending \( \tau \) are related (Bean and Dutton, 1966) by
\[
\epsilon = \arctan \left[ \frac{\cos \tau - \sin \tau \tan \phi - (1 + \mu)}{\tan \phi (1 + \mu) - \cos \tau \tan \phi - \sin \tau} \right].
\]
(35)

Since \( \tau \) is always a small angle, if we make the customary small-angle approximations and keep up to second-order terms, we have
\[
\epsilon \approx (\tau^2/2 + \tau \tan \phi + \mu)/(\tau + \tan \phi - \tan \phi_0).
\]
(36)

Furthermore, Weishbrod and Anderson (1959) have
shown that over short ranges
\[ \tau = 2 \times 10^{-8} (N_0 - N) / (\tan \phi + \tan \phi_0) \]
\[ \approx K (1 - E) / \tan \phi \approx -\mu / \tan \phi. \] (37)

Thus, for short ranges the numerator of (36) reduces to \( \tau^2 / 2 \). At the same time, \((\tan \phi - \tan \phi_0) \ll \tau\) so the denominator tends to \( \tau \). Hence,
\[ \epsilon \rightarrow \tau / 2 \ (\text{short ranges}), \]
and this approximation will undercorrect at long ranges, since as the range increases, \( \tan \phi \) increases without bounds and
\[ \epsilon \rightarrow \tau \ (\text{long ranges}), \]
which is a well-known result (Bean and Dutton, 1966). Thus, if we apply the correction \( \tau / 2 \) to the initial radar elevation angle \( \phi_0 \) only once, there will be a tendency for proper compensation at all ranges, since this correction undercompensates at the longer ranges, which is exactly what (34) needs. This compensation mechanism indeed works, so a simple iteration is possible by letting \( \beta_0 = \phi_0 \) as the initial guess and computing an approximate bending \( \tau_1 \) using (34). A second improved elevation angle is \( \beta_0 = \beta_0 + \tau_1 / 2 \), and the final bending \( \tau \) is obtained using (34) again.

Computer requirements are very small and only an exponential subroutine is needed. Once the bending \( \tau \) has been obtained, all other refraction corrections can be obtained from it.

d. Range error

The elementary retardation error is (Weisbrod and Anderson, 1959)
\[ d(RT) = (n - 1) dS, \] (38)
where RT is the total retardation. Introducing (21) and (22),
\[ d(RT) = K \exp(w_0^2 - w^2) / (A \cos \beta_0) d\omega, \] (39)
which can be integrated to yield
\[ RT \approx K \sec \beta_0 \left[ F(w_0) - EF(w) \right] / (2A^2). \] (40)
Comparing (40) and (34) and substituting \( \phi_0 \) for \( \beta_0 \), since the secant varies very slowly when the arc is small, yields
\[ RT = r_0 \tau \sec \phi_0 / (2A^2), \] (41)
where we have multiplied by \( r_0 \) to express the retardation in conventional units.

For radar ranges under 600 km, we can approximate the path by an equivalent circular arc, and the curvature error CE is given by the difference in length between this equivalent arc and the chord:
\[ CE = 2R_c [\tau/2 - \sin(\tau/2)] = R_c \tau (\tau/24) = r_0 S \tau/24. \] (42)
The final range error RE is given by the sum of the retardation and curvature errors:
\[ RE = r_0 \tau [\sec \phi_0 / (2A^2) + S \tau/24]. \] (43)

e. Elevation error

From the geometry of Fig. 1 we see that
\[ \theta = \phi - \phi_0 + \tau, \] (44)
\[ S^2 = 1 + (1 + \rho)^2 - 2(1 + \rho) \cos \theta. \] (45)
Combining these two equations with (14), (15), (17) and (33), we have a system of six equations with six unknowns \((\rho, \beta_0, \mu, \phi, \epsilon, \theta)\) which could be solved for the elevation error \( \epsilon \). An iterative computing procedure similar to the one used to obtain the bending, however, is preferable. Let
\[ \begin{aligned}
\rho &= (w_0^2 - w^2) / (2A^2) \\
\mu &= K (1 - E) \\
\delta &= \rho + \mu \\
\tan \phi &= [\tan \phi_0 + \delta (2 - \delta)] - \tan \phi_0,
\end{aligned} \] (46)
where \( w_0 \) and \( E \) are obtained from (20) and (26) by setting \( \beta_0 = \phi_0 \). A first iterated value \( \epsilon_1 \) is now obtained by using (36). A second iteration is now made using the improved value
\[ \beta_0 = \beta_0 - \epsilon_1, \] (47)
and this second iteration yields our final estimation for \( \epsilon \). No further meaningful improvement in \( \epsilon \) can be expected from further iterations.

f. Doppler velocity angle error

Millman (1961) has shown that the Doppler velocity error in the radial direction which can be attributed to refraction is, to a first-order approximation, measured by the angle \( \delta \) between the ray direction and the slant true path. From Fig. 1 we see that
\[ \delta = \tau + \phi_0 - \beta_0 = \tau - \epsilon, \] (48)
which shows that a Doppler correction angle is immediately available from the estimates for the total bending and the elevation angle error.

