Implications of a Quadratic Stream Definition In Radiative Transfer Theory[1]

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ABSTRACT

An explicit definition of the radiation-stream concept is stated and applied to approximate the integro-differential equation of radiative transfer with a set of twelve coupled differential equations. Computational efficiency is enhanced by distributing the corresponding streams in three-dimensional space in a totally symmetric way. Polarization is then incorporated in this model. A computer program based on the model is briefly compared with a Monte Carlo program for simulation of horizon scans of the earth's atmosphere. It is found to be considerably faster.

1. Introduction

This work was undertaken as part of a research effort directed at the computerized modeling of light scattering in the earth's atmosphere, where there are important polarization effects in the presence of multiple scattering involving both Rayleigh and aerosol scatterers, as well as absorption. Basic theories required to handle such a problem are well known; one of the best known references is due to Chandrasekhar (1950). He shows the description of scattering with the equation of radiative transfer, in which appears a function that transforms unit incident irradiance into scattered radiance (irradiance per solid angle) that integrates to unity. This function is called the phase function. The theory for generating phase functions has been described by Mie (1908) and applied by many others. In principle, specifying the phase functions and formulating the equation of radiative transfer are straightforward. But in practice, scattering is often difficult to describe even numerically. Problems arise because the phase functions of large particles constitute extremely complicated functions of scattering angle and because the spatial distributions of particles can be quite arbitrary.

This paper approaches the various problems by making an explicit definition for superposition of radiances to form the irradiances of the radiation streams that are used in a discretized model of radiative transfer. Although streams have frequently been used before, the problem of explicitly forming them from radiances seems not to have attracted adequate attention. The explicit definition provided here determines the number of streams for a discretized model of radiative transfer and limits the complexity of phase function inputs actually required.

The theory is first stated in terms of scalars, and later extended to include polarization. That extension has to be expressed either with real 4×4 and 4×1 matrices, or with complex 2×2 matrices. We choose the latter formalism because we have found that ultimately it leads to more efficient computer simulations of radiative transfer. The programming of scattering with polarization is discussed in Section 9.

The general ideas presented in this paper should be useful for scattering problems with any geometric configuration, including a completely arbitrary one. We do not require a plane parallel atmosphere, or a spherically symmetric one, or any other specific configuration. This is the case because the array of streams that is obtained is distributed uniformly in three-dimensional space. It has no preferred direction that should be oriented along some preferred direction in the scattering medium. Thus, the scattering medium need not have any special configuration that would define a preferred direction.

At the present time, one of the possible applications is being exploited; namely, the one required by the research effort that supported this work. The requirement was for horizon profiles of the earth's atmosphere as seen by a spacecraft several hundreds of kilometers up. The horizon profiles are generated by scattering from typical atmospheres containing Mie scatterers, Rayleigh scatterers, and absorbers. A computer program based on the model in this paper and addressed to this application is being developed at the Charles Stark Draper Laboratory (CSDL). It is of great interest to assess the accuracy and efficiency of this program in comparison to programs based on other techniques. The

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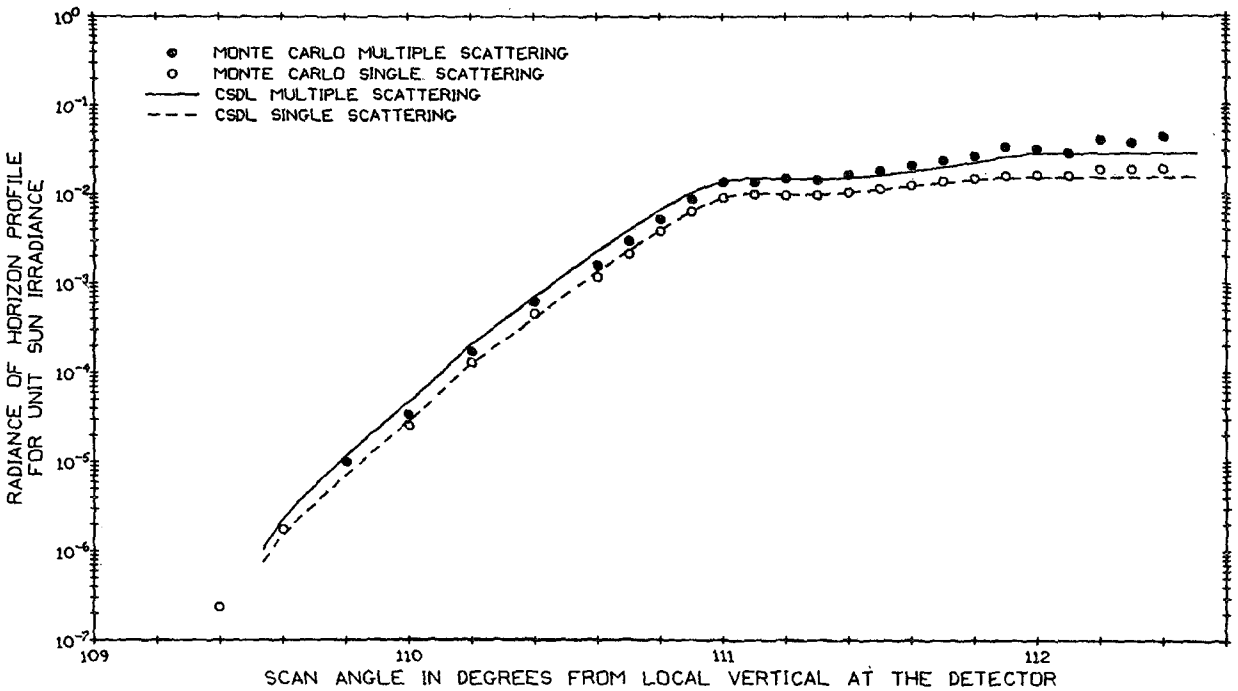

FIG. 1. Horizon profiles generated by CSDL and by Monte Carlo methods.

simulation problem is quite difficult, and at the present time we know of only one other technique that has successfully been adapted to its solution. That is the Monte Carlo technique, which underlies a computer program developed by Radiation Research Associates (RRA). The two programs have been compared in one case for which RRA results were available to us. Details concerning inputs used, assumptions made, etc., are recorded in Appendix A. The results are superposed in Fig. 1. For simplicity, only total radiances are plotted on the graph, but both programs did process polarization as well. Notice that the results of the two programs are quite similar. Unfortunately, at present this comparison of results, and the fact that the Monte Carlo program processed altogether 10,000 photons, are the only guides to assessing the absolute accuracy of either program. In principle, it is also difficult to compare the efficiencies of different programs written in different languages and run on different machines. But in this case the comparison seems significant. The CSDL program computes the horizon profile in just over half a minute on an IBM 360/75, while the Monte Carlo run time was 20 min even on the somewhat faster CDC 6600 computer. A time saving of this sort was anticipated, and was the major motivation for the work reported in this paper.

2. Combination of propagation directions in a superposition integral

It is evident that scattering generally involves propagation of light in all directions. The situation can be described by a radiance function of propagation direction $\hat{\mathbf{k}}$, location $\mathbf{r}$, and frequency ω, which we denote by $S(\hat{\mathbf{k}},\mathbf{r},\omega)$. Often, however, one would be satisfied with a few less detailed irradiance functions of $\mathbf{r}$ and ω, with different propagation directions appropriately combined. In radiative transfer theory, one frequently finds the radiances corresponding to all propagation directions whose unit vectors point within some nonzero solid angle on a unit sphere characterized by a single radiation stream whose propagation direction is some nominal $\hat{\mathbf{p}}$. Usually, however, the stream is not explicitly related to $S(\hat{\mathbf{k}},\mathbf{r},\omega)$. Here we shall offer an explicit relationship and explore its consequences.

