The Propagation of Acoustic-Gravity Waves in a Moist Atmosphere

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ABSTRACT

The nonlinear equations governing the propagation of acoustic-gravity waves in a moist atmosphere are derived following the theory of mixtures. The water droplets are assumed to be very small compared to the characteristic scales of the system and numerous enough to treat them as a fluid. The atmosphere is assumed saturated at all heights, and the equations are linearized about a background with no shear flow. The WKB solution reveals that the local properties of the atmosphere are considerably affected by the exchange of heat due to condensation or evaporation. In particular, for a given frequency and horizontal wavelength, the vertical wavelength increases compared to the case of a dry atmosphere. The treatment applies to all frequencies less than the audible range.

1. Introduction

The dynamics of air, water vapor mixtures has received a lot of attention over the years, because of its many manifestations and applications. In the present paper we are concerned with the role played by water vapor and water droplets on the dynamics of acoustic and internal gravity waves in the atmosphere, i.e., on those waves whose periods and wavelengths are small enough to justify the neglect of the Coriolis force.

When the humidity and temperature distributions in the atmosphere are such that condensation doesn't take place, internal gravity waves behave as if the air were dry but characterized by different physical constants, i.e., ratio of specific heats, scale heights, etc., appropriate to the new mixture of constituent perfect gases. However, when the background is such that changes of phase are possible, then the situation is drastically altered because of the enormous amount of latent heat that can be released. This energy release can be important in sustaining wave motions in the troposphere, in particular in the earth's boundary layer, as well as at heights between 80 and 85 km where ice particles are thought to play a significant role in the appearance of noctilucent clouds.

Two main approaches are available in studying the problem. The first one is the statistical approach which derives the governing equations from the Boltzmann equation. Culick (1964), Pai (1970) and others have followed this method. This approach, although conceptually clear, encounters the usual mathematical difficulties and complexities familiar from kinetic theory plus additional complications due to the new particle-particle mass exchange interaction term.

The second approach assumes that the water droplets are very small compared to characteristic scales of the system and numerous enough so as to treat them as a pseudo-fluid. Thus, the system can be considered to be a mixture of three gases, dry air, water vapor, and water droplet gas with the possibility of a phase change between the last two. Truesdell (1957, 1969), Green and Naghdi (1965, 1969), Müller (1968), DeSilva and Kaloni (1969), Allen and Kline (1969), Drew and Segel (1971), Soo (1967) and others have discussed various properties of mixtures and derived general governing conservation equations. The latter are necessary to compensate for our inability, or even lack of interest because of disproportionate effort, to solve completely and exactly the problem of N spheres moving in and interacting with a mixture of air and water vapor. For example, the lack of knowledge of the detailed velocity, density and pressure fields surrounding a particular droplet implies that we cannot describe exactly the momentum and energy exchange due to the evaporation of part of the droplet. We circumvent the difficulty by introducing characteristic exchange velocities and energies which cannot be completely arbitrary but which must satisfy a number of constraints leading to the constitutive relations. The justification and the nature of the constitutive relations in a multi-phase system are by no means a settled matter, and significant philosophical and practical differences exist among various authors.

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The approach just described has been followed by Marble (1969), Marble and Wooten (1970), and Cole and Dobbins (1970, 1971) in studying the effects of condensation on the propagation of sound waves in the atmosphere. They limit themselves to sufficiently small periods, so the effect of gravity is neglected.

Marble (1969) first derived the equations for a droplet-vapor mixture where the droplets were treated as a continuum. The constitutive equations were obtained from phenomenological considerations without comments as to their more general form which could introduce important changes when nonlinear processes are discussed or a background velocity is introduced. Marble and Wooten (1970) extended the linearized form of these equations to include dry air and discussed the relaxation times and the attenuation of sound waves giving numerical results for a variety of frequency ranges. Cole and Dobbins (1970, 1971), starting with similar equations, considered in detail the case of atmospheric fog and using appropriate parameters gave attenuation rates for sound propagation and compared them with experiments where possible.

The effect of gravity has been included by Lalas (1972) who derived the dispersion relation for internal gravity waves under a number of simplifying assumptions and for a specific background temperature and density profile.

In the present paper, we use the energy postulate approach along with invariance considerations to derive a general system of equations for the air, water vapor, water droplet mixture. The atmosphere is assumed saturated at all heights and the droplet number density is taken high enough so that one can treat the collection of droplets as a continuum, with detailed exchange mechanisms based on an average droplet size. The introduction of a droplet size distribution function, although not carried out here, appears straightforward.

The governing equations are derived in Section 2, while the constitutive equations and the final system of equations are given in Section 3. These equations are linearized in Section 4 where the background quantities are also discussed. In Section 5 the system of linear equations is solved for a slowly varying background consistent with the thermodynamic conditions imposed by the coexistence of the liquid and gaseous phase of water. The background is assumed stationary. A WKB approximation is invoked to solve the second-order ordinary differential equation with variable coefficients that results from the assumption of an $\exp[i(\omega t-k_x x)]$ dependence on time and the horizontal coordinate x. A height-dependent local dispersion relation is obtained which enables us to calculate attenuation and phase velocity at each height. As an illustration, a particular humidity distribution is assumed and plots of phase speed and attenuation are calculated for various values of the parameters, i.e., average radius of the droplets, specific humidity, etc.

Finally, we mention that in a companion paper, Lalas and Einaudi (1973) apply the equations obtained here to determine the effect of water vapor and water droplets on the stability of an atmosphere with a height-dependent background wind.

2. Governing equations for the mixture

In this section we outline the mathematical theory of mixtures; the selection and justification of the various constitutive relations for the specific problem at hand are left for Section 3.

Let us write the conservation of energy statement for an arbitrary volume τ enclosed by a surface σ. Since the rate of increase of internal plus kinetic energy has to equal the sum of (i) the rate of work done on τ by body forces and surface stresses, (ii) the rate of inward transport of heat via thermal conduction through σ, and (iii) the rate of generation of energy through production of species within τ, we can write (see Nachbar *et al.*, 1959):

$$\frac{\partial}{\partial t}\int_\tau \rho^\alpha E^\alpha d\tau+\int_\sigma n_k v_k^\alpha \rho^\alpha E^\alpha d\sigma$$
$$=\int_\tau [\rho^\alpha F_i^\alpha v_i^\alpha+\theta_i^\alpha \hat{v}_i^\alpha]d\tau+\int_\sigma [t_{ij}^\alpha v_i^\alpha-q_i^\alpha-\bar{q}_i^\alpha]n_i d\sigma$$
$$+\int_\tau [\rho^\alpha r^\alpha+\psi^\alpha]d\tau, \quad (2.1)$$

where:

- ρ^α partial density of constituent α, i.e., the mass of α per unit volume of the mixture
- v_i^α velocity of α
- t_{ij}^α partial stress tensor of α
- F_i^α external body forces acting on α (gravity for example)
- θ_i^α internal (interspecies) body forces acting on α (drag force, for example)
- E^α total energy of α $[=\epsilon^\alpha+\frac{1}{2}v_i^\alpha v_i^\alpha]$
- ϵ^α partial specific internal energy of α
- q_i^α heat flux vector to α from α and outside sources
- $\bar{q}_i^\alpha$ heat flux vector to α from remaining species
- ψ^α volume contribution to energy of α via interaction with other species
- $\hat{v}_i^\alpha$ a velocity, to be specified later, which depends on the detailed flow field around each droplet
- r^α an external heat source term to α
- n_i ith component of the outward normal to σ.

