# Cumulus Parameterization and CISK[1]

M. ISRAELI[2] AND E. S. SARACHIK

*Dept. of Meteorology, Massachusetts Institute of Technology, Cambridge 02139*

ABSTRACT

Arakawa's recent parameterization of the effects of a cumulus ensemble on the large-scale environment is applied to the problem of conditional instability of the second kind (CISK). In particular, Charney's linear, two-level, line-symmetry CISK model of the ITCZ is re-examined using a simplified non-entraining cloud version of the Arakawa scheme. It is found that the growth rate is maximum, in fact infinite, at some reasonable mesoscale rather than at cumulus scale as is characteristic of Charney's solution. A more accurate semi-analytic model of CISK is considered and it is found that a separable, line-symmetric CISK solution is always possible under very general conditions. In both the two-level and semi-analytic models of CISK, it is proved that a necessary condition for the existence of a growing solution is that the mass flux into the clouds exceeds the Ekman pumping out of the boundary layer, or equivalently, that the air between the clouds must subside and therefore heat the environment by adiabatic compression.

## 1. Introduction

Since it is generally believed that tropical disturbances are driven by the heat released by cumulonimbus convection, the understanding of the interaction between the cumulus scale and the disturbance scale has been of particular concern to meteorologists. An important first step was taken by Charney and Eliassen (1964), Ogura (1964) and Ooyama (1964), who noted that the heat released in the upward motion (caused by large-scale cyclonic vorticity through frictional convergence in the Ekman layer) in a conditionally unstable, moist tropical atmosphere can in turn organize and intensify the large-scale motion; the cooperative growth that results is called conditional instability of the second kind (CISK). The parameterization of this heat release, which Charney (1971) took to be proportional to the latent heat released by the condensation of moisture in the upward motion, is the critical link in this CISK process. Modified forms of Charney and Eliassen's parameterization have been widely used in numerical models of tropical disturbances. [See Bates (1972) for a review of cumulus parameterization schemes.] Recently, however, Arakawa (1971a) has pointed out that the mechanism of heating by clouds is more subtle than previously believed [see also Gray (1970)]. The environment is not heated directly by the release of latent heat, but rather by the subsidence that necessarily accompanies the large upward mass fluxes in the clouds. The role of condensational heating is simply the maintenance of the upward mass flux which then induces the large-scale environmental subsidence. This upward mass flux in the clouds then becomes the key quantity in cumulus parameterization, and a method for its determination was given by Arakawa (1971b) and Arakawa and Schubert (1973).

It is the purpose of this paper to explore this new method of cumulus parameterization in the old problem of CISK. We will explore, in particular, Charney's (1971) perturbation model of the ITCZ as a linear zonally-symmetric CISK process and show that this new method of cumulus parameterization gives reasonable results with a minimum of *ad hoc* assumptions.

We believe that recent results from multi-level linear CISK models of Hayashi (1971) and Yamasaki (1969, 1971) are to a large extent invalidated by the non-physical linearity assumption, which makes their computational approach feasible but predicts diabatic cooling (negative convective activity) in regions where there is descending motion at the top of the boundary layer.

The plan of the paper is as follows. The second section summarizes the Arakawa scheme, presents some simplifications, and derives necessary conditions for CISK growth. Two-level models of CISK will be treated in the third section and it is shown that the Arakawa scheme cures some old diseases in linear two-level CISK models. The fourth section introduces a semi-analytic model (equations of motion treated exactly but with an integrated heat equation), and it is shown that separable CISK solutions are always possible. Section 5 summarizes our results.

[1] A version of this paper was reported at the GATE Workshop, 10–14 July 1972, held at the National Center for Atmospheric Research, Boulder, Colo.

[2] Present affiliation: Department of Mathematics, Massachusetts Institute of Technology.

## 2. Arakawa's parameterization of cumulus convection

We consider a conditionally unstable moist tropical atmosphere whose moist static energy $\bar{h}(z)=\bar{s}(z)+L\bar{q}(z)$ is as in Yanai (1971b), where $\bar{s}(z)$ is the dry static energy, $C_p\bar{T}+gz$. The action of cumulonimbus convection is to change the environment according to the equations (Yanai, 1971b; Arakawa, 1971b; Ooyama, 1971):

$$\bar{\rho}\frac{\partial\bar{s}}{\partial t}=-LDl_D+(M_c-\bar{\rho}\bar{w})\frac{\partial\bar{s}}{\partial z}-\bar{\rho}\bar{\mathbf{v}}\cdot\nabla\bar{s}+Q_R, \quad (1a)$$

$$\bar{\rho}\frac{\partial\bar{q}}{\partial t}=D(\overline{q^*}-\bar{q}+l_D)+(M_c-\bar{\rho}\bar{w})\frac{\partial\bar{q}}{\partial z}-\bar{\rho}\bar{\mathbf{v}}\cdot\nabla\bar{q}, \quad (1b)$$

where $M_c(x,y,z)$ is the total upward mass flux in the clouds, $(\bar{\