ON THE RELATION BETWEEN THE WIND FIELD AND PRESSURE CHANGES

By B. Haurwitz

Massachusetts Institute of Technology

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ABSTRACT

It is shown on the evidence of observational material that the simplifications necessary in order to derive the equation for the isallobaric wind are not justified, because the neglected terms in the equations of motion, viz., the convective terms and the local derivatives of the geostrophic deviation, are of the same order of magnitude as the terms retained in the equations. Hence the concept of the isallobaric wind has to be abandoned. Consequently the convergence ahead of moving cyclones and the divergence behind, in the lower troposphere, cannot be explained by means of the isallobaric-wind relation. It is shown that the distribution of the acceleration of motion in moving pressure systems offers an explanation of the observed distribution of convergence and divergence.

1. Introduction

Since a close connection exists between the fields of pressure and wind, the changes of pressure can be expected to bear a marked relation to the wind. An equation showing the modification of the wind by the gradient of the pressure tendency has been derived by Hesselberg (1915) and Brunt and Douglas (1928). According to the last two authors

$$u = -\frac{1}{l\rho}\frac{\partial p}{\partial y} - \frac{1}{l^2\rho}\frac{\partial \dot{p}}{\partial x}$$
$$v = \frac{1}{l\rho}\frac{\partial p}{\partial x} - \frac{1}{l^2\rho}\frac{\partial \dot{p}}{\partial y}.$$

Here u and v are the velocity components in the horizontal x- and y-directions, ρ is the density, p is the pressure, $\dot{p} = \partial p/\partial t$ is the local pressure change, and $l = 2\,\omega \sin \phi$ is the Coriolis parameter. According to these equations the wind velocity consists of two parts, viz., the geostrophic wind, which is proportional to the pressure gradient, and an additional term which is called the *isallobaric wind* because of its proportionality to the gradient of the pressure tendency.

However, the derivation of these equations for the isallobaric wind involves simplifying assumptions which make its validity questionable. The purpose of the following remarks is to discuss these simplifications, which will be seen to be not justified by the observational data.

Another formula for the isallobaric wind has been derived by Ertel (1938). It is based on the supposition that in a changing pressure field geostrophic balance is not established immediately, so that the Coriolis acceleration, and hence the wind, correspond to the pressure gradient at a somewhat earlier time. It follows from this hypothesis that the wind component which is produced by the changes of the pressure field should be parallel to the isallobars, not to the isallobaric gradient as stated by Brunt and Douglas. Statistical checks of both hypotheses, made at the Department of Meteorology, Massachusetts Institute of Technology, have been inconclusive because of the difficulties involved in determining the geostrophic deviation. Only the Brunt-Douglas formula is investigated here. Ertel's formula, in view of the assumptions on which its derivation is based, does not appear to be in more satisfactory agreement with the facts.

The basic material for the present discussion is a part of the computations and map analyses prepared by staff members of the Department of Meteorology in connection with a project to find more objective and quantitative forecasting methods. The results of this study to date are discussed and interpreted in a paper by Houghton and Austin (1946).

2. Derivation of the equation for the isallobaric wind

The derivation of the isallobaric-wind equation given here follows Sutcliffe (1938), but all terms will be retained in order to study the legitimacy of the simplifications.

Let x and y be axes which point towards east and north, respectively. It will be assumed that the motion of the atmosphere is horizontal and frictionless. The equations of motion are then

$$\left.\begin{aligned} \frac{du}{dt} - lv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} = -lv_{gs} \\ \frac{dv}{dt} + lu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} = lu_{gs}. \end{aligned}\right\} \quad (1)$$

The components of the geostrophic wind are denoted by u_{gs} and v_{gs}. If the components of the geostrophic deviation, which are defined as

$$\left.\begin{aligned} u' &= u - u_{gs} \\ v' &= v - v_{gs}, \end{aligned}\right\} \tag{2}$$

are introduced in equation (1), the result is obtained that

$$\left.\begin{aligned} du/dt &= lv' \\ dv/dt &= -\,lu'. \end{aligned}\right\} \tag{3}$$

Thus, the geostrophic deviation is proportional to the acceleration and makes an angle of 90° with it. In the northern hemisphere the geostrophic deviation is directed to the left of the acceleration.