Using Gauss's theorem, we can rewrite (2.1) as follows:

$$\int_\tau \frac{\partial}{\partial t}(\rho^\alpha E^\alpha)d\tau$$
$$+\int_\tau \left[\frac{\partial}{\partial x_i}(\rho^\alpha E^\alpha v_i^\alpha)-\frac{\partial}{\partial x_i}(t_{ij}^\alpha v_j^\alpha-q_i^\alpha-\bar{q}_i^\alpha)\right.$$
$$\left.-(\rho^\alpha F_i^\alpha v_i^\alpha+\theta_i^\alpha \hat{v}_i^\alpha)-(\rho^\alpha r^\alpha+\psi^\alpha)\right]d\tau=0. \quad (2.2)$$

We now postulate that

$$\psi^\alpha \frac{\partial \bar{q}_i^\alpha}{\partial x_i} = \Psi^\alpha + \tfrac{1}{2}\gamma^\alpha w_i^\alpha \hat{w}_i^\alpha, \tag{2.3}$$

i.e., we assume that the amount of energy given to (or taken from) α by the remaining species can be written as the sum of two terms: Ψ^α due to all contributions independent of velocities, and a term that depends on the relative velocities of different species. The quantities γ^α are mass fluxes with dimensions of mass per unit volume and time, while w_i^α and $\hat{w}_i^\alpha$ are characteristic velocities, and they all still need constitutive relations and physical interpretation.

Since we assume that all species occupy the same volume $\tau^\alpha = \tau$ simultaneously and since τ is an arbitrary control volume, we conclude, from (2.2), that

$$\rho^\alpha r^\alpha + (\pi_i^\alpha - \tfrac{1}{2}m^\alpha v_i^\alpha)v_i^\alpha + \theta_i^\alpha \hat{v}_i^\alpha - m^\alpha \epsilon^\alpha - \rho^\alpha \frac{d\epsilon^\alpha}{dt^\alpha} - \frac{\partial q_i^\alpha}{\partial x_i} + \Psi^\alpha + \tfrac{1}{2}\gamma^\alpha w_i^\alpha \hat{w}_i^\alpha + t_{ij}^\alpha \frac{\partial v_j^\alpha}{\partial x_i} = 0, \tag{2.4}$$

where

$$\frac{d}{dt^\alpha} = \frac{\partial}{\partial t} + v_i^\alpha \frac{\partial}{\partial x_i}, \quad m^\alpha = \frac{d\rho^\alpha}{dt^\alpha} + \rho^\alpha \frac{\partial v_i^\alpha}{\partial x_i}, \tag{2.5}$$

$$\pi_i^\alpha = \rho^\alpha F_i^\alpha - \rho^\alpha \frac{dv_i^\alpha}{dt^\alpha} + \frac{\partial t_{ki}^\alpha}{\partial x_k}. \tag{2.6}$$

After noticing that m^α and π_i^α are invariant under a uniform rigid-body translation, we assume that r^α, ρ^α, ϵ^α, θ_i^α, q_i^α, t_{ij}^α, Ψ^α are also invariant under such a translation. Now the energy conservation equation (2.4) must be invariant under the transformation

$$v_i^\alpha = \tilde{v}_i^\alpha + b_i, \tag{2.7}$$

with b_i a constant (see, e.g., Green and Rivlin, 1964; Green and Naghdi, 1965). Let us limit ourselves to the choice of w_i^α, $\hat{w}_i^\alpha$ and $\hat{v}_i^\alpha$ that satisfies a similar invariance condition, that is, $w_i^\alpha = \tilde{w}_i^\alpha + b_i$, $\hat{w}_i^\alpha = \tilde{\hat{w}}_i^\alpha + b_i$, $\hat{v}_i^\alpha = \tilde{\hat{v}}_i^\alpha + b_i$. This is not a very restrictive assumption since θ_i^α and γ^α are still in need of a constitutive equation. Substitution of (2.7) in (2.4) immediately shows that the conditions for invariance are that the coefficients of b_i and b_ib_i in (2.4) be zero. These conditions yield the equations

$$m^\alpha = \gamma^\alpha, \tag{2.8}$$

$$\pi_i^\alpha - m^\alpha v_i^\alpha + \theta_i^\alpha + \frac{\gamma^\alpha}{2}(w_i^\alpha + \hat{w}_i^\alpha) = 0, \tag{2.9}$$

which can be rewritten as

$$\frac{\partial \rho^\alpha}{\partial t} + \frac{\partial}{\partial x_i}(\rho^\alpha v_i^\alpha) = \gamma^\alpha, \tag{2.10}$$

$$\rho^\alpha \frac{dv_i^\alpha}{dt^\alpha} - \left(\frac{\partial}{\partial x_j} t_{ij}^\alpha + \rho^\alpha F_i^\alpha\right) = \theta_i^\alpha + \frac{\gamma^\alpha}{2}(w_i^\alpha + \hat{w}_i^\alpha) - \gamma^\alpha v_i^\alpha = \hat{m}_i^\alpha. \tag{2.11}$$

These are the usual equations of conservation of mass and momentum with extra terms on the right-hand side to account for the exchanges between species. The physical meaning of γ^α and $(w_i^\alpha + \hat{w}_i^\alpha)/2$ introduced in (2.3) is now apparent. They represent the net production of mass of species α per unit volume and time, and the average velocity of the newly created mass of species α per unit mass of species α. The quantity $\hat{m}_i^\alpha$ represents the growth of linear momentum per unit volume for the species α, through interaction with other species.

We now assume that the energy exchange term Ψ^α is made of a term Φ^α due to conduction and of a term that accounts for the possibility of the newly created mass added to species α having internal energy other than ϵ^α:

$$\Psi^\alpha = \Phi^\alpha + \gamma^\alpha \hat{\epsilon}^\alpha, \tag{2.12}$$

so that (2.4) becomes

$$\rho^\alpha \frac{d\epsilon^\alpha}{dt^\alpha} - t_{ij}^\alpha \frac{\partial v_j^\alpha}{\partial x_i} - \rho^\alpha r^\alpha + \frac{\partial q_i^\alpha}{\partial x_i} = (\pi_i^\alpha - \tfrac{1}{2}m^\alpha v_i^\alpha)v_i^\alpha + \theta_i^\alpha \hat{v}_i^\alpha - m^\alpha(\epsilon^\alpha - \hat{\epsilon}^\alpha) + \Phi^\alpha + \tfrac{1}{2}\gamma^\alpha w_i^\alpha \hat{w}_i^\alpha = \mathcal{E}^\alpha, \tag{2.13}$$

where $\mathcal{E}^\alpha$ is the growth of energy per unit volume for species α via interaction with other species. The quantities γ^α, $\hat{m}_i^\alpha$ and $\mathcal{E}^\alpha$ would all be zero if component α were isolated from the other species.

The equations (2.10), (2.11) and (2.13) so far derived contain quantities such as r^α, t_{ij}^α, θ_i^α, F_i^α, $\hat{v}_i^\alpha$, w_i^α, $\hat{w}_i^\alpha$, q_i^α, Φ^α, γ^α, $\hat{\epsilon}^\alpha$ for which we have to provide constitutive relations. However, additional relations among the above variables can be obtained using phenomenological considerations concerning the behavior of a mixture. By assuming 1) that any property of the mixture is a mathematical consequence of the properties of the various components, and 2) that equations governing the motion of the mixture have the same form as for a single constituent (provided the diffusion effects are properly taken into account), Truesdell (1957) first derived the following three relations:

$$\sum_\alpha \gamma^\alpha = 0, \tag{2.14}$$

$$\sum_\alpha (\hat{m}_i^\alpha + \gamma^\alpha u_i^\alpha) = 0, \tag{2.15}$$

$$\sum_\alpha [\mathcal{E}^\alpha + \gamma^\alpha(\epsilon^\alpha + \tfrac{1}{2}u_i^\alpha u_i^\alpha) + \hat{m}_i^\alpha u_i^\alpha] = 0, \tag{2.16}$$

where $u_i^\alpha = v_i^\alpha - v_i^0$, v_i^0 being the velocity of the center of mass defined as

$$v_i^0=(\sum_\alpha \rho^\alpha v_i^\alpha)/(\sum_\alpha \rho^\alpha).$$

Relation (2.14) expresses the well-accepted concept that mass is not created or destroyed by chemical reactions, but only converted from one species to another. Eqs. (2.15) and (2.16), expressing the same idea of redistribution between species rather than creation of momentum and energy, were first introduced by Truesdell in his 1957 papers. The first equation states that the sum of all partial increments of momentum generation due to (i) species interaction ($\hat{m}_i^\alpha$), and (ii) diffusion of the generated mass ($\gamma^\alpha u_i^\alpha$), must be zero. The second states that the sum of the partial increments of energy generation due to (i) species interaction ($\mathcal{E}^\alpha$), (ii) work against diffusion of the partial momentum generation due to species interactions ($\hat{m}_i^\alpha u_i^\alpha$), and (iii) energy exchanged through mass generation in a center of mass frame, must be zero.