In general, superposition of nearby $\hat{\mathbf{k}}$ can be described as integration of $S(\hat{\mathbf{k}},\mathbf{r},\omega)$ with some weighting function that is large for $\hat{\mathbf{k}}$ parallel to $\hat{\mathbf{p}}$ and negligible for $\hat{\mathbf{k}}$ perpendicular to $\hat{\mathbf{p}}$. Computational problems that commonly occur in radiative transfer cause us to seek, in particular, a weighting function such that the superposition is not unduly sensitive to small changes in the choice of $\hat{\mathbf{p}}$. These properties are provided with a weighting function that is a positive power n of the vector dot product $\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}$. The choice $n=2$ suggests itself because it matches the weighting function to the radiance, the essentially quadratic nature of which has been remarked by Wiener (1929), Wolf (1954), and others. This paper explores that choice in detail. The basic techniques can be adapted to arbitrary n, and a brief exploration of some general results that emerge is provided in Appendix B.

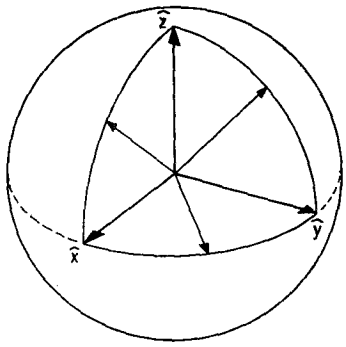

FIG. 2. Octant basis.

Given the choice $n=2$, the definition of superposition is

$$S_{\hat{\mathbf{p}}}(\mathbf{r},\omega)=\int_{\hat{\mathbf{k}}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2 S(\hat{\mathbf{k}},\mathbf{r},w)d\Omega. \tag{2.1}$$

The $\int_{\hat{\mathbf{k}}}$ is accomplished in the conventional way with differential solid angles on the surface of a unit sphere. In the special case $\hat{\mathbf{p}}=\hat{\mathbf{z}}$, this superposition reduces to the K integral defined by Chandrasekhar (1950).

3. Expansion of the superposition integral in terms of a basis set of integrals

There naturally arises now the question of how many superposition integrals are really needed to provide a complete description of a radiation field. We need only enough so that a linear combination can be formed to represent $S_{\hat{\mathbf{p}}}$ for any arbitrary $\hat{\mathbf{p}}$. The required number can be discovered as follows: The dot product $\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}$ can be expanded as

$$\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}=\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}\,\hat{\mathbf{x}}\cdot\hat{\mathbf{k}}+\hat{\mathbf{p}}\cdot\hat{\mathbf{y}}\,\hat{\mathbf{y}}\cdot\hat{\mathbf{k}}+\hat{\mathbf{p}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{z}}\cdot\hat{\mathbf{k}},$$

and it follows that $S_{\hat{\mathbf{p}}}(\mathbf{r},\omega)$ can be expanded as

$$S_{\hat{\mathbf{p}}}=(\hat{\mathbf{p}}\cdot\hat{\mathbf{x}})^2S_{\hat{\mathbf{x}}}+(\hat{\mathbf{p}}\cdot\hat{\mathbf{y}})^2S_{\hat{\mathbf{y}}}+(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})^2S_{\hat{\mathbf{z}}}+R_{\hat{\mathbf{x}}\hat{\mathbf{y}}}+R_{\hat{\mathbf{y}}\hat{\mathbf{z}}}+R_{\hat{\mathbf{z}}\hat{\mathbf{x}}}. \tag{3.1}$$

Here we have dropped the explicit dependence on $\mathbf{r}$ and ω and introduced symbols such as

$$R_{\hat{\mathbf{z}}\hat{\mathbf{x}}}=2\hat{\mathbf{p}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}S_{\hat{\mathbf{z}}\hat{\mathbf{x}}},$$

with

$$S_{zx}=\int_{\hat{\mathbf{k}}}\hat{\mathbf{z}}\cdot\hat{\mathbf{k}}\,\hat{\mathbf{x}}\cdot\hat{\mathbf{k}}S(\hat{\mathbf{k}},\mathbf{r},\omega)d\Omega.$$

The general behavior of the R's can be induced from consideration of the special case in which $\hat{\mathbf{p}}$ lies in the z,x plane. Using for $\hat{\mathbf{p}}$ the particular vector

$$\hat{\mathbf{p}}=\hat{\mathbf{u}}=2^{-\frac{1}{2}}(\hat{\mathbf{z}}+\hat{\mathbf{x}}),$$

we find from (3.1) that

$$S_{\hat{\mathbf{u}}}=\tfrac{1}{2}S_{\hat{\mathbf{x}}}+\tfrac{1}{2}S_{\hat{\mathbf{z}}}+S_{\hat{\mathbf{z}}\hat{\mathbf{x}}},$$

and it follows that generally

$$R_{\hat{\mathbf{z}}\hat{\mathbf{x}}}=\hat{\mathbf{p}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}\,(2S_{\hat{\mathbf{u}}}-S_{\hat{\mathbf{x}}}-S_{\hat{\mathbf{z}}}), \tag{3.2}$$

which is a linear combination of subscripted S's. Naturally, the other R's are subject to similar remarks, and so altogether $S_{\hat{\mathbf{p}}}$ is a linear combination of six other subscripted S's. In effect, it is as if $S_{\hat{\mathbf{p}}}$ were a vector in a six-dimensional vector space whose basis comprised the six other $S_{\hat{\mathbf{p}}'}$'s that occur in the expansion.

4. An alternate basis for expansion

The procedure described in Section 3 for expanding an arbitrary $S_{\hat{\mathbf{p}}}$ in terms of six other $S_{\hat{\mathbf{p}}'}$'s requires in particular $\hat{\mathbf{p}}'=\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}}$ and $\hat{\mathbf{p}}'=2^{-\frac{1}{2}}(\hat{\mathbf{x}}+\hat{\mathbf{y}})$, $2^{-\frac{1}{2}}(\hat{\mathbf{y}}+\hat{\mathbf{z}})$, $2^{-\frac{1}{2}}(\hat{\mathbf{z}}+\hat{\mathbf{x}})$. Since these six vectors all lie on boundaries of an octant of a sphere, it is convenient to refer to them as the octant basis (see Fig. 2).

From the integral expression (2.1), it is evident that any $S_{\hat{\mathbf{p}}}$ primarily contains contributions from $\hat{\mathbf{k}}$ within two angular regions centered on $\pm\hat{\mathbf{p}}$. The six basis integrals $S_{\hat{\mathbf{p}}'}$ correspond to twelve such regions centered on the twelve $\pm\hat{\mathbf{p}}'$. Symmetry is best served if the twelve $\pm\hat{\mathbf{p}}'$'s are distributed uniformly in space instead of on the boundaries of two octants. This can be accomplished as shown in Fig. 3, by pointing the $\pm\hat{\mathbf{p}}'$'s to the faces of a regular dodecahedron. The six basis $\hat{\mathbf{p}}'$'s can be taken as the six vectors in the upper hemisphere of Fig. 3. For convenience, let this basis be called the hemisphere basis.