By partial differentiation of equations (2) with respect to time, one obtains, on assuming the density to be constant,

$$\frac{\partial u}{\partial t} = \frac{\partial u'}{\partial t} + \frac{\partial u_{gs}}{\partial t} = \frac{\partial u'}{\partial t} - \frac{1}{\rho l}\frac{\partial \dot{p}}{\partial y}$$

$$\frac{\partial v}{\partial t} = \frac{\partial v'}{\partial t} + \frac{\partial v_{gs}}{\partial t} = \frac{\partial v'}{\partial t} + \frac{1}{\rho l}\frac{\partial \dot{p}}{\partial x}.$$

Expansion of the individual time derivatives in equations (3) gives the following equations:

$$\frac{\partial u}{\partial t} = lv' - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y}$$

$$\frac{\partial v}{\partial t} = -\,lu' - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y}.$$

If these values for $\partial u/\partial t$ and $\partial v/\partial t$ are substituted in the preceding equations and if the latter are solved with respect to u' and v', it follows that

$$\left.\begin{aligned} lu' &= -\frac{1}{\rho l}\frac{\partial \dot{p}}{\partial x} - u\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial y} - \frac{\partial v'}{\partial t} \\ lv' &= -\frac{1}{\rho l}\frac{\partial \dot{p}}{\partial y} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{\partial u'}{\partial t}. \end{aligned}\right\} \tag{4}$$

These equations are identical with those derived by Brunt and Douglas if

$$\frac{\partial v'}{\partial t} = \frac{1}{l}\frac{\partial}{\partial t}\left(\frac{du}{dt}\right)$$

$$\frac{\partial u'}{\partial t} = -\frac{1}{l}\frac{\partial}{\partial t}\left(\frac{dv}{dt}\right),$$

based on equations (3), are substituted in them. The form (4) is preferable for a determination of the order of magnitude of the various terms from the observations.

Equations (4) are unsuitable for the practical determination of u' and v' because the three last terms on the right-hand side of each equation cannot readily be determined. Brunt and Douglas discuss the order of magnitude of these terms and conclude that they can be neglected compared with the other two terms. Then equations (4) are reduced to relations between the geostrophic deviation and the isallobaric gradient and become identical with the relations which are quoted in the introduction.

3. Magnitude of the different terms

The equations of Brunt and Douglas for the isallobaric wind contain only quantities which can be determined from weather maps. However, the validity of the isallobaric-wind concept cannot be considered as established; there is reason to doubt the arguments by which Brunt and Douglas attempt to justify the omission of the convective terms and of the local derivatives of the geostrophic deviations. The arguments need not be reproduced here. Instead, the orders of magnitude of the various terms appearing on the right-hand side of (4) are discussed on the basis of new determinations made from synoptic maps.

The data chosen for this study are the observed pressure and wind fields at 10,000 feet for 1600Z on 30 March 1945 and for 0400Z and 1600Z on 31 March 1945 over the North American continent. These maps were selected because the same period was used for another investigation, as mentioned previously.

If the isallobaric-wind relation is to hold, either the isallobaric terms

$$\frac{1}{\rho l}\frac{\partial \dot{p}}{\partial x},\quad \frac{1}{\rho l}\frac{\partial \dot{p}}{\partial y}$$

in equations (4) must be appreciably larger than the other terms on the right-hand side, or the sum of the other terms must equal zero, i.e.,

$$\left.\begin{aligned} \Delta(lv') &\equiv \frac{\partial v'}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = 0 \\ \Delta(lu') &\equiv \frac{\partial u'}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = 0. \end{aligned}\right\} \tag{5}$$

To investigate the first alternative the magnitude of the isallobaric term as well as of the quantities

$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)u,\quad \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)v,\quad \frac{\partial u'}{\partial t},\quad \frac{\partial v'}{\partial t}$$

were determined by Miss Hogan of the staff at Massachusetts Institute of Technology. Time derivatives, including the pressure tendency, have to be found from two consecutive maps 12 hours apart. The use of such large time intervals precludes an accurate determination of the time derivatives. Therefore, the quantities $(u\,\partial/\partial x + v\,\partial/\partial y)u$ and $(u\,\partial/\partial x + v\,\partial/\partial y)v$ for 0400Z 31 March 1945 have been compared with the values of $\partial u'/\partial t$, $\partial v'/\partial t$, and the isallobaric term computed for the 12-hour periods preceding and following this time (subscripts 1 and 2, respectively). The isallobars for both time intervals were found by graphical subtraction.

The individual values of the various quantities to be compared are, of course, subject to large errors, but, if the orders of magnitude of the various terms were very different, the absolute means should give an indication of this fact. Altogether 48 points were selected. The following absolute means were obtained (in cm sec^{-2}):

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)v,\quad 22.4\cdot 10^{-3};$$

$$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u,\quad 12.7\cdot 10^{-3};$$

$$\left(\frac{\partial v'}{\partial t}\right)_1,\quad 10.6\cdot 10^{-3};\qquad \left(\frac{\partial v'}{\partial t}\right)_2,\quad 10.6\cdot 10^{-3};$$

$$\left(\frac{\partial u'}{\partial t}\right)_1,\quad 8.1\cdot 10^{-3};\qquad \left(\frac{\partial u'}{\partial t}\right)_2,\quad 9.4\cdot 10^{-3};$$

$$\frac{1}{\rho l}\left(\frac{\partial \dot{p}}{\partial s}\right)_1,\quad 10\ \cdot 10^{-3};\qquad \frac{1}{\rho l}\left(\frac{\partial \dot{p}}{\partial s}\right)_2,\quad 18\ \cdot 10^{-3}.$$

The isallobaric term was not separated into components (s denotes distance in the direction of the isallobaric gradient). Nevertheless, it is, of course, comparable with the other terms. It is seen to be not appreciably larger than the other terms, so that they can hardly be neglected.