While we recognize that (2.15) and (2.16) are somewhat axiomatic, we shall use them together with (2.10), (2.11), (2.13) and (2.14) to form our basic set of governing equations to be supplemented by additional constitutive relations in the following section.

3. The constitutive equations and the final set of equations

a. *The constitutive equations and assumptions*

We now focus our attention to the case of three species in the atmosphere and we use superscripts "1", "2" and "3" to indicate dry air, water vapor and droplet gas, respectively. We also assume that

$$r^\alpha=q^\alpha=0,\quad F_i^\alpha=-g\delta_{i3},\quad t_{ij}^\alpha=-p^\alpha\delta_{ij}, \tag{3.1}$$

where p^α is the partial pressure of species α, g the acceleration of gravity, and δ_{ij} the Kronecker delta with $x_3\equiv z$ the vertical axis whose positive direction is upward. Here (3.1) implies that viscous and conduction effects are unimportant except close to the interfaces and that radiation or other sources of energy are absent. Furthermore, following Eliassen and Kleinschmidt (1957), we define

$$\epsilon^\alpha=c_v^\alpha T^\alpha+L^\alpha, \tag{3.2}$$

where c_v^α is the specific heat at constant volume, T^α the temperature of species α, and L^α a constant which depends on the origin of the energy scale.

Both the dry air and the water vapor are assumed to behave as perfect gases so that

$$p^{(1)}=\rho^{(1)}R^{(1)}T^{(1)},\quad p^{(2)}=\rho^{(2)}R^{(2)}T^{(2)}, \tag{3.3}$$

where $R^{(1)}$ and $R^{(2)}$ are the gas constants for air and water vapor, respectively. In addition, we assume that they are well mixed so that one can take

$$v_i^{(1)}=v_i^{(2)}=v_i,\quad T^{(1)}=T^{(2)}=T. \tag{3.4}$$

The fact that the dry air does not change phase allows us to write

$$\gamma^{(1)}=w_i^{(1)}=\hat{w}_i^{(1)}=0, \tag{3.5}$$

and therefore (2.14) yields

$$\gamma^{(2)}=-\gamma^{(3)}=\Gamma. \tag{3.6}$$

We now proceed to analyze gas "3" and in particular its microstructure. If all the droplets are considered the same with average radius a, and if ρ_w is the density of liquid water and n_p the number of droplets per unit volume of the mixture, then

$$\rho^{(3)}=\tfrac{4}{3}\pi a^3\rho_w n_p=m_p n_p, \tag{3.7}$$

m_p being the mass of the average droplet. We also set the partial pressure $p^{(3)}=0$ on the basis that the volume fraction of water droplets is negligible. Furthermore, even though no meaningful temperature of the droplet gas is introduced in the usual statistical mechanical sense, internal energy and exchange processes can be described by utilizing the well-defined droplet temperature T_d, which is assumed constant throughout the droplet.

The microstructure of the droplet gas involves the region close to the droplet surface where all the detailed phenomena occur that affect the local vapor parameters through the exchange terms Φ^α and θ_i^α. Within this region that extends only a few droplet diameters, the local average values of $p^{(2)}$, $\rho^{(2)}$ and T differ from the vapor pressure, density and temperature at the droplet surface (p_ω, ρ_ω and $T_\omega=T_d$). In turn, p_ω, ρ_ω and T_ω are related to each other by the equation of state for ideal gases and by the Clausius-Clapeyron equation:

$$p_\omega=R^{(2)}\rho_\omega T_\omega, \tag{3.8}$$

$$\frac{dp_\omega}{p_\omega}=\frac{L_v}{R^{(2)}T_\omega^2}dT_\omega, \tag{3.9}$$

where L_v is the latent heat of evaporation equal to 596 cal gm^{-1} at 300K. Since the atmosphere is assumed saturated, (3.9) holds throughout. Eq. (3.9) represents our situation only approximately since it was derived to describe the chemical equilibrium of two phases divided by a plane interface.

We now express the production of water vapor per unit volume and unit time Γ, and the total heat per unit volume $\Phi^{(3)}$ transferred from the air, water vapor system to the droplet gas, via conduction, in terms of the gradients of pressure, density and temperature surrounding each droplet. $\Phi^{(3)}$ is equal to the heat transferred to one droplet [$q=4\pi a^2(\kappa/a)(T-T_d)$] multiplied by n_p, i.e.,

$$\Phi^{(3)}=n_p m_p\frac{3\kappa}{a^2\rho_w}(T-T_d),\quad \kappa=\frac{\kappa^{(1)}\rho^{(1)}+\kappa^{(2)}\rho^{(2)}}{\rho^{(1)}+\rho^{(2)}}, \tag{3.10}$$

where κ is the effective thermal conductivity of the air-vapor gas mixture expressed in terms of the conductivities $\kappa^{(1)}$ and $\kappa^{(2)}$ of dry air and water vapor. In describing the heat transfer by (3.10), use was made of the low Reynolds number assumption, i.e., convective effects due to the motion of each droplet relative to the air are neglected, since the slip velocity is negligible as will be shown later. Furthermore, since there are no other forms of heat exchange that we consider here, we can write

$$\Phi^{(1)}+\Phi^{(2)}=-\Phi^{(3)}. \tag{3.11}$$

In order to calculate Γ, we assume that the rate-controlling mechanism of mass exchange between droplet and ambient vapor is diffusion; following Byers (1965), we can then write

$$\frac{d}{dt_p}m_p=4\pi\rho_w a^2\frac{da}{dt_p}=4\pi Da[\rho^{(2)}-\rho_\omega], \tag{3.12}$$

where $d/dt_p=d/dt^{(3)}$ and D is the diffusivity of water vapor. Eq. (3.12) implies that the radius a is always larger than the critical value below which processes of nucleation are important. Γ therefore has the form

$$\Gamma=-n_p\frac{d}{dt_p}m_p=-4\pi n_p aD[\rho^{(2)}-\rho_\omega]. \tag{3.13}$$

In (3.13) we have excluded growth by collision and coalescence which would have introduced an additional term involving the time rate of change of n_p.

Let us now calculate the interspecies force θ_i^α. Since the only dynamic interaction between the droplet gas and the moist air is via the drag force acting on each droplet, we write the total body force $\theta^{(3)}$ acting on gas "3" as

$$\left.\begin{aligned}\theta_i^{(3)}&=\frac{9}{2}\frac{\mu}{\rho_w a^2}n_p m_p[v_i-v_i^{(3)}]\\ \mu&=\frac{\mu^{(1)}\rho^{(1)}+\mu^{(2)}\rho^{(2)}}{\rho^{(1)}+\rho^{(2)}}\end{aligned}\right\}, \tag{3.14}$$

where μ is the viscosity coefficient for the air-vapor mixture expressed in terms of the viscosity coefficients $\mu^{(1)}$ and $\mu^{(2)}$ for dry air and water vapor. The above equation is valid in the Stokes regime, i.e., for Reynolds' numbers $\mathrm{Re}\approx(2a/\mu)\rho^{(1)}V_d$ up to 10 (Soo, 1967); V_d is the velocity of the droplet relative to the fluid. For typical conditions in clouds (see Ludlam and Mason, 1957), the predominant radius is $\sim10^{-3}$ cm and $\mu\approx\mu^{(1)}=1.82\ 10^{-4}$ gm sec^{-1} cm^{-1}, so that $\mathrm{Re}\approx0.015\ V_d$ (V_d in cm sec^{-1}). Therefore, even for velocities of 10–30 m sec^{-1} we would still be in the Stokes regime. We also note that

$$\theta_i^{(3)}=-[\theta_i^{(1)}+\theta_i^{(2)}]. \tag{3.15}$$

Finally, we have to specify w_i^α, $\hat{w}_i^\alpha$, $\hat{v}_i^\alpha$ and $\hat{\epsilon}^\alpha$. We are free to choose them in any way that is consistent with (2.14)–(2.16), and our physical intuition, since we lack detailed solutions of the coupled mass and heat exchange hydrodynamic problems. By explicitly introducing these exchange terms, the present formulation provides an assurance of completeness and a knowledge of the assumptions made.