4. Comparison of refraction corrections at low angles of elevation using the iteration technique with those obtained by numerical integration

Tables 1 and 2 present a comparison between the exact solutions for the refraction corrections at low angles of elevation, using numerical integration, and those using the technique developed in this report. More concisely, we refer to the exact and the iteration results. The reference atmosphere used is a standard CRPL exponential atmosphere with a surface refractive index of 344.5 N units and a decay parameter of 0.1568 km⁻¹.
### Table 1. Comparison between refraction-induced range errors (m) and bending errors (mils) in a standard CRPL atmosphere for different ranges and radar elevation angles, and the values obtained by the iteration technique.

<table>
<thead>
<tr>
<th>( \phi_e ) (mrad)</th>
<th>Error evaluation technique</th>
<th>Range error Range (km)</th>
<th>Total bending Range (km)</th>
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<tbody>
<tr>
<td></td>
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<td>10</td>
<td>50</td>
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<td>8.7</td>
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</table>

* E, exact value of error obtained by using numerical integration; I, error computed using the iteration technique.

### Table 2. Comparison between refraction-induced elevation and Doppler velocity angle errors (mils) in a standard CRPL atmosphere for different ranges and radar elevation angles, and the values obtained by the iteration technique.

<table>
<thead>
<tr>
<th>( \phi_e ) (mrad)</th>
<th>Error evaluation technique</th>
<th>Elevation error Range (km)</th>
<th>Doppler error Range (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>10</td>
<td>50</td>
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<td>0.9</td>
</tr>
</tbody>
</table>

* E, exact value of error obtained by using numerical integration; I, error computed using the iteration technique.

The tables illustrate the remarkably good fit between the iteration and the exact solutions. The fit is particularly good at medium and short ranges, an area where Rowlandson's (1968) approximations start losing validity. The largest absolute discrepancy in range error is under 3 m in the worst case, and bending and elevation error discrepancies do not exceed 0.2 mrad. This is specially impressive if we realize that the iteration technique is perfectly general and uses no "ad hoc" parameters to improve the fit. The algorithm is much faster than Rowlandson’s, since there is no need to integrate numerically for the elevation error. The com-
putational discrepancies observed between the exact and iterated techniques are, in general, smaller than those that may occur due to our imperfect knowledge of the refractive profile.

5. A quadratic model—Recapitulation

The estimation of the best parameters for fitting an exponential model to a refractive profile is handled by standard curve-fitting techniques. For the simpler linear model that we have been considering so far, this involves fitting a least-squares line to the plot of \( H_i \) vs \( \ln(N_i) \). As soon as several of these plots are made, however, one is immediately tempted to improve the fit by going to a higher order polynomial, especially when sharp changes of refractive gradient are encountered. If the monomial in (9) is substituted by a binomial, we obtain a quadratic model where

\[
N = N_0 \exp(-\alpha_1 \Delta h - \alpha_2 \Delta h^2),
\]

and we would fit a parabola to the plot of \( H_i \) vs \( \ln(N_i) \) instead of a straight line.

If we make the appropriate substitutions in (9), (15) and (16) and neglect terms of higher than second order in the subsequent derivations, we find that we only have to make

\[
A = (r_0 c_1/2)^{1/2} (1 + r_0 c_2 \tan \beta_0 / r_0 c_1/2)^{-1}
\]

instead of the value given in (20) to arrive at the same solution (34) for \( \tau \). This makes the quadratic model almost as easy to use as the linear, since fitting a parabola to a set of data points is still a very simple computational procedure.

The gain, however, may be illusory. As pointed out by Vickers and López (1969), the uncertainties in our knowledge of the state of the atmosphere might cause changes in the refraction corrections greater than the changes brought about by using a more sophisticated fit, especially when dealing with low angles of elevation and refractive profiles derived from single-point sources such as radiosonde data.

In recapitulation, the procedure for obtaining high-speed refraction corrections is as follows: A least-squares straight line is fitted to the plot of \( h_i \) vs \( \ln(N_i) \). Suppose the best-fitting line is

\[
\ln(N_i) = a + bh_i.
\]

The parameters for the exponential model are then

\[
\begin{align*}
\lambda &= -br_0 \\
N_0 &= \exp(a + bh_0)
\end{align*}
\]

where, as before, \( h_0 \) is the radar elevation and \( r_0 = R + h_0 \).

Suppose now that \( SR \) is the measured radar range and \( \phi_0 \) the antenna elevation. We next compute \( S = SR/r_0 \), set \( \beta_0 = \phi_0 \), and compute a first iterated value for \( \tau \) from (20) and (34), using (26) and (28) to obtain the values of \( E \) and \( F(x) \), respectively. A second and final iteration for \( \tau \) is made by letting \( \beta_0 = \beta_0 + \tau / 2 \) and recomputing (20) and (34).

To obtain the elevation error \( \epsilon \) we set again \( \beta_0 = \phi_0 \) and obtain a first iterated value for \( \epsilon \) from (46) and (36), using (20) and (26) to compute \( w_0, w \) and \( E \), respectively. A final value for \( \epsilon \) is obtained by letting \( \beta_0 = \beta_0 - \epsilon \) and repeating the calculations just once more. The range error and Doppler velocity error angle are obtained from (43) and (48), respectively.

REFERENCES


Rowlandson, L. L., 1968: Simple analytical functions which provide magnitude of range and angle errors for propagation in an exponential atmosphere. ESD Contract F19(628)-68-C-0209, Aerospace Instrumentation Program Office, Syracuse University.