The value of the isallobaric-wind component which would follow from isallobaric terms of the magnitude given above is only between 1 and 2 m sec^{-1}. The isallobaric gradients found on surface maps often are larger. But it is reasonable to assume that in such cases the other terms, which are due to the various parts of the acceleration, would also be larger.

The concept of the isallobaric wind thus cannot be justified from considerations of orders of magnitude, and the only remaining possibility is that the conditions stated in (5) are satisfied. In order to check this, the quantities $\Delta(lv')$ and $\Delta(lu')$ have been computed using for $\partial u'/\partial t$ and $\partial v'/\partial t$ the values obtained for the preceding (subscript 1) and the following (subscript 2) 12 hours. The following values were obtained for the absolute means of these quantities (in cm sec^{-2}):

$$\Delta_1(lv'),\quad 23.7\cdot 10^{-3};\qquad \Delta_2(lv'),\quad 24.0\cdot 10^{-3};$$
$$\Delta_1(lu'),\quad 13.2\cdot 10^{-3};\qquad \Delta_2(lu'),\quad 14.9\cdot 10^{-3}.$$

In view of the inaccuracies inherent in the determination of the numerical values, on no hypothesis could these means be expected to be exactly equal to zero; but they are not even smaller than the terms of which they are composed.

It can be shown in yet another manner that the conditions of (5) are not satisfied. In order for these relations to hold, the sign of $(u\,\partial/\partial x + v\,\partial/\partial y)u$ must be different from the sign of $\partial u'/\partial t$, and the sign of $(u\,\partial/\partial x + v\,\partial/\partial y)v$ must be different from the sign of $\partial v'/\partial t$. A check of the 48 selected points gave the following figures:

Quantities compared		Identical signs	Opposite signs
$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u$,	$\left(\frac{\partial u'}{\partial t}\right)_1$	17 (5)	31 (7)
$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)u$,	$\left(\frac{\partial u'}{\partial t}\right)_2$	18 (3)	27 (10)
$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)v$,	$\left(\frac{\partial v'}{\partial t}\right)_1$	18 (6)	25 (8)
$\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)v$,	$\left(\frac{\partial v'}{\partial t}\right)_2$	19 (7)	25 (9)

The figures in parentheses show the frequencies of pairs with identical and opposite signs when only terms larger than 10^{-2} cm sec^{-2} are considered. In these cases of larger values the signs would presumably be more reliable. Opposite signs are more frequent than identical signs, but the difference of the frequencies is definitely not pronounced enough to permit the assertion that the terms cancel each other in general.

Since only horizontal motion is considered here the terms $w\,\partial u/\partial z$ and $w\,\partial v/\partial z$ are omitted in the preceding discussion. These terms appear in the expansion of the expressions for the individual acceleration. They are of the same or even larger magnitude than the other terms in the expansion. Consequently, they impair the validity of the equation for the isallobaric wind still more.

4. Regions of convergence and divergence in a moving pressure system

The theoretical as well as the observational basis of the isallobaric wind is so unsatisfactory that this concept has to be abandoned. On the other hand, regions of convergence in the lower troposphere are observed ahead of moving cyclones, regions of divergence behind, apparently closely related to isallobaric lows and highs, respectively. With the isallobaric-wind relation this observed distribution could easily be explained. However, since the isallobaric-wind relation is shown to be invalid, another explanation for the distribution of regions of convergence and divergence will be offered here. This explanation is based on the assumption that a pressure center, for instance a cyclone, moves through the atmosphere in a manner somewhat analogous to that of a wave traveling through water. As the front of the wave arrives at a given spot the water is set in motion, while in the rear of the wave the motion subsides. Similarly, new air is drawn into a moving cyclone on the front side, while air is left behind in the rear. The proposed explanation thus assumes that the pressure field moves relatively faster than the air. It is not claimed that this is necessarily true in all or most cases, even near the ground, where friction is effective. The purpose of the following discussion is merely to show that other explanations besides the isallobaric-wind relation can be found for the observed distribution of convergence and divergence around a cyclone.