Reasoning that the kinetic energy exchanged via mass exchange is evenly split between species, we take

$$w_i^{(2)}=\hat{w}_i^{(2)}=w_i^{(3)}=\hat{w}_i^{(3)}=\frac{u_i+v_i}{2}, \tag{3.16}$$

where $u_i=v_i^{(3)}$. The above choices, together with (3.5), identically satisfy (2.15). Furthermore, we choose

$$\hat{v}_i^{(1)}=\hat{v}_i^{(2)}=\hat{v}_i^{(3)}=u_i, \tag{3.17}$$

since the work done by the internal body forces is $\theta_i^\alpha u_i$.

We now observe that the rate of generation of nonmechanical energy of species "2" is due to the exchange of mass of internal energy $\epsilon^{(2)}(T_\omega)$ and to the work done as the water vaporizes p_ω/ρ_ω, and so we write

$$\hat{\epsilon}^{(2)}=\epsilon^{(2)}(T_\omega)+p_\omega/\rho_\omega, \tag{3.18}$$

and by similar argument

$$\hat{\epsilon}^{(3)}=\epsilon^{(3)}(T_\omega)+L_v. \tag{3.19}$$

We may note that $L_v=\hat{\epsilon}^{(2)}-\epsilon^{(3)}$ by definition. Finally, we point out that the above choice of constitutive relations satisfies (2.16) identically.

b. The final set of equations

Using the various equations introduced above and in Section 2, Eqs. (2.10), (2.11) and (2.13) can be written in the following form:

Conservation of mass

$$\frac{\partial}{\partial t}[\rho^{(1)}+\rho^{(2)}]+\frac{\partial}{\partial x_i}\{[\rho^{(1)}+\rho^{(2)}]v_i\}=\Gamma \tag{3.20}$$

$$\frac{\partial\rho^{(2)}}{\partial t}+\frac{\partial}{\partial x_i}[\rho^{(2)}v_i]=\Gamma \tag{3.21}$$

$$\frac{\partial}{\partial t}(n_p m_p)+\frac{\partial}{\partial x_i}(n_p m_p u_i)=-\Gamma \tag{3.22}$$

Conservation of momentum

$$[\rho^{(1)}+\rho^{(2)}]\left(\frac{\partial v_i}{\partial t}+v_j\frac{\partial v_i}{\partial x_j}\right)+\frac{\partial}{\partial x_i}[p^{(1)}+p^{(2)}]$$
$$+[\rho^{(1)}+\rho^{(2)}]g\delta_{i3}=\theta_i^{(1)}+\theta_i^{(2)}+\frac{\Gamma}{2}(u_i-v_i) \tag{3.23}$$

$$n_p m_p\left(\frac{\partial u_i}{\partial t}+u_j\frac{\partial u_i}{\partial x_j}\right)+n_p m_p g\delta_{i3}=\theta_i^{(3)}+\frac{\Gamma}{2}(u_i-v_i) \tag{3.24}$$

Conservation of energy

$$[\rho^{(1)}c_v^{(1)}+\rho^{(2)}c_v^{(2)}]\left(\frac{\partial T}{\partial t}+v_i\frac{\partial T}{\partial x_i}\right)+[p^{(1)}+p^{(2)}]\frac{\partial v_i}{\partial x_i}$$
$$=\Phi^{(1)}+\Phi^{(2)}+(u_i-v_i)[\theta_i^{(1)}+\theta_i^{(2)}]$$
$$+\Gamma[c_v^{(2)}(T_\omega-T)+R^{(2)}T_\omega]+\frac{\Gamma}{2}\left(\frac{v_i-u_i}{2}\right)^2 \quad (3.25)$$

$$n_pm_pc_v^{(3)}\left(\frac{\partial T_\omega}{\partial t}+u_i\frac{\partial T_\omega}{\partial x_i}\right)$$
$$=\Phi^{(3)}-\frac{\Gamma}{2}\left(\frac{v_i-u_i}{2}\right)^2-\Gamma L_v. \quad (3.26)$$

In the above equations Γ is the rate of mass production of water vapor per unit volume; θ_i^α is the drag force per unit volume acting on the constituent α and due to its dynamic interactions with the remaining species; and Φ^α is the rate of heat conduction per unit volume to α from the remaining species.

Eqs. (3.20), (3.23) and (3.25) were obtained by adding the corresponding equations for species "1" and "2". The system of equations for the 16 unknowns, $[\rho^{(1)}+\rho^{(2)}]$, $\rho^{(2)}$, n_p, m_p, ρ_ω, $[p^{(1)}+p^{(2)}]$, p_ω, T, T_ω, v_i, u_i, Γ, is closed by the equations of state (3.3) and (3.8), the Clausius-Clapeyron equation (3.9), the mass generation equation (3.12), and an externally specified n_p. The auxiliary variables a, Φ^α and θ_i^α are specified by the constitutive equations (3.7), (3.10), (3.11), (3.14) and (3.15). The quantity n_p is simply the number of nuclei per unit volume around which the water droplets have developed.

We finally point out again that the above system of equations assumes that the atmosphere is saturated at all heights and at all times: if this situation ceases to be true, then the quantities p_ω, ρ_ω and T_ω become arbitrary and have to be redefined.

4. The background equations and the linearized equations

a. The background equations

We derive here the background equations for an atmosphere saturated at all heights. Horizontal variations are neglected and no background winds are considered. Hence, $v_i=\partial/\partial t=\partial/\partial x=\partial/\partial y=0$. Using the suffixes 0 and g to denote background quantities at an arbitrary height z and at $z=0$, respectively, we have

$$\rho_0^{(2)}=\rho_{\omega 0},\quad T_0=T_{\omega 0},\quad p_0^{(2)}=p_{\omega 0}, \quad (4.1)$$

$$\frac{d}{dz}[p_0^{(1)}+p_0^{(2)}]=-g[\rho_0^{(1)}+\rho_0^{(2)}+\rho_0^{(3)}], \quad (4.2)$$

$$p_0^{(1)}+p_0^{(2)}=[\rho_0^{(1)}R^{(1)}+\rho_0^{(2)}R^{(2)}]T_0, \quad (4.3)$$

$$p_0^{(2)}/p_g^{(2)}=\exp\left[\frac{L_v}{R^{(2)}T_g}\left(1-\frac{T_g}{T_0}\right)\right]. \quad (4.4)$$

In the above equations the effect of the falling terminal velocity u_{z0} of the droplets has been neglected. An estimate on the value of u_{z0} can be obtained by setting the Stokes drag equal to the gravitational body force, so

$$\theta_z^{(3)}=-\frac{9}{2a_0^2}\frac{\mu}{\rho_w}\rho_0^{(3)}u_{z0}\approx\rho_0^{(3)}g, \quad (4.5)$$

where a_0 is the average radius of the droplets in the absence of any disturbance. For typical values of $a_0=10^{-3}$ cm, $g=981$ cm sec^{-2}, $\mu\approx\mu^{(1)}=1.82\ 10^{-4}$ gm sec^{-1} cm^{-1}, and $\rho_w=1$ gm cm^{-3}, one gets $u_{z0}\approx-1$ cm sec^{-1}. This value is negligible compared to either the speed of sound ($c_0\approx3.2\ 10^4$ cm sec^{-1}) or typical gravity wave speeds of the order of a few meters per second. By neglecting u_{z0}, we in effect ignore the very long time scale interaction between the water droplets and the background medium.