Our problem now is to demonstrate that one can indeed use the hemisphere basis instead of the octant basis to expand, for arbitrary $\hat{\mathbf{p}}$, the superposition $S_{\hat{\mathbf{p}}}$. To do this, we must provide for the hemisphere basis results comparable to those provided for the octant basis in Section 3; that is, specification of the actual expansion coefficients. There are two procedures that suggest themselves, and both are discussed below. Both require that we know first of all exactly what the vectors in the hemisphere basis are. We can begin by arbitrarily setting $\hat{\mathbf{p}}_0=\hat{\mathbf{z}}$ and arbitrarily requiring $\hat{\mathbf{p}}_1$ to lie in the z,x plane. Then we can use the symmetry of the dodecahedron, which guarantees that all inner products $(\hat{\mathbf{p}}_i\cdot\hat{\mathbf{p}}_j)^2$ for $j\neq i$ should have the same value. A numerical iteration serves to fix this value as $1/5$, and this in turn completely determines all the vectors.

The first procedure for then specifying hemisphere expansion coefficients is basically numerical. We suppose that the octant expansion coefficients are evaluated as indicated by formulas (3.1) and (3.2) and then arranged in a 1×6 matrix B. Then let each element of the hemisphere basis be expanded in terms of the octant basis, using (3.1) and (3.2). Let the six sets of expansion coefficients provide the rows of a 6×6 matrix A. The matrix A is not singular, so the matrix

$$C=BA^{-1} \tag{4.1}$$

can be evaluated numerically to provide the hemisphere expansion coefficients.

The second procedure for specifying the hemisphere expansion coefficients is more formal. We note the functional form of the quadrant expansion coefficients and postulate a similar functional form for the hemisphere expansion coefficients. We then use some obvious special cases to solve for some undetermined constants which finally fix the hemisphere expansion coefficients. This procedure is carried out below.

In the octant basis, the expansion coefficients contain quadratic self products like $(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})^2$ and cross products like $2\hat{\mathbf{p}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}$. The cross products can all be regarded as linear combinations of other self products; for instance,

$$2\hat{\mathbf{p}}\cdot\hat{\mathbf{z}}\,\hat{\mathbf{p}}\cdot\hat{\mathbf{x}}=2(\hat{\mathbf{p}}\cdot\hat{\mathbf{u}})^2-(\hat{\mathbf{p}}\cdot\hat{\mathbf{x}})^2-(\hat{\mathbf{p}}\cdot\hat{\mathbf{z}})^2,$$

with $\hat{\mathbf{u}}=2^{-\frac{1}{2}}(\hat{\mathbf{z}}+\hat{\mathbf{x}})$. Therefore, we shall postulate that only quadratic self products, and not cross products, need be involved in the hemisphere expansion coefficients.

When we seek the coefficient of $S_{\hat{\mathbf{p}}_i}$ for a particular i, we refer to $(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_i)^2$ as the primary self product. Since the hemisphere vectors are uniformly distributed in space, no one being fundamentally different from another, we further postulate that all but the primary self product enter the coefficient in exactly the same way. We are therefore postulating expansion coefficients of the simple general form

$$c(\hat{\mathbf{p}}_i)=a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_i)^2+b\sum_{i'=0}^{5}(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_{i'})^2.$$

The sum on i' here can be scrutinized with regard to the symmetry of the dodecahedron. One would suspect, and it can readily be confirmed by calculating numerous examples, that the sum should be independent of the orientation of $\hat{\mathbf{p}}$. The sum always comes out to be 2. Thus, we have

$$c(\hat{\mathbf{p}}_i)=a(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_i)^2+2b.$$

Consideration of two special cases will fix the a and b. The $c(\hat{\mathbf{p}}_i)$ should be 1 if $\hat{\mathbf{p}}=\hat{\mathbf{p}}_i$, and it should be 0 if $\hat{\mathbf{p}}=\hat{\mathbf{p}}_j$ with $j\neq i$. That fixes $a=5/4$ and $b=-1/8$, and

$$c(\hat{\mathbf{p}}_i)=(5/4)(\hat{\mathbf{p}}\cdot\hat{\mathbf{p}}_i)^2-1/4. \tag{4.2}$$

This functional form has been found to give correct numerical values for the elements of the matrix C in (4.1).

5. Equation of radiative transfer

The symmetry of the hemisphere basis makes it a desirable set of stream directions to use in the discretization of the equation of radiative transfer for $S(\hat{\mathbf{k}},\mathbf{r},\omega)$. Neglecting the explicit $\mathbf{r}$ and ω dependence of all the variables, the equation can be written

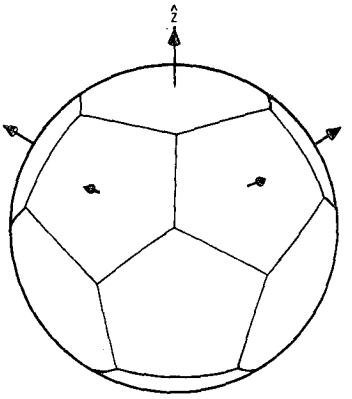

FIG. 3. Hemisphere basis.

$$dS(\hat{\mathbf{k}})/dr=-\epsilon S(\hat{\mathbf{k}})+\sigma\int_{\hat{\mathbf{k}}'}P(\psi)S(\hat{\mathbf{k}}')d\Omega', \tag{5.1}$$

where dr is an increment of propagation path along $\hat{\mathbf{k}}$, ϵ the total extinction coefficient, σ the scattering coefficient, $P(\psi)$ the phase function, and ψ the scattering angle from $\hat{\mathbf{k}}'$ to $\hat{\mathbf{k}}$ (see Chandrasekhar, 1950). Using one of the hemisphere $\hat{\mathbf{p}}$, let the superposition operation $\mathcal{J}_{\hat{\mathbf{k}}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2$ be applied to the $dS(\hat{\mathbf{k}})$ in (5.1). We have

$$dS_{\hat{\mathbf{p}}}=-dr\Big[\epsilon S_{\hat{\mathbf{p}}}-\sigma\int_{\hat{\mathbf{k}}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2\times\int_{\hat{\mathbf{k}}'}P(\mathbf{k}'\rightarrow\hat{\mathbf{k}})S(\hat{\mathbf{k}}')d\Omega'd\Omega\Big]. \tag{5.2}$$

For clarity, the phase function has been expressed explicitly in terms of propagation directions. The double integration on $\hat{\mathbf{k}}$ and $\hat{\mathbf{k}}'$ can be re-expressed on the basis of analogies with the well-known theory of Fourier transforms [see, for instance, Lee (1960)]. In general, one can regard superposition with a $(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^n$ weighting function as an integral-transform. The transformed quantity in (5.2) is the integral on $\hat{\mathbf{k}}'$. That integral is the analog of a convolution, where the convolved functions are the input $S(\hat{\mathbf{k}}')$ and the phase function $P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})$. The transform of the convolution will comprise products of the transforms of the individual functions; that is, products of subscripted P's and S's. The most general result imaginable would involve products with differing subscripts. We can allow for this possibility by letting the coefficients of the products be represented by undetermined c's. For convenience, we use primes and double primes (rather than indices) to distinguish different $\hat{\mathbf{p}}$'s. We have formally

$$dS_{\hat{\mathbf{p}}}=-dr(\epsilon S_{\hat{\mathbf{p}}}-\sigma\Sigma_{\hat{\mathbf{p}}'}\Sigma_{\hat{\mathbf{p}}''}c_{\hat{\mathbf{p}}''\hat{\mathbf{p}}'}P_{\hat{\mathbf{p}}''}S_{\hat{\mathbf{p}}'}). \tag{5.3}$$

Although (5.3) is stated simply on the basis of analogy

to Fourier transform theory, its sufficiency can readily be verified by somewhat tedious calculation (see Appendix C).

