A Comparison Between the Effects of Fourier Truncation and a Class of Linear Digital Filters

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In a recent paper (Shapiro, 1971) the desirable properties of a certain class of linear digital filters were discussed and compared with the use of both linear and nonlinear diffusion operators as parameterizations of sub-grid-scale atmospheric diffusion. These filters were characterized as "ideal" in the sense that for any order

n, the filter removes the smallest resolvable wave component (the two grid-interval wave) but produces a minimum of damping of all longer waves without introducing any phase shifts or extraneous wave components. Such filters are simple to use and can easily be designed to suit a variety of purposes. Nevertheless, depending upon the order of the filter and the size of the domain in which it is applied, it may actually entail a greater number of product operations than similar types of filtering obtained by Fast Fourier Transform (FFT). For example, in the above-mentioned paper, sea-level pressure data at 5° longitude intersections around latitude circles were used for illustrative purposes. The stencil of the $n=8$, one-dimensional ideal filter has a length of $(2n+1)$ or 17 grid points. Therefore, to filter the data around the entire latitude circle requires 1224 operations consisting of a multiplication and an addition. The number of operations required for the FFT is equal to $N(r_1+r_2+\cdots+r_m)$ where N, the total number of data points, equals $r_1 r_2 \cdots r_m$ (Cooley and Tukey, 1965). Thus, to determine the spectrum of the above sea-level pressure data, to modify each component as desired, and to reconstruct the modified pressure distribution around the latitude circle by means of the FFT requires about 1700 operations. It is apparent that the computational efforts of both procedures are comparable, especially since under some circumstances the efficiency of the FFT can be increased by about a factor of 2 (Bingham *et al.*, 1967). However, if N were larger or if n were smaller, there would be even fewer operations with the ideal filter, relative to the FFT. Nevertheless, since the computational advantage of the ideal filter is not likely to exceed a factor of 2 or 3, the choice of which procedure is most appropriate for a particular application would most likely be based upon other considerations. It is, therefore, of interest to compare some of the properties of the Fourier and ideal filter approaches to the modification of meteorological data. For this purpose we shall make use of the same sea-level pressure data.

One difference between the two approaches, as has been pointed out (Shapiro, 1971), pertains to their treatment of sharp spikes in the data. Since the stencil of the Fourier approach is in effect equal to the length of the entire domain, the influence of a sharp spike at one point is immediately felt throughout the entire latitude circle. On the other hand, the stencil length of the ideal filter will, in general, be considerably smaller than the entire domain and thus the influence of a sharp spike will be limited to the vicinity of the spike. Since the modification of each Fourier component by the ideal filter is described by the filter's response function, this response can be exactly duplicated by a Fourier analysis. If the amplitude of each wavenumber k determined by Fourier analysis in a periodic domain of length $N\Delta X$ is weighted by the response function of the ideal filter $[1-\sin^{2n}(k\Delta X/2)]$, and the pressure distribution is reconstructed using the modified amplitudes with the

TABLE 1. Mean square differences (mb²) between the original pressure distribution around three latitude circles on two separate dates and various modifications of the pressure distribution.

Latitude N	Ideal filter 9-point	Ideal filter 17-point	Fourier reconstruction using wavenumbers 1–12	1–18	1–24
December					
35	0.13	0.08	1.11	0.51	0.23
45	0.13	0.09	3.06	0.44	0.22
55	0.13	0.07	1.79	0.73	0.23
January					
35	0.46	0.31	4.09	1.19	0.75
45	0.38	0.22	4.88	1.77	0.61
55	0.22	0.12	2.10	0.59	0.36
Average	0.24	0.15	2.84	0.87	0.40

original phases, then the resulting pressure distribution is identical to that obtained by applying the ideal filter directly to the data. This correspondence, however, is not maintained for any other set of weights, such as a truncated Fourier spectrum.

Inasmuch as Fourier truncation is often used as a filtering mechanism, a variety of Fourier truncation strategies have been compared with the use of both the $n=4$ and $n=8$ ideal filters on the sea-level pressure distribution given at 5° longitude intervals around latitude circles. The results are presented in Table 1. The first two columns show the mean squared differences between the original pressures at the 72 grid points and the pressures after one application of the 9-point and 17-point filters, respectively. The remaining columns show the mean square differences between the same original pressure data and three different strategies for truncating the Fourier spectrum. In these columns, the headings 1–12, 1–18 and 1–24 indicate that only wavenumbers 1–12, 1–18 and 1–24 have been retained with their original amplitude and phase and that all higher wavenumbers in each category have been completely eliminated.

Both in the use of the ideal filter and in the use of Fourier truncation it is assumed that we wish to remove the two-grid-interval wave and strongly damp the high wavenumber components. All of the filtering strategies in Table 1 accomplish this purpose, but the ideal filters do so with a minimum modification of the original pressure distribution. The lower order of filtering produces considerably less modification than the highest order of truncation. In and of itself, however, this feature of the digital filter does not show that its use is preferable to that of Fourier truncation. If wavenumbers 1–35 had been retained, the Fourier modification of the pressure distribution would have been less than that produced by the 17-point ideal filter. In fact, in terms of mean square differences, the 17-point filter corresponds to Fourier truncation at around wavenumber 31 or 32 of the spectra of these pressure data. However, truncation at wavenumber 31 or 32 would remove only waves shorter than about 2.3

grid intervals and might not produce sufficient filtering of the short wavelengths.

It appears, therefore, that the ideal filter would be preferable to Fourier truncation for most applications. However, inasmuch as there is little or no difference between the filter and a suitably weighted Fourier reconstruction, the basis of a choice between them depends only upon the nature of the application. If we know what kind of filtering we wish to apply to the data we can design a procedure, either digitally or spectrally, which produces the filtering most efficiently. In some circumstances, however, such as where the domain is non-periodic, the digital filter will be the probable choice because of the simplicity of its application. In such cases it is often very convenient to apply the ideal filter as a sequence of n, 3-point complex operators rather than as a single $(2n+1)$ real operator. In other circumstances, such as where the filtering desired departs appreciably from that produced by the ideal filter (for example, spectral truncation), the spectral approach will probably be preferable.

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