When the atmosphere is not saturated, (4.4) disappears and we can choose T_0 arbitrarily and obtain $p_0^{(1)}+p_0^{(2)}$ and $\rho_0^{(1)}+\rho_0^{(2)}$ in terms of T_0. In particular, we can still choose $T_0=$constant and have an exponentially decaying $\rho_0^{(2)}(z)$. This freedom is reduced when we are dealing with a saturated atmosphere since now we have an additional relation between T_0 and $\rho_0^{(2)}$. Thus, if $T_0=$constant, from (4.4) and (3.3) we conclude that $\rho_0^{(2)}=$constant too.

If we want to express the various quantities in terms of T_0, then we can take

$$p_0^{(1)}=p_g^{(1)}\exp\left(-\int_0^z\frac{dz}{H}\right), \quad (4.6)$$

and derive an expression for H, the usual scale height in the atmosphere, as a function of T_0. Using (4.3) and (4.4) in (4.2), we obtain

$$\left[\frac{g}{R^{(1)}T_g}-\frac{1}{H}\right]p_g^{(1)}\exp\left(-\int_0^z\frac{dz}{H}\right)$$
$$+\left(\frac{L_v}{T_0^2}\frac{dT_0}{dz}+\frac{g}{T_g}\right)\frac{p_g^{(2)}}{R^{(2)}}\exp\left[\frac{L_v}{R^{(2)}T_g}\left(1-\frac{T_g}{T_0}\right)\right]$$
$$=-g\rho_0^{(3)}. \quad (4.7)$$

If we specify $\rho_0^{(3)}(z)$, then (4.7) determines H as a function of T_0.

Of particular interest is the case in which

$$p_0^{(i)}=p_g^{(i)}\exp\left(-\int_0^z\frac{dz}{H}\right),\qquad i=1,2, \quad (4.8)$$

$$\rho_0^{(i)}=\rho_g^{(i)}\frac{H_g}{H}\exp\left(-\int_0^z\frac{dz}{H}\right),\quad i=1,2,3. \quad (4.9)$$

From the equation of state and (4.4) and (4.8), one obtains

$$\frac{T_0}{T_g}=\frac{H}{H_g},\quad T_g=\frac{p_g^{(1)}+p_g^{(2)}}{[R^{(1)}\rho_g^{(1)}+R^{(2)}\rho_g^{(2)}]},\tag{4.10}$$

$$\int_0^z \frac{dz}{H}=\frac{-L_v}{R^{(2)}T_g}\left(1-\frac{T_g}{T_0}\right).\tag{4.11}$$

Substituting (4.10) into (4.11) and differentiating with respect to z, we obtain the z dependence of T_0:

$$T_0/T_g=\exp\left[-\frac{R^{(2)}T_g}{L_v}\frac{z}{H_g}\right],\tag{4.12}$$

consistent with $p_0^{(i)}$ and $\rho_0^{(i)}$ being of the form (4.8) and (4.9). The value of H_g is derived by substituting (4.9) and (4.12) into (4.7):

$$H_g=\frac{1}{g}[p_g^{(1)}+p_g^{(2)}]/[\rho_g^{(1)}+\rho_g^{(2)}+\rho_g^{(3)}].\tag{4.13}$$

This particular background distribution is mentioned because of the similarity with the standard solution of gravity waves in a dry atmosphere with a varying background temperature. Accordingly, it will be used later in calculating representative numerical results.

b. The linearized equations

We first introduce and justify two basic simplifications, namely:

$$v_i\approx u_i,\quad T\approx T_\omega.\tag{4.14}$$

The condition $v_i\approx u_i$ follows from the strong coupling between the droplet gas and the air-vapor mixture via the viscous drag force. From (3.24), the ratio of the inertia to the drag term is approximately given by

$$\omega|u_i|\bigg/\left[\frac{9}{2a_0^2}\frac{\mu}{\rho_w}|v_i-u_i|\right]\approx\omega\frac{|u_i|}{|v_i-u_i|}10^{-3},\tag{4.15}$$

where ω is the characteristic frequency of the time-dependent phenomenon which, for internal gravity waves, is of the order of 10^{-2} sec^{-1}. For the above ratio to be of the order of unity, we must have $v_i\approx u_i$. The equivalent ratio in (3.26) gives

$$\rho_0^{(3)}c_v^{(3)}\omega T_\omega/\Phi^{(3)}\approx\frac{c_v^{(3)}\omega T_\omega}{3\kappa^{(1)}(T-T_\omega)}a_0^2\rho_w,$$

$$\approx 3.5\frac{T_\omega}{T-T_\omega}10^{-3},\tag{4.16}$$

since $\kappa^{(1)}\approx 6.2$ kcal m^{-1} sec^{-1} (°K)$^{-1}$ and $c_v^{(3)}=c_p^{(3)}=1.0$ kcal kg^{-1} (°K)$^{-1}$. Thus, in the low-frequency range one can assume $T=T_\omega$. In conclusion, Eq. (4.14) holds as long as the frequency of the disturbance does not reach into the audible range.

We now proceed to linearize the equations of motion by assuming that the amplitude of the disturbance is infinitesimal compared with the background quantities. If we use the subscript "1" to describe the disturbance, then each variable can be written as the sum of two terms: the background value and a fluctuating one. For example: $a=a_0+a_1$ with $a_1/a_0\approx O(\epsilon)$, if ϵ is the smallness parameter describing the disturbance. Since we assume no background wind, the velocities v_i will contain only a term v_{1i} of order ϵ compared to some typical velocity such as the speed of sound. Similarly, $\Gamma=\Gamma_1$ will only have a fluctuating part.

By equating $O(\epsilon)$ terms the nonlinear system of equations derived in the previous section reduces to the following linear system:

$$\frac{\partial\rho_1^{(2)}}{\partial t}+\rho_0^{(2)}\frac{\partial v_i}{\partial x_i}+v_z\frac{d\rho_0^{(2)}}{dz}=\Gamma_1,\tag{4.17}$$

$$\frac{\partial}{\partial t}\rho_{M1}+\rho_{M0}\frac{\partial v_i}{\partial x_i}+v_z\frac{d\rho_{M0}}{dz}=0,\tag{4.18}$$

$$\rho_{M0}\frac{\partial v_i}{\partial t}+\frac{\partial p_{M1}}{\partial x_i}+\rho_{M1}g\delta_{i3}=0,\tag{4.19}$$

$$\rho_{M0}c_{Mv}\left(\frac{\partial T_1}{\partial t}+v_z\frac{d}{dz}T_0\right)+p_{M0}\frac{\partial v_i}{\partial x_i}=-R^{(2)}T_0\left[\frac{L_v}{R^{(2)}T_0}-1\right]\Gamma_1,\tag{4.20}$$

$$\Gamma_1=[\rho_{\omega 1}-\rho_1^{(2)}]/\tau_m,\quad 1/\tau_m=4\pi n_p a_0 D,\tag{4.21}$$

$$\rho_{\omega 1}=\rho_0^{(2)}\left[\frac{L_v}{R^{(2)}T_0}-1\right]\frac{T_1}{T_0},\tag{4.22}$$

$$\frac{\partial p_{M1}}{\partial t}=R^{(1)}T_0\frac{\partial\rho_{M1}}{\partial t}+[\rho_0^{(1)}R^{(1)}+\rho_0^{(2)}R^{(2)}]\frac{\partial T_1}{\partial t}+T_0R^{(2)}\frac{\partial\rho_1^{(2)}}{\partial t}+R^{(1)}T_0\left\{[\rho_0^{(2)}+\rho_0^{(3)}]\frac{\partial v_i}{\partial x_i}+v_z\frac{d}{dz}[\rho_0^{(2)}+\rho_0^{(3)}]\right\},\tag{4.23}$$

where

$$\left.\begin{aligned}&\rho_{M0}=\sum_1^3\rho_0^{(i)},\quad \rho_{M1}=\sum_1^3\rho_1^{(i)},\quad p_{M0}=\sum_1^2 p_0^{(i)}\\ &p_{M1}=\sum_1^2 p_1^{(i)},\quad c_{Mv}=\frac{1}{\rho_{M0}}\sum_1^3\rho_0^{(i)}c_v^{(i)}\end{aligned}\right\}.\tag{4.24}$$

Eqs. (4.17), (4.21), (4.22) are derived from (3.21), (3.13), (3.9), respectively; (4.18), (4.19), (4.20) are obtained from (3.20) and (3.22)–(3.26) by summing over all three species; and (4.23) is obtained by manipulating the equation of state and using (3.21) and (3.22). For simplicity we have also used v_i for v_{1i}.