The coefficients $c_{\hat{\mathbf{p}}''\hat{\mathbf{p}}'}$ in (5.3) now have to be evaluated. This can be done in a straightforward way shown in Appendix C. That method does not, however, readily extend to the ultimate case in which polarization is a part of the problem. So we here develop another method that *does* so extend, saving the method in Appendix C mainly to verify the results.

Consider the physical significance of the sympol $P_{\hat{\mathbf{p}}''}$. By definition, each $P_{\hat{\mathbf{p}}''}$ is an angular superposition of outputs generated from a unit irradiance input propagating along the z axis; that is, an input of the form

$$S(\hat{\mathbf{k}}')=\delta(\hat{\mathbf{k}}'-\hat{\mathbf{z}}),$$

where δ denotes a three-dimensional Dirac delta function. This is equivalent to *six* superposition inputs, including

$$S_{\hat{\mathbf{z}}}=1,$$

and five others of the form

$$S_{\hat{\mathbf{p}}\neq\hat{\mathbf{z}}}=\tfrac{1}{5}.$$

The problem is to extract the outputs generated by a *single* superposition input $S_{\mathbf{p}'}=1$, with all the other input $S_{\hat{\mathbf{p}}'}=0$. To do this, we first form a 6×6 matrix $M_{\hat{\mathbf{p}}'\hat{\mathbf{p}}}$ whose $\hat{\mathbf{p}}'$ row specifies the irradiances of superpositions equivalent to a unit irradiance δ function input along $\hat{\mathbf{p}}'$. It follows that the *inverse* of M has $\hat{\mathbf{p}}'$ *column* that gives the δ function irradiances equivalent to a unit irradiance *superposition* input along $\hat{\mathbf{p}}'$. In forming M, we use diagonal entries that are 1 and off-diagonal entries that are 1/5. Inverting M, we find the δ function irradiances are 9/8 for $\hat{\mathbf{p}}=\hat{\mathbf{p}}'$, $-1/8$ for $\hat{\mathbf{p}}\neq\hat{\mathbf{p}}'$.

The coefficients 9/8 and $-1/8$ can be assigned to appropriate terms in (5.3) as follows. Suppose first that $\hat{\mathbf{p}}'=\hat{\mathbf{p}}$. Then the dominant scattering corresponds to $\hat{\mathbf{p}}''=\hat{\mathbf{z}}$, so that term has the coefficient 9/8, while the five other $\hat{\mathbf{p}}''$ terms have coefficients $-1/8$. Similarly, for $\hat{\mathbf{p}}'\neq\hat{\mathbf{p}}$, the dominant scattering is away from the original direction. A term with $\hat{\mathbf{p}}''\neq\hat{\mathbf{z}}$ has the coefficient 9/8. (Since scalar phase functions are axially symmetric, all of the $P_{\hat{\mathbf{p}}''}$ with $\hat{\mathbf{p}}''\neq\hat{\mathbf{z}}$ are equal, so it doesn't matter here which receives the 9/8.) The remaining five $\hat{\mathbf{p}}''$ terms have coefficients $-1/8$. These results are compared in Appendix C with those obtained by a more straightforward but less extendable method. Agreement is demonstrated.

At this point, Eq. (5.3) still has not been brought to a form useful for simulating radiative transfer. Recall that each $S_{\hat{\mathbf{p}}}$ reflects primarily radiances corresponding to $\hat{\mathbf{k}}$ in *two* angular regions centered on $\pm\hat{\mathbf{p}}$. The superposing of two angular regions is inconvenient. Let us split $S_{\hat{\mathbf{p}}}$ into forward traveling and backward traveling components, $S_{+\hat{\mathbf{p}}}$ and $S_{-\hat{\mathbf{p}}}$. Let the definition of $S_{+\hat{\mathbf{p}}}$ differ from (2.1) by the inclusion of different weight factors on the two hemispheres of integration, $\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}>0$ and $\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}<0$. That is, let

$$S_{+\hat{\mathbf{p}}}=\left[(1+a)\int_{\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}>0}-a\int_{\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}<0}\right](\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2S(\hat{\mathbf{k}})d\Omega, \quad (5.4)$$

where a is an arbitrary constant yet to be evaluated. The splitting generally requires twelve basis $\hat{\mathbf{p}}$'s distributed formly in space instead of just the six in the upper hemisphere of Fig. 3. Splitting $P_{\hat{\mathbf{p}}}$ in the same way as $S_{\hat{\mathbf{p}}}$, we obtain the equation of radiative transfer in the form

$$dS_{+\hat{\mathbf{p}}}=-dr(\epsilon S_{+\hat{\mathbf{p}}}-\sigma\Sigma_{+\hat{\mathbf{p}}''}\Sigma_{+\hat{\mathbf{p}}'}c_{+\hat{\mathbf{p}}''+\hat{\mathbf{p}}'}P_{+\hat{\mathbf{p}}''}S_{+\hat{\mathbf{p}}'}). \quad (5.5)$$

The coefficients $c_{+\hat{\mathbf{p}}''+\hat{\mathbf{p}}'}$ in (5.5) can be evaluated and assigned in the way the $c_{\hat{\mathbf{p}}''\hat{\mathbf{p}}'}$ in (5.3) were, by a matrix inversion. The numbers that appear in the 12×12 matrix $M_{+\hat{\mathbf{p}}'+\hat{\mathbf{p}}}$ are

$$\left.\begin{aligned}\alpha&=1+a\\ \beta&=(1+a)/5\\ \gamma&=-a/5\\ \delta&=-a\end{aligned}\right\}.$$

The explicit form of the matrix depends on the scheme by which the streams are numbered. We let $\hat{\mathbf{p}}_0=\hat{\mathbf{z}}$, $\hat{\mathbf{p}}_i$ for $i=1,\ldots,5$ lie in the upper hemisphere, $\hat{\mathbf{p}}_i$ for $i=6,\ldots,10$ lie in the lower hemisphere, and $\hat{\mathbf{p}}_{11}=-\hat{\mathbf{z}}$. For $i=1,\ldots,10$, each stream is located by a positive rotation about $\hat{\mathbf{z}}$ from the one before. Then the matrix looks like

$$M_{+\hat{\mathbf{p}}'+\hat{\mathbf{p}}}=\begin{pmatrix}\alpha&\beta&\beta&\beta&\beta&\beta&\gamma&\gamma&\gamma&\gamma&\gamma&\delta\\ \beta&\alpha&\beta&\gamma&\gamma&\beta&\beta&\beta&\gamma&\delta&\gamma&\gamma\\ \beta&\beta&\alpha&\beta&\gamma&\gamma&\gamma&\beta&\beta&\gamma&\delta&\gamma\\ \beta&\gamma&\beta&\alpha&\beta&\gamma&\delta&\gamma&\beta&\beta&\gamma&\gamma\\ \beta&\gamma&\gamma&\beta&\alpha&\beta&\gamma&\delta&\gamma&\beta&\beta&\gamma\\ \beta&\beta&\gamma&\gamma&\beta&\alpha&\beta&\gamma&\delta&\gamma&\beta&\gamma\\ \gamma&\beta&\gamma&\delta&\gamma&\beta&\alpha&\beta&\gamma&\gamma&\beta&\beta\\ \gamma&\beta&\beta&\gamma&\delta&\gamma&\beta&\alpha&\beta&\gamma&\gamma&\beta\\ \gamma&\gamma&\beta&\beta&\gamma&\delta&\gamma&\beta&\alpha&\beta&\gamma&\beta\\ \gamma&\delta&\gamma&\beta&\beta&\gamma&\gamma&\gamma&\beta&\alpha&\beta&\beta\\ \gamma&\gamma&\delta&\gamma&\beta&\beta&\beta&\gamma&\gamma&\beta&\alpha&\beta\\ \delta&\gamma&\gamma&\gamma&\gamma&\gamma&\beta&\beta&\beta&\beta&\beta&\alpha\end{pmatrix};$$