If the air in front of the moving cyclone is thus subjected to an increasing pressure-gradient force, it acquires a certain velocity; but, since it has newly arrived in the low-pressure system, the velocity is not yet large enough for the Coriolis and centrifugal forces to counterbalance the pressure-gradient force. The last force acts towards the center, while the other two act away from the center. Because of the predominance of the pressure-gradient force the air will have a component towards the center in the front part of the cyclone. In the rear of the cyclone the pressure-gradient force decreases and the air motion is still strong, so that the sum of Coriolis force and centrifugal force overbalances the pressure-gradient force. Hence, outflow takes place.

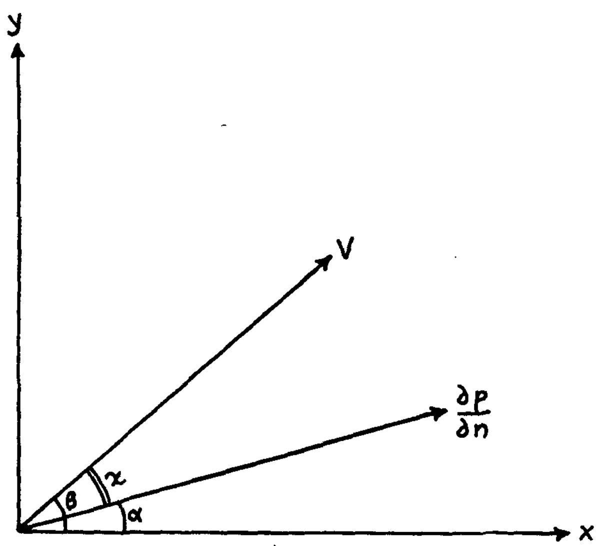

FIG. 1. Orientation of wind and pressure gradient.

The same result may be derived more rigorously as follows: Multiplication of the two equations (1) by u and v, respectively, and addition give

$$u\frac{du}{dt} + v\frac{dv}{dt} = -\frac{1}{\rho}\left(u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right).$$

Let the angle (Fig. 1) between the wind velocity **V** and the x-axis be β, between the pressure ascendant and the x-axis α, further let $\beta - \alpha = \chi$. Then

$$u = V\cos\beta, \qquad v = V\sin\beta,$$

and

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial n}\cos\alpha, \quad \frac{\partial p}{\partial y} = \frac{\partial p}{\partial n}\sin\alpha.$$

By substitution in the equation above,

$$V\frac{dV}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial n}V\cos(\beta - \alpha)$$

and

$$\cos\chi = -\frac{dV}{dt}\Big/\left(\frac{1}{\rho}\frac{\partial p}{\partial n}\right). \qquad (6)$$

Since $\partial p/\partial n$ is positive by definition, it follows that

$$\chi \gtreqless 90° \quad \text{if} \quad dV/dt \gtreqless 0.$$

For $\chi > 90°$, the angle between the pressure ascendant and the wind is larger than a right angle, so that the air has a component of inflow towards low pressure. This inflow is connected with the increasing wind velocities on the front side of the low according to the above inequalities. Behind the low-pressure system, conditions are reversed.

For purposes of orientation numerical values may be substituted in equation (6). Let

$$dV/dt = 10^{-2}\ \text{cm sec}^{-2} \sim 1\ \text{m sec}^{-1}\ (3\ \text{hr})^{-1}$$
$$\partial p/\partial n = 1\ \text{mb}\ (100\ \text{km})^{-1}.$$

Then, at latitude 43°, $V_{gs} = 10$ m sec^{-1} and $\cos\chi = -0.1 = \cos 96°$. In other words, the wind makes an angle of 6° with the isobars, towards lower pressure. If the acceleration were four times larger, the angle between wind and isobars would be 24°.

It must be pointed out that all the preceding statements refer to flow across the isobars but do not necessarily imply an actual divergence and convergence, although the two phenomena will in general be closely connected.

Another way to establish a relation between divergence and acceleration would be by means of the well known formula for the change of vorticity,

$$\frac{d}{dt}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = -\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + 2\omega\sin\phi\right)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

Ahead of the cyclone, cyclonic (positive) vorticity presumably increases, so that there is negative divergence, or convergence. Behind the cyclone the cyclonic vorticity decreases, so that the divergence should be positive. Thus the generally observed distribution of divergence and convergence around a cyclone in the lower part of the atmosphere can be explained by the field of acceleration without the isallobaric wind.

While it has been shown that the simple relation between the isallobaric gradient and the geostrophic deviation derived by Hesselberg and by Brunt and Douglas does not hold, it is, of course, not suggested that the two quantities are entirely unrelated. But the actual connection is evidently influenced considerably by a number of other factors, such as the local change of the geostrophic deviation, the horizontal variations of the field of motion, and the vertical motion. The last has not been considered here, but its influence must be pronounced since it strongly affects atmospheric convergence and divergence, which constitute an important factor in the mechanism of pressure variations.

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