In deriving the above set of equations we have neglected the background velocity u_{z0} which in the context of linearization is equivalent to assuming that $u_{z0}/c_z < O(\epsilon^2)$, when c_r is some characteristic velocity. On the other hand, we have retained terms of the order of δ and Δ with

$$\delta=\rho_g^{(2)}/\rho_{Mg},\quad \Delta=\rho_g^{(3)}/\rho_{Mg}, \tag{4.25}$$

which are indicative of the moisture present in the system and which are never larger than 0.03–0.04 in the atmosphere (Tverskoi, 1965). It follows that some of the nonlinear terms which we have neglected would be important if $\epsilon \gg \delta$ and/or $\epsilon \gg \Delta$. Finally, it should be pointed out that in linearizing the Clausius-Clapeyron equation we have assumed that the quantity $[L_v/R^{(2)}T_0]T_1/T_0 \approx 20T_1/T_0$ is much less than unity; this condition could in some circumstances severely limit the maximum value that the ratio T_1/T_0 can reach for the linearization to still hold. In that case, where $\epsilon \approx O(1/20)$, the Clausius-Clapeyron equation would indicate that $T_1 \approx$ constant, and the waves are isotermal waves since any disturbance in temperature is quickly compensated for by heat released in condensation.

Eqs. (4.17)–(4.23) form a set of nine linear equations in nine unknowns: $\rho_1^{(2)}$, v_i, Γ_1, ρ_{M1}, p_{M1}, T_1, $\rho_{\omega 1}$. We can use transform techniques in solving them or equivalently assume that all dependent variables have an $\exp[i(\omega t-k_x x)]$ dependence on t and x [$k_x=2\pi/\lambda_x$ and λ_x are the wavenumber and wavelength in the horizontal direction]. Furthermore, we non-dimensionalize dependent and independent variables as follows:

$$\left.\begin{aligned}
&x_i^*=x_i/H_g,\quad t^*=tc_r/H_g,\quad k_x^*=k_xH_g\\
&\omega^*=\omega H_g/c_r,\quad \rho_0^{(1)*}=\frac{\rho_0^{(1)}}{\rho_{Mg}},\quad \overset{*}{\rho}_{M0}=\rho_{M0}/\rho_{Mg}\\
&\overset{*}{\rho}_{M1}=\rho_{M1}/\rho_{Mg},\quad \delta\rho_0^{(2)*}=\rho_0^{(2)}/\rho_{Mg}\\
&\delta\rho_1^{(2)*}=\rho_1^{(2)}/\rho_{Mg},\quad \delta\overset{*}{\rho}_{\omega 1}=\rho_{\omega 1}/\rho_{Mg}\\
&\Delta\rho_0^{(3)*}=\rho_0^{(3)}/\rho_{Mg},\quad \Delta\rho_1^{(3)*}=\rho_1^{(3)}/\rho_{Mg}\\
&T_0^*=T_0/T_g,\quad T_1^*=T_1/T_g,\\
&\overset{*}{p}_{M0}=p_{M0}/(\rho_{Mg}c_r^2),\quad \overset{*}{p}_{M1}=p_{M1}/(\rho_{Mg}c_r^2)\\
&g^*=gH_g/c_r^2,\quad \tau_m^*=\tau_m c_r/H_g,\quad v_i^*=v_i/c_r\\
&\overset{*}{c}_{Mv}=c_{Mv}/c_{Mvg},\quad L_v^*=L_v/[R^{(2)}T_g]\\
&\gamma^{(1)*}=c_r^2/[R^{(1)}T_g],\quad \gamma^{(2)*}=c_r^2/[R^{(2)}T_g]\\
&\gamma_M^*=c_r^2[\rho_0^{(1)}+\rho_0^{(2)}]/\\
&\qquad\{[\rho_0^{(1)}R^{(1)}+\rho_0^{(2)}R^{(2)}]T_g\}
\end{aligned}\right\}. \tag{4.26}$$

Using the above definitions and eliminating Γ_1, we rewrite (4.17) through (4.23) as follows:

$$i\omega^*\rho_1^{(2)*}+\rho_0^{(2)*}\frac{\partial}{\partial x_i^*}v_i^*+v_z^*\frac{d}{dz^*}\rho_0^{(2)*} =\frac{1}{\tau_m^*}[\overset{*}{\rho}_{\omega 1}-\rho_1^{(2)*}], \tag{4.27}$$

$$i\omega^*\overset{*}{\rho}_{M1}+\overset{*}{\rho}_{M0}\frac{\partial}{\partial x_i^*}v_i^*+v_z^*\frac{d}{dz^*}\overset{*}{\rho}_{M0}=0, \tag{4.28}$$

$$i\omega^*\overset{*}{\rho}_{M0}v_x^*-ik_x^*\overset{*}{p}_{M1}=0, \tag{4.29}$$

$$i\omega^*\overset{*}{\rho}_{M0}v_z^*+\frac{\partial}{\partial z^*}\overset{*}{p}_{M1}+g^*\overset{*}{\rho}_{M1}=0, \tag{4.30}$$

$$\overset{*}{\rho}_{M0}\overset{*}{c}_{Mv}\left(i\omega^*T_1^*+v_z^*\frac{d}{dz^*}T_0^*\right)+\left(\frac{c_r^2}{T_gc_{Mvg}}\right)\overset{*}{p}_{M0}\frac{\partial}{\partial x_i^*}v_i^* =\frac{\delta}{\tau_m^*}\frac{R^{(2)}}{c_{Mvg}}T_0^*\left(1-\frac{L_v^*}{T_0^*}\right)[\overset{*}{\rho}_{\omega 1}-\rho_1^{(2)*}], \tag{4.31}$$

$$\overset{*}{\rho}_{\omega 1}=\rho_0^{(2)*}\left(\frac{L_v^*}{T_0^*}-1\right)(T_1^*/T_0^*), \tag{4.32}$$

$$\overset{*}{p}_{M1}=T_0^*\left[\frac{\overset{*}{\rho}_{M1}}{\gamma^{(1)*}}+\delta\frac{\rho_1^{(2)*}}{\gamma^{(2)*}}+\frac{1}{i\omega^*\gamma^{(1)*}}\left\{[\delta\rho_0^{(2)*}+\Delta\rho_0^{(3)*}]\frac{\partial}{\partial x_i^*}v_i^*+v_z^*\frac{d}{dz^*}[\delta\rho_0^{(2)*}+\Delta\rho_0^{(3)*}]\right\}\right] +\frac{1}{\gamma_M^*}[\rho_0^{(1)*}+\delta\rho_0^{(2)*}]T_1^*. \tag{4.33}$$

The above set of equations forms our basic result; it reduces, in the limit of $g\rightarrow 0$, to the equations of Cole and Dobbins (1971) and of Marble and Wooten (1970) with the difference that our equation of state [(4.33)] contains additional terms, which, being proportional to δ and Δ, are relatively small.

5. Propagation of infinitesimal disturbances in a slowly varying atmosphere

We now derive a WKB solution for the system of linear equations obtained in Section 4. The background temperature is assumed to vary slowly and horizontal variations of the parameters are neglected. The same kind of conceptual difficulties associated with the WKB approximation in application to acoustic-gravity waves in a dry atmosphere as discussed by Hines (1965) and Einaudi and Hines (1970) are present here, so that we assume to be dealing with height ranges where no reflection points are involved.