The inverse matrix has exactly the same form, though of course with different values for α, β, γ and δ. Fig. 1 was done with the choice $a=3/14$. For this choice, the input values of α, β, γ and δ are

$$\left.\begin{aligned}\alpha&=17/14\\ \beta&=17/70\\ \gamma&=-3/70\\ \delta&=-3/14\end{aligned}\right\},$$

and the inverted values

$$\left.\begin{aligned}\alpha&=1\\ \beta&=-3/20\\ \gamma&=1/40\\ \delta&=1/8\end{aligned}\right\}.$$

The question naturally arises as to whether a can be chosen arbitrarily, as we have indicated. The justification lies in another use to which $M_{+\hat{\mathbf{p}}'+\hat{\mathbf{p}}}$ and its inverse are put in the model. Recall that the streams appearing above have been regarded as analogs of Fourier transforms of the radiance $S(\hat{\mathbf{k}},\hat{\mathbf{r}},\omega)$. The streams correspond neither to the given input nor the desired output in any real problem. The given input has to be converted to streams and the output has to be inverted back to radiances again. The input operation involves the numbers in $M_{+\hat{\mathbf{p}}'+\hat{\mathbf{p}}}$, while the output operation involves the numbers in the inverse matrix. Thus, the value of a appears only in the interior workings of the model, and tends to cancel out between the input and the output.

The interpretation of dr as an increment of distance in the direction of $+\hat{\mathbf{p}}$ is the final step that makes (5.5) a set of twelve coupled differential equations for streams. This approximation for the integro-differential equation of radiative transfer offers some unique advantages over other similar models. First, the simulation illustrated in Fig. 1 for a problem involving a very complicated radiation field is achieved with only twelve streams. The number of streams greatly influences the run time of any radiative transfer program, and most models cannot handle aerosol scattering without many more than twelve streams. For instance, Gaussian approximate quadrature for a field with azimuthal symmetry uses a number of directions determined as the order of whatever polynomial adequately represents the radiation field as a function of polar angle! Second, with the dodecahedral arrangement of the twelve directions that occur in (5.5), the angle between pre-scattering and post-scattering streams can only be one of four possible angles. This fact allows time-saving through precomputing many functions and calling them with subscripts as needed. Precomputation is especially useful where aerosols are involved. All necessary integrals of the phase function can be precomputed and stored for use in a whole family of scattering problems. Third, the small number of streams and their symmetric distribution combine to minimize the burden of this precomputation. The aerosol phase functions can be quite hard to integrate because of their highly oscillatory, ill-behaved nature. So a model that only needs integrals about four scattering angles has a distinct advantage over any model that needs more. Fourth, the phenomenon of forward scattering, which is usually a computational plague with aerosols, can be turned to advantage. This is the case because forward scattering has a well-defined meaning here. It is scattering involving the phase function integral $P_{+\hat{\mathbf{z}}}$. Since forward scattering does not remove radiation from the stream, we can accommodate it by simply scaling optical depths by a factor of $(1-P_{+\hat{\mathbf{z}}})$. The result is the program will converge faster the larger $P_{+\hat{\mathbf{z}}}$ is; that is, the more forward scattering there is. The lack of a well defined meaning for "forward" is, of course, the primary feature inhibiting other models from doing the same thing.

It seems worthwhile here to introduce a name for the twelve $S_{\hat{\mathbf{p}}}$'s. Following the suggestion of D. Deirmendjian, we call an $S_{\hat{\mathbf{p}}}$ a radiation dodecaton. The description of scattering in terms of the twelve radiation dodecata seems to offer much convenience.

6. Description of polarization states

We come now to the problem of incorporating in the theory the potentially influential phenomenon of polarization. The full description of radiation propagating in a given direction $\hat{\mathbf{k}}$ at location $\mathbf{r}$ with frequency ω really requires a construct with four polarization parameters. The two constructs most commonly used in the literature are the 4×1 matrix $\mathcal{S}$ of the real parameters introduced by Stokes (1852) and the Hermitian 2×2 coherency matrix J introduced by Parrent and Roman (1960). There are, however, others in use. Several are discussed and related to $\mathcal{S}$ and J by Sekera (1966), and additional relationships are given by Schmieder (1969). This author has noted elsewhere (1971) that several formal advantages can be gained by introducing yet another 2×2 construct slightly different from those already commonly used. The advantages include 1) a relationship to $\mathcal{S}$ that is simpler than those mentioned, and 2) a special dynamic interpretation for the polarized states on which the 2×2 construct is based. Polarization state will be described with the latter construct, which is developed below.

Rossi (1957) has illustrated the existence of definite, positive and negative spin-density for polarized plane waves having positive and negative helicity, or left and right circular polarization. From the amplitudes of two such waves, let a 2×2 matrix S be formed by correlation, in the same manner as J is formed from amplitudes of linearly polarized waves. Most generally, the correlations can be performed with an arbitrary time delay and followed by Fourier decomposition to produce a Hermitian matrix S.

Because S is Hermitian, it can be written in the form

$$S=\begin{bmatrix} s_0+s_3 & s_1-is_2 \\ s_1+is_2 & s_0-s_3 \end{bmatrix}, \tag{6.1}$$

where each lower case s is a real number. It then becomes convenient to express S with the notation

$$S=s_0\sigma_0+\mathbf{s}\cdot\boldsymbol{\sigma}, \tag{6.2}$$

where σ_0 is the identity matrix

$$\sigma_0=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$\mathbf{s}$ is the vector $\mathbf{s}=\{s_1,s_2,s_3\}$, and $\boldsymbol{\sigma}$ is a vector with matrix entries

$$\boldsymbol{\sigma}=\{\sigma_1,\sigma_2,\sigma_3\},$$

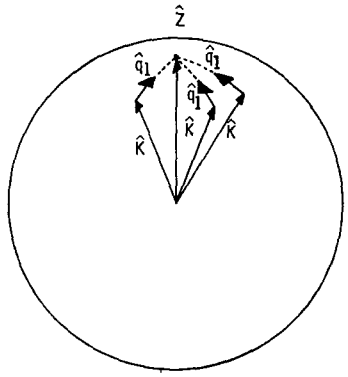

FIG. 4. Usual polarization reference $\hat{\mathbf{q}}_1$ for three propagation directions $\hat{\mathbf{k}}$ in the vicinity of $\hat{\mathbf{z}}$.

where

$$\sigma_1=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\quad \sigma_2=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix},\quad \sigma_3=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}.$$

The basis matrices $\boldsymbol{\sigma}$ are the well-known Pauli spin matrices. We call the four numbers $\{s_0, \mathbf{s}\}$ the matrix components of S. Similar notation, with matrices denoted by capital letters and components by the corresponding lower-case letters, is used throughout the paper. Here the s_0 component can be interpreted as total radiance of signal. Further, let the vector $\mathbf{s}$ be expressed as

$$\mathbf{s}=s\hat{\mathbf{s}},$$

where s is a scalar and $\hat{\mathbf{s}}$ is a unit-length three-vector. The s can be interpreted as the radiance of the polarized constituent in the signal. The work of Perrin (1942) suggests that $\hat{\mathbf{s}}$ be used to determine a point on a sphere. Apart from reversal of the vertical poles, the sphere is the same as that used by Poincaré (1892). It follows that apart from a minus sign on s_3, the matrix components $\{s_0, \mathbf{s}\}$ of S are equivalent to the well-known Stokes parameters. The minus sign arises only because the basis of S is $+$, $-$ spin density instead of right, left circular polarization.

The incorporation of polarization in the radiative transfer model derived in this paper begins with the reinterpretation of the symbol $S(\hat{\mathbf{k}},\mathbf{r},\omega)$ as a 2×2 complex matrix, like S above.

7. Combination of propagation directions with polarization

This section discusses the application of the superposition operation in Section 2 to the case where polarization is included in $S(\hat{\mathbf{k}},\mathbf{r},\omega)$.

Superposing S matrices for different $\hat{\mathbf{k}}$ means superposing polarization parameters for different $\hat{\mathbf{k}}$. A problem arises as follows. For propagation along $\hat{\mathbf{z}}$, there is a commonly used association of $\hat{\mathbf{s}}=\hat{\mathbf{1}}$ with linear polarization in the x direction. But that association is entirely arbitrary, and the same arbitrariness occurs for every other propagation direction as well. This means that the polarization of a superposition will depend on what arbitrary convention is used to assign the Poincaré state $\hat{\mathbf{1}}$ to a physical direction $\hat{\mathbf{q}}_1$ perpendicular to each $\hat{\mathbf{k}}$. Meaningful superposition requires that the $\hat{\mathbf{q}}_1$ vary smoothly over the region of a unit sphere containing the tips of the $\hat{\mathbf{k}}$ that are superposed. Unfortunately, the topology of a sphere defies a continuous, uniformly varying assignment of $\hat{\mathbf{q}}_1$ over all its surface. For example, Fig. 4 shows the arbitrary convention described by Chandrasekhar and most commonly used in the literature. Notice that the polarization reference $\hat{\mathbf{q}}_1$ varies wildly from point to point in the vicinity of the $\pm\hat{\mathbf{z}}$ poles. These singularities make the convention inapplicable to a definition of superposition. It is really necessary to formulate superposition in a way that circumvents these difficulties.

The above-mentioned difficulties can be avoided by formulating a convention that depends on the particular direction $\hat{\mathbf{p}}$ for which we wish to form a stream. Imagine that the x,y plane, and its convention $\pm\hat{\mathbf{x}}=\hat{\mathbf{1}}$, is tipped to align with the plane perpendicular to $\hat{\mathbf{k}}$ by the following operations. First the $\hat{\mathbf{z}}$ axis is rotated in the z,p plane into the nearer of $+$ and $-\hat{\mathbf{p}}$, then in the p,k plane into the nearer of $+$ and $-\hat{\mathbf{k}}$. The rotated $\hat{\mathbf{x}}$ vector is then used as $\hat{\mathbf{q}}_1$ for $\hat{\mathbf{k}}$. In this way, it is guaranteed that the $\hat{\mathbf{q}}_1$'s for $\hat{\mathbf{k}}$'s near $\hat{\mathbf{p}}$ are not very different from each other. We call the above a floating convention, relative to $\hat{\mathbf{p}}$. The convention has no singular points, so for any $\hat{\mathbf{p}}$, it is meaningful to superpose the S matrices of $\hat{\mathbf{k}}$'s near each other and $\hat{\mathbf{p}}$ (see Fig. 5).

The second step toward the incorporation of polarization in the radiative transfer model derived in this paper is the reinterpretation of $S_{\hat{\mathbf{p}}}(\mathbf{r},\omega)$ as an integral of a matrix function $S(\hat{\mathbf{k}},\mathbf{r},\omega)$ with polarization defined relative to the $\hat{\mathbf{p}}$.

8. Phase function integrals with polarization

When polarization is introduced, the phase-function symbol $P(\psi)$ in (5.1) has to be understood as a rather abbreviated notation. In general, scattering has to be described by a matrix operation that affects the polarization state while preserving the Hermiticity of S. In particular, both the propagation direction $\hat{\mathbf{k}}$ after scattering and the direction of incidence $\hat{\mathbf{k}}'$ influence the effect on polarization state. Thus, the phase function symbol $P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})$ in (5.2) is more appropriate, but still not sufficient. In detail, the phase-function operation has the form

$$S(\hat{\mathbf{k}})=V(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})S(\hat{\mathbf{k}}')V^{\dagger}(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}}),\qquad(8.1)$$

where V is a 2×2 complex matrix and † denotes Hermitian adjoint. Thus, the symbol $P(\psi)$ or $P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})$

indicates an operation which could be written in more detail as

$$P(\psi)=P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})=V(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})\cdots V^{\dagger}(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}}). \quad (8.2)$$

Forming any of the $P_{\hat{\mathbf{p}}}$ that appear in (5.4) requires applying the additional operation of superposition to (8.2). The result is not generally expressable in terms of pre- and post-matrix multiplication on $S_{\hat{\mathbf{p}}'}$. However, superposing different $\hat{\mathbf{k}}$ here is analogous to superposing scatters from different particles in a symmetric medium, a situation discussed by Perrin (1942). Thus, the information required to perform the resultant operation does generally comprise at most eight independent real scalars. This is the same number of real scalars that it takes to specify a completely *arbitrary* 2×2 matrix. These scalars have been provided to the CSDL radiative transfer model by a subordinate computer program, so that the phase function integrations are only done once for a whole family of horizon profiles.

9. Equation of radiative transfer with polarization

When polarization is introduced, it is no longer true of (5.3) that the five $P_{\hat{\mathbf{p}}''}$ with $\hat{\mathbf{p}}''\neq\hat{\mathbf{z}}$ are identical. They differ from each other by rotations of $4\pi/5$ in Poincaré space. Care must therefore be taken to assign the coefficients 9/8 and −1/8 to exactly the correct terms in (5.3). Similarly, as the separation between forward and backward traveling components is made in (5.5), the coefficients 1, 1/8, −3/20 and 1/40 must be properly assigned. Furthermore, the fact that the polarization reference for $S_{\hat{\mathbf{p}}'}$ differs from that for $dS_{\hat{\mathbf{p}}}$ must be taken into account. The simplest way to achieve these goals is through a sequence of operations written into the computerization of the phase function integrals and radiative transfer model.

First, the phase function integrator evaluates $P_{\hat{\mathbf{p}}}$ for $\hat{\mathbf{p}}$ in the z,x plane. Considering the separation between forward-traveling and backward-traveling streams in (5.5), there are four such integrals. Then, taking account of the rotations that relate the other $P_{\hat{\mathbf{p}}}$, the program evaluates for each basic scattering angle a stream transformation of the form

$$T_{+\hat{\mathbf{p}}}=P_{+\hat{\mathbf{p}}}+\tfrac{1}{8}P_{-\hat{\mathbf{p}}}+\sum_{\substack{\hat{\mathbf{p}}'\neq\hat{\mathbf{p}},\text{ in}\\ \text{hemisphere}\\ \text{centered}\\ \text{on }+\hat{\mathbf{p}}}}\left(-\frac{3}{20}P_{+\hat{\mathbf{p}}'}+\frac{1}{40}P_{-\hat{\mathbf{p}}'}\right), \quad (9.1)$$

and provides this to the radiative transfer model.

Now when the radiative transfer model encounters a scatter from a particular $\hat{\mathbf{p}}'$ to a particular $\hat{\mathbf{p}}$, it first rotates the $S_{\hat{\mathbf{p}}'}$ so that polarization in the scattering plane acquires the name 1 in Poincaré space. This makes the scattering plane equivalent to the z,x plane, for which stream transformations have been provided. Depending on the angle between $\hat{\mathbf{p}}$ and $\hat{\mathbf{p}}'$, the program applies the proper stream transformation. Then finally it rotates $dS_{\hat{\mathbf{p}}}$ so that the scattering plane acquires the correct name for reference relative to $\hat{\mathbf{p}}$.

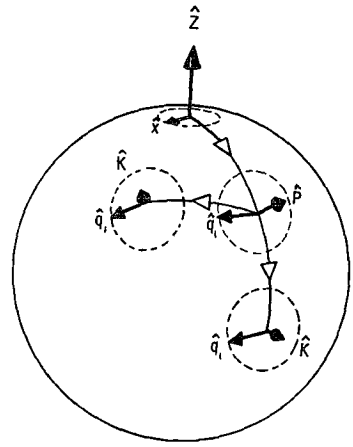

FIG. 5. Floating polarization reference relative to a nominal propagation direction $\hat{\mathbf{p}}$.

10. Summary

The concept of superposing different propagation directions plays a fundamental role in producing simple descriptions of radiative transfer phenomena. A quadratic definition of superposition is introduced here. The number of basis superpositions required to expand an arbitrary superposition is found to be six. When forward and backward traveling streams are separated, the six superpositions become twelve traveling streams. An approximate description of radiative transfer in terms of twelve coupled differential equations, instead of an exact integro-differential equation, is obtained. Computational efficiency is enhanced by distributing the corresponding twelve streams in three-dimensional space in a totally symmetric way. Polarization is then incorporated in this model. With polarization included, superposition is seen to be meaningful only if the definition of linear reference state varies smoothly with propagation direction in the region superposed. It is found that smoothness can be achieved with a *floating* reference. Use of the floating reference specifies refinements required for the incorporation of polarization in the computerization of the twelve coupled differential equations of radiative transfer. A computer program based on the model is briefly compared with a Monte Carlo program for simulation of horizon scans of the earth's atmosphere. It is found to be considerably faster.

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investigation of the computational advantages to the description of multiple scattering resulting from symmetries of a dodecahedral partition of solid angles gave impetus to the study. Peter Vernam of MIT has helped immeasurably to bring about the realization of the CSDL computer program that generates horizon profiles. Additional computational assistance was given by Kathy Tompkins of MIT. The Monte Carlo data points with which the CSDL program is compared in Fig. 1 were provided by Air Force Cambridge Research Laboratories through the courtesy of R. Fenn.

APPENDIX A

Comparison of CSDL and Monte Carlo Methods

This Appendix compares the physical assumptions and mathematical approaches underlying the two methods used for generating the horizon profiles in Fig. 1.

For both programs, the wavelength is $0.32\,\mu$. Aerosols have a refractive index of 1.5, range in radius r from 0.01 to $1\,\mu$, and are distributed according to r^{-3}. The surface of the earth is described by Lambert's law, with an albedo of 0.6. The detector is 500 km above the surface, scanning to the left. Sunlight enters from the right, traveling in a direction lying 201.4° from the detector local vertical (see Fig. 6).

For the Monte Carlo program, the atmosphere is modeled as a set of 22 concentric spherical layers of 5 km thickness. At the top boundary of each layer is specified the total optical thickness from ground up, the local ratios of scattering to absorption and Rayleigh scattering to total scattering, and the local refractive index. These quantities are also specified at ground, but these values have little influence because there is a virtually impenetrable cloud at 4 km. Between the layer boundaries, the various quantities are assumed to vary linearly.

For the CSDL program, the atmosphere is modeled as a set of 51 concentric spherical layers of 2 km thickness. The bottom of the first layer is at 4 km, because of the cloud. Each layer is characterized by extinction coefficients for Rayleigh scatterers, aerosols and ozone, and by an index of refraction, all presumed constant within the layer. The aerosols present have a rather complicated phase function. For this reason, single scattering is handled exactly. Only multiple scattering is modeled with the discretization described in this paper. Most of the multiple scattering that occurs is nearly forward, and is conveniently handled with the same spherical geometry as single scattering. For the small amount of multiple scattering involving several large scattering angles, additional simplifications are introduced: larger, 20 km layers are used, and in some noncritical parts of the calculation they are imagined to be flat.

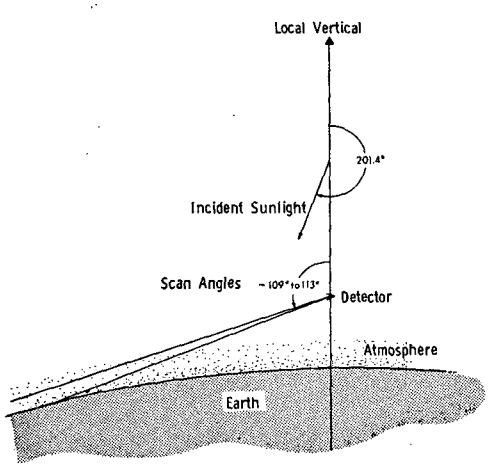

FIG. 6. Horizon scan geometry.

APPENDIX B

Some Consequences of a General Power Law Stream Definition

This paper began with a general weighting function $(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^n$, and then explored the consequences of choosing $n=2$. We return now to the general n. The most important influence that n has is on the number and distribution of streams that occurs in the radiative transfer model. This Appendix shows how to find the number of streams, and how to distribute them in a symmetric pattern so that the computational burden will be minimized.

The number of superpositions (N) that results from using power n in the weighting function equals the number of distinguishable expressions of the form $x^a y^b z^c$ with $a+b+c=n$. This number can be evaluated as follows. For any choice of a, there are $n+1-a$ choices possible for b: $b=0, 1, \ldots, n-a$. With b chosen, c is fixed as $n-a-b$. Thus,

$$N=\sum_{a=0}^{N}(n+1-a)\equiv(n+1)(n+2)/2.$$

The number of traveling streams is twice this. For the first few values of n we have:

Power n	Number N of $S_{\hat{\mathbf{p}}}$'s	Number $2N$ of $S_{\pm\hat{\mathbf{p}}}$'s
1	3	6
2	6	12
3	10	20
4	15	30
5	21	42
6	28	56
7	36	72
8	45	90

The way in which the traveling streams are distributed on a sphere is of particular importance for computational purposes. The streams should be

distributed in such a way as to minimize the total number of allowed scattering angles, and hence the total number of phase function integrals required by the radiative transfer model. This minimization is achieved by distributing the streams as uniformly as possible so that the sphere is invariant to a large number of rotations; that is, so that many streams are basically indistinguishable in terms of their relation to neighboring streams. To illustrate the extent to which uniformity is indeed possible, suitable distributions for $n=1$–8 are described below.

For $n=1$, one can point the 6 streams to the faces of a cube; for $n=2$, the 12 streams can point to the faces of a dodecahedron; for $n=3$, the 20 streams can point to the faces of an icosahedron. For $n=4$, the 30 streams can point to the mid points of the edges of the dodecahedron (or, equivalently, to faces of an almost-regular figure constructed by covering each dodecahedron edge with a rhombus whose points lie at the dodecahedron corners and face midpoints). For $n=5$, one could superpose the distributions for $n=2$ and $n=4$, and the result would be almost uniform. The figure to whose faces the 42 streams point is made up entirely of pentagons and hexagons. It looks much like a soccer ball, although it has altogether more faces. The next case, $n=6$, is particularly interesting because evidently one can obtain fewer scattering angles by using a redundant set of 60 streams instead of just the 56 demanded. To distribute the 60, we begin with the figure of 30 rhomboidal faces first introduced for $n=4$. We inscribe squares in the rhombi, and use their 60 corner points. The figure to whose faces these 60 streams point is made up of identical quadrillaterals. For $n=7$, we can superpose the distributions obtained for $n=2$ and $n=6$. Again, the result is almost uniform. The associated figure has 72 pentagonal faces, 12 regular, and the rest irregular but identical. For $n=8$, we could begin with the soccer-ball like figure obtained for $n=5$, and use the 60 pentagon corners and 30 hexagon centers. The figure that results has 60 distorted hexagonal faces and 30 square faces.

It is hard to imagine that anything beyond this level would be useful at present, but presumably one could proceed to larger n if desired. The fundamental technique is to use the symmetry of the dodecahedron.

APPENDIX C

Detailed Derivation of the Discretized Equation of Radiative Transfer (5.3) and its $c_{\hat{\mathbf{p}}''\hat{\mathbf{p}}'}$

This Appendix records for the reader's convenience the somewhat tedious calculation that can be done to verify the formal sufficiency of the discretized equation of radiative transfer (5.3) and to evaluate its coefficients, the $c_{\hat{\mathbf{p}}''\hat{\mathbf{p}}'}$.

In concept, the derivation of (5.3) borrows heavily from the usual derivation of the Fourier transform of a convolution [see, for instance (Lee, 1960]].

The basic problem is to evaluate the double integral

$$D=\int_{\hat{\mathbf{k}}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2\int_{\hat{\mathbf{k}}'}P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})S(\hat{\mathbf{k}}')d\Omega'd\Omega$$

that appears in (5.2). We begin by reversing the order of integrations to

$$D=\int_{\hat{\mathbf{k}}'}\int_{\hat{\mathbf{k}}}(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}})^2P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})S(\hat{\mathbf{k}}')d\Omega d\Omega'.$$

The interior integral affects only the $P(\hat{\mathbf{k}}'\rightarrow\hat{\mathbf{k}})$, and produces a superposition of the sort we have denoted throughout the paper with a subscript $\hat{\mathbf{p}}$, but with $\hat{\mathbf{k}}'$ playing the role of $\hat{\mathbf{z}}$. We denote this superposition by $P_{\hat{\mathbf{p}}|\hat{\mathbf{k}}'}$. In terms of that, the double integral reduces to

$$D=\int_{\hat{\mathbf{k}}'}P_{\hat{\mathbf{p}}|\hat{\mathbf{k}}'}S(\hat{\mathbf{k}}')d\Omega'.$$

Now using the octant basis, $P_{\hat{\mathbf{p}}|\hat{\mathbf{k}}'}$ can be expanded in the usual way as

$$P_{\hat{\mathbf{p}}|\hat{\mathbf{k}}'}=(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}')^2P_{\hat{\mathbf{z}}}+[(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}_1')^2+(\hat{\mathbf{p}}\cdot\hat{\mathbf{q}}_{-1}')^2]P_{\hat{\mathbf{x}}},$$

where $\hat{\mathbf{q}}_1'$ and $\hat{\mathbf{q}}_{-1}'$ are orthogonal vectors perpendicular to $\hat{\mathbf{k}}$. With no change in its numerical value, the quantity in brackets can be replaced by

$$(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{k}}')^2+(\hat{\mathbf{q}}_{-1}\cdot\hat{\mathbf{k}}')^2,$$

where $\hat{\mathbf{q}}_1$ and $\hat{\mathbf{q}}_{-1}$ are orthogonal vectors perpendicular to $\hat{\mathbf{p}}$, and independent of $\hat{\mathbf{k}}'$. Now substituting back in D, we have

$$D=\int_{\hat{\mathbf{k}}'}\{(\hat{\mathbf{p}}\cdot\hat{\mathbf{k}}')^2P_{\hat{\mathbf{z}}}+[(\hat{\mathbf{q}}_1\cdot\hat{\mathbf{k}}')^2+(\hat{\mathbf{q}}_{-1}\cdot\hat{\mathbf{k}}')^2]P_{\hat{\mathbf{x}}}\}S(\hat{\mathbf{k}}')d\Omega'.$$

The integration of $\hat{\mathbf{k}}'$ affects only $S(\hat{\mathbf{k}}')$, and produces superpositions $S_{\hat{\mathbf{p}}}$, $S_{\hat{\mathbf{q}}_1}$, and $S_{\hat{\mathbf{q}}_{-1}}$. Thus

$$D=P_{\hat{\mathbf{z}}}S_{\hat{\mathbf{p}}}+P_{\hat{\mathbf{x}}}[S_{\hat{\mathbf{q}}_1}+S_{\hat{\mathbf{q}}_{-1}}].$$

The quantities $P_{\hat{\mathbf{x}}}$, $S_{\hat{\mathbf{q}}_1}$ and $S_{\hat{\mathbf{q}}_{-1}}$ can all be expanded in the hemisphere basis, and in so doing, we bring D to the form shown in (5.3).

We can carry out the above operation explicitly with the help of (4.2) for the expansion coefficients. The sum $S_{\hat{\mathbf{q}}_1}+S_{\hat{\mathbf{q}}_{-1}}$ is particularly easy to expand because the two contributions combine in such a way that the coefficient of $S_{\hat{\mathbf{p}}}$ is $-\frac{1}{2}$ and the coefficient of all $S_{\hat{\mathbf{p}}'\neq\hat{\mathbf{p}}}$ is $+\frac{1}{2}$. The $P_{\hat{\mathbf{x}}}$ is only slightly more complicated to

expand. The coefficient of $P_{\hat{z}}$ is $-\frac{1}{4}$. For the $P_{\hat{p}'\neq\hat{z}}$, there is one coefficient equal to 3/4, two equal to $\cos^2(2\pi/5)-\frac{1}{4}$, and two equal to $\cos^2(2\pi/10)-\frac{1}{4}$. Thus, we have

$$D=P_{\hat{z}}S_{\hat{p}}+\left(-\tfrac{1}{4}P_{\hat{z}}+\left\{\frac{3}{4}+2\left[\cos^2\left(\frac{2\pi}{5}\right)-\frac{1}{4}\right]\right.\right.$$
$$\left.\left.+2\left[\cos^2\left(\frac{2\pi}{10}\right)-\frac{1}{4}\right]\right\}P_{\hat{p}''\neq\hat{z}}\right)\left(-\tfrac{1}{2}S_{\hat{p}}+\sum_{\hat{p}'\neq\hat{p}}^{5\text{ terms}}\tfrac{1}{2}S_{\hat{p}'}\right).$$

These results are to be compared with the ones obtained in the main text. The coefficient of $P_{\hat{z}}S_{\hat{p}}$ is exactly 9/8 and that of $P_{\hat{z}}S_{\hat{p}'\neq\hat{p}}$ is exactly $-1/8$, as obtained in the text. At first glance there seems to be a difference between this Appendix and the main text concerning the coefficients of terms involving the $P_{\hat{p}''\neq\hat{z}}$. But scalar phase functions are axially symmetric, and for all $\hat{\mathbf{p}}''\neq\hat{\mathbf{z}}$, the $P_{\hat{p}''}$ are numerically the same. If for numerically identical terms we *sum* the coefficients, we obtain the same results in both cases: The coefficient of $P_{\hat{p}''\neq\hat{z}}S_{\hat{p}}\equiv-5/8$ and the coefficient of $P_{\hat{p}''\neq\hat{z}}S_{\hat{p}'\neq\hat{p}}\equiv+5/8$. It is this axial symmetry that disappears when polarization is brought into the problem, making the results in this Appendix worthless and the results in the main text uniquely usable.

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