Eliminating all the variables in (4.27)–(4.33) in favor of v_z^*, one obtains

$$A_1\frac{d^2v_z^*}{dz^{*2}}+A_2\frac{dv_z^*}{dz^*}+A_3v_z^*=0, \tag{5.1}$$

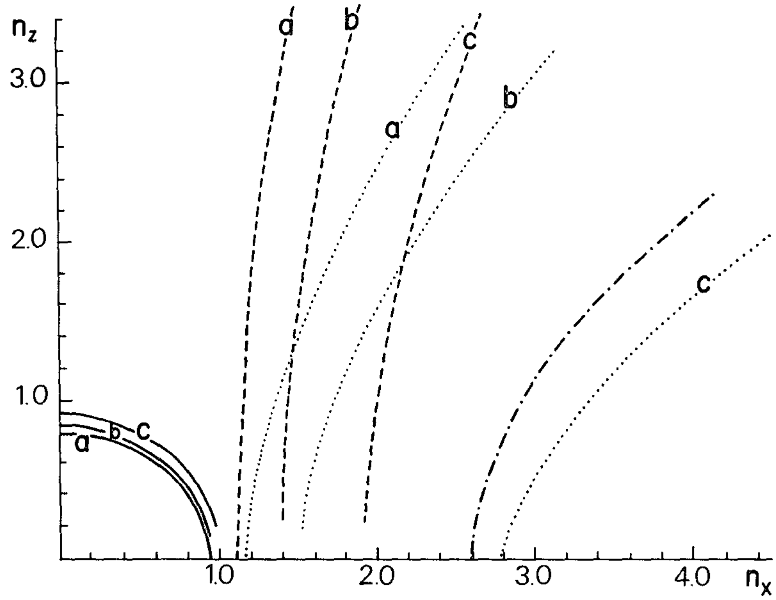

FIG. 1. Phase velocity diagram, i.e., $n_z=q/\omega^*$ vs $n_x=k_x^*/\omega^*$ for real k_x^* and ω^*. Solid lines correspond to 3 min period, dotted lines to 10 min period, and dashed lines to 18 min period. Lines labeled (a) correspond to $\Delta=\delta=0.0$, i.e., dry air, (b) to $\Delta=\delta=0.001$ and (c) to $\Delta=\delta=0.01$. All are for $\tau_m=1$ sec. The dot-dashed line is the $\delta=0.01$ curve for 10 min period using the results of Lalas (1972) and corresponds to the dotted (c) curve.

where A_1, A_2 and A_3 are functions of k_x^*, ω^* and z^* given by (A1), (A2) and (A3) in the Appendix. The above equations may be converted to canonical form

$$\frac{d^2\psi}{dz^{*2}}+q^2\psi=0,\quad q^2=\frac{A_3}{A_1}-\frac{1}{2}\frac{d}{dz^*}\left(\frac{A_2}{A_1}\right)-\frac{1}{4}\left(\frac{A_2}{A_1}\right)^2, \tag{5.2}$$

by the change of variable

$$v_z^*=\psi(z^*)\exp\left[-\int^{z^*}\frac{A_2}{2A_1}dz^*\right]. \tag{5.3}$$

We write the WKB approximate solutions of (5.2) as the pair

$$\psi^{\pm}=\psi_0^{\pm}q^{-\frac{1}{2}}\exp\left(\pm i\int^{z^*}qdz^*\right), \tag{5.4}$$

with $\psi_0^{\pm}$ independent of z^*. Any linear combination of ψ^+ and ψ^- will provide an accurate solution of (5.2) if

$$\left|\frac{1}{2q^3}\frac{d^2q}{dz^{*2}}-\frac{3}{4q^4}\left(\frac{dq}{dz^*}\right)^2\right|\ll 1. \tag{5.5}$$

Simplified expressions for A_1, A_2 and A_3 are derived by using in (A1), (A2) and (A3) the simplified form for $\tilde{F}_1$ through $\tilde{F}_6$ given by (A5); the simplification is obtained by neglecting terms of order δ and Δ unless these quantities are simultaneously multiplied by $\{[L_v/R^{(2)}T_0]-1\}^2$, which is of the order of 400. A detailed justification for this approximation in also given in the Appendix.

The above WKB solution is of limited value if one wants to study the stability of the system, because of the difficulties involved in imposing the boundary conditions. A stability analysis leading to a numerical calculation of damping or growth rates would require the introduction of a multi-layer atmospheric model with the appropriate matching conditions at the boundaries as done by Gossard (1962), Hines (1965), Pearce and White (1967) and others. In particular, one could consider a moist layer of finite depth in contact with unsaturated or even dry air above and the ground below. Useful information can nevertheless be obtained from the WKB solution for an infinite medium concerning the local properties of the medium itself. In particular, the quantity $q(z^*)$ defined by (5.2) can be thought of as the local wavenumber in the z direction and by plotting q as a function of k_x^* and ω^* at different heights one can compare the properties of a saturated atmosphere with those of either a dry or an unsaturated one.

To demonstrate the quantitative effect of condensation we have carried out some numerical calculations of the local vertical wavenumber q for the case in which the imaginary part of ω is zero and for a background atmosphere described by Eqs. (4.8)–(4.13).

The results are shown in Figs. 1 and 2. Fig. 1 is a plot of local phase velocity vs k_x in units of ω/c_0 for various values of δ and Δ, equivalent to Fig. 10 of Hines (1960). The local vertical wavelength for given real k_x and ω increases and the gap of prohibited frequencies for which there is no propagation also increases. The latter result should be taken cautiously since in and near the frequency gap the validity of the WKB approximation is in doubt. Fig. 2 demonstrates the effect of the char-

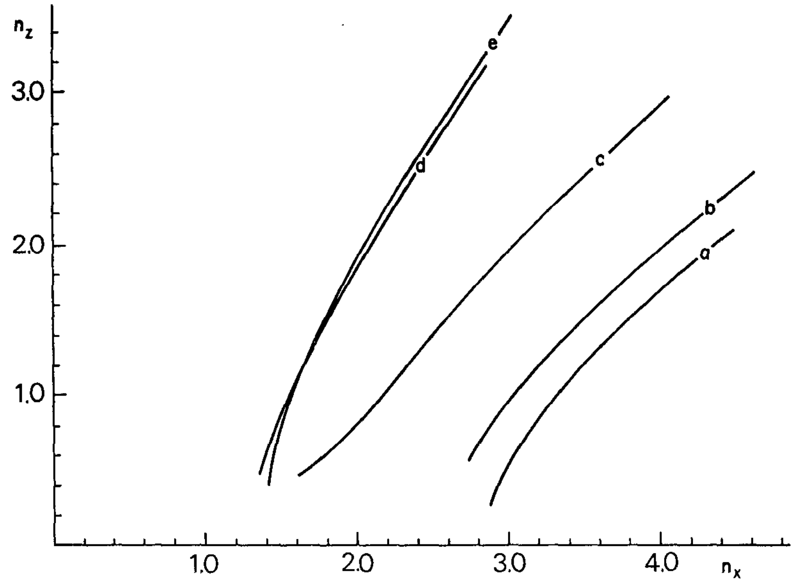

FIG. 2. Effect of diffusion time τ_m on the phase plot, n_z vs n_x, of a wave with real k_x^* and real ω^* corresponding to a 10 min period, with (a) $\tau_m=1.0$, (b) $\tau_m=10.0$, (c) $\tau_m=10^2$, (d) $\tau_m=10^3$ and (e) $\tau_m=10^8$: $\delta=0.01$ and Δ is the appropriate one for each value of τ_m.

acteristic mass diffusion time on the local vertical wavelength for a given real period of 10 min. For large mass diffusion times, there is no mass exchange occurring so the difference between curves (a) and (d) is due exclusively to condensation or evaporation.

The numerical calculations also show that the attenuation due to phase change, per local vertical wavelength, is negligible, in the internal gravity wave range. For example, with the same q and real ω, the difference between the cases of high condensation (a) and no condensation (d) is less than 2%. One should notice that in both figures the curves that correspond to condensation are not continued to the k_x axis because the WKB solution breaks down for small values of q.

6. Conclusions

The set of nonlinear equations governing the propagation of gravity waves in a stratified atmosphere with saturated water vapor has been derived. The water droplets are treated as a fluid so that the system is equivalent to a three-fluid mixture whose properties are described by equations resulting from basic postulates and a minimum of ad hoc assumptions and constitutive relations. Since the basic simplifying assumption that velocities and temperatures are the same for the three fluids is invoked, our treatment is limited to frequencies below the audible range. The equations are then linearized about a background without wind.

An explicit solution is obtained in the WKB form which allows us to compare the local properties of a saturated atmosphere with a dry one.

The gravity waves are found to be slightly attenuated because of condensation. Large changes are seen in their phase velocity. In particular, for a given frequency and horizontal wavelength, the effect of condensation is to increase the vertical wavelength. At the same time the range of prohibited frequencies is enlarged, so that gravity waves will exhibit wave ducting when propagating in a three-layer (moist unsaturated, moist saturated, dry) atmosphere. The latter result deserves further analysis since in and near the gap the validity of the WKB approximation is in doubt. Specific values are calculated for a particular moisture distribution to demonstrate quantitatively the changes in gravity wave propagation characteristics created by condensation.

It should be pointed out that the present treatment does *not* include propagation through an atmosphere near but not at saturation; many of the equations remain the same but some changes are needed which are far from obvious. The subject is under investigation. Finally, we anticipate here the result of a companion paper: if shear is introduced in the background atmosphere, condensation tends to make the atmosphere less stable. The proof follows closely the stability analysis of a dry atmosphere given by Chimonas (1970).

APPENDIX

Auxiliary Calculations

The coefficients of (5.1) are given by

$$A_1=[-(iF_8\overset{*}{\rho}_{M0}/\omega^*)-F_9 + ik_x^*F_{12}(F_9+iF_8\overset{*}{\rho}_{M0}/\omega^*)/F_{11}]/F_7, \quad \text{(A1)}$$

$$A_2=\frac{d}{dz^*}A_1+\left[-i\frac{F_8}{\omega^*}\frac{d}{dz^*}\overset{*}{\rho}_{M0}-F_{10}+ik_x^*(F_9+iF_8\overset{*}{\rho}_{M0}/\omega^*)F_{13}/F_{11}\right]\Big/F_7+\frac{\overset{*}{\rho}_{M0}}{\omega^*g^*}(i+k_x^*F_{12}/F_{11}),\qquad \text{(A2)}$$

$$A_3=i\omega^*\overset{*}{\rho}_{M0}+\frac{1}{\omega^*g^*}\left(i\frac{d}{dz^*}\overset{*}{\rho}_{M0}+k_x^*\overset{*}{\rho}_{M0}F_{13}/F_{11}\right)+\frac{d}{dz^*}\left\{\left[-i\frac{F_8}{\omega^*}\frac{d\overset{*}{\rho}_{M0}}{dz^*}-F_{10}+ik_x^*(F_9+iF_8\overset{*}{\rho}_{M0}/\omega^*)F_{13}/F_{11}\right]\Big/F_7\right\},\qquad \text{(A3)}$$

where:

$$\left.\begin{aligned}
F_1&=\frac{\overset{*}{\rho}_{M0}}{\gamma_M^*}+\delta\rho_0^{(2)*}b(L_v^*/T_0^*-1)/\gamma^{(2)*},\\
&\qquad b=1/(1+i\omega^*\tau_m^*)\\
F_2&=T_0^*\{-[\delta\rho_0^{(2)*}+\Delta\rho_0^{(3)*}]/[i\omega^*\gamma^{(1)*}]+\delta\rho_0^{(2)*}\tau_m^*b/\gamma^{(2)*}\}\\
F_3&=T_0^*\left\{-[1/i\omega^*\gamma^{(1)*}]\frac{d}{dz^*}[\delta\rho_0^{(2)*}+\Delta\rho_0^{(3)*}]+\frac{d\rho_0^{(2)*}}{dz^*}[\delta b\tau_m^*/\gamma^{(2)*}]\right\}\\
F_4&=i\omega^*[\overset{*}{\rho}_{M0}\overset{*}{c}_{Mv}+\delta\rho_0^{(2)*}bR^{(2)}(L_v^*/T_0^*-1)^2/c_{Mvg}]\\
F_5&=\overset{*}{\rho}_{M0}\overset{*}{c}_{Mv}\frac{d}{dz^*}T_0^*+\delta bT_0^*\frac{R^{(2)}}{c_{Mvg}}\left(\frac{L_v^*}{T_0^*}-1\right)\frac{d\rho_0^{(2)*}}{dz^*}\\
F_6&=T_0^*\{R^{(1)}[\overset{*}{\rho}_{M0}-\delta\rho_0^{(2)*}-\Delta\rho_0^{(3)*}]/c_{Mvg}+\delta\rho_0^{(2)*}R^{(2)}/c_{Mvg}\}+\delta b\rho_0^{(2)*}(L_v^*/T_0^*-1)T_0^*R^{(2)}/c_{Mvg}\\
F_7&=F_4/F_1,\quad F_8=-F_7T_0^*/\gamma^{(1)*}\\
F_9&=F_6+F_2F_7,\quad F_{10}=F_5+F_3F_7\\
F_{11}&=\omega^*\overset{*}{\rho}_{M0}+k_x^{*2}(\overset{*}{\rho}_{M0}F_8/\omega^*-iF_9)/F_7\\
F_{12}&=-k_x^*(F_9+i\overset{*}{\rho}_{M0}F_8/\omega^*)/F_7\\
F_{13}&=-k_x^*\left(F_{10}+i\frac{F_8}{\omega^*}\frac{d}{dz^*}\overset{*}{\rho}_{M0}\right)\Big/F_7
\end{aligned}\right\}.\qquad \text{(A4)}$$

The simplified form of A_1, A_2 and A_3 discussed in Section 5 is obtained by using the following simplified form for the functions F_1–F_6:

$$\left.\begin{aligned}
&\tilde{F}_1\approx\overset{*}{\rho}_{M0}/\gamma_M^*,\quad \tilde{F}_2=\tilde{F}_3\approx 0,\quad \tilde{F}_4=F_4\\
&\tilde{F}_5=\overset{*}{\rho}_{M0}\overset{*}{c}_{Mv}\frac{d}{dz^*}T_0^*,\quad \tilde{F}_6=T_0^*R^{(1)}\overset{*}{\rho}_{M0}/c_{Mvg}
\end{aligned}\right\}.\qquad \text{(A5)}$$

The justification of the above approximation is based on the fact that δ and Δ are always less than 0.03–0.04 and more usually of the order 0.001. In some terms of the energy equation, however, δ and Δ are simultaneously multiplied by parameters which are in some sense large. Let us rewrite the energy equation (4.20), expressing Γ_1 in terms of the temperature and velocities and setting $\rho_{M0}\approx\rho_0^{(1)}$, $c_{Mv}\approx c_v^{(1)}$, $p_{M0}\approx p_0^{(1)}$ in its left-hand side:

$$\rho_0^{(1)}c_v^{(1)}\left[i\omega T_1+v_z\frac{d}{dz}T_0\right]+p_0^{(1)}\frac{\partial v_i}{\partial x_i}=-\frac{R^{(2)}T_0}{1+i\omega\tau_m}\left[\frac{L_v}{R^{(2)}T_0}-1\right]\left\{i\omega\rho_0^{(2)}\left[\frac{L_v}{R^{(2)}T_0}-1\right]\frac{T_1}{T_0}+v_z\frac{d\rho_0^{(2)}}{dz}-\frac{\rho_0^{(2)}}{\rho_0^{(1)}}\left[v_z\frac{d}{dz}\rho_0^{(1)}+i\omega\rho_1^{(1)}\right]\right\}.\qquad \text{(A6)}$$

The ratio of the first terms of both sides of (A6) reads

$$\frac{\rho_0^{(2)}}{\rho_0^{(1)}}\frac{R^{(2)}}{c_v^{(1)}}\left|\frac{1}{1+i\omega\tau_m}\right|\left|\left[\frac{L_v}{R^{(2)}T_0}-1\right]^2\right|\approx 400\frac{\rho_0^{(2)}}{\rho_0^{(1)}},$$

so that for $\rho_0^{(2)}/\rho_0^{(1)}\approx 10^{-3}$ such a ratio is of the order 0.4 and cannot be neglected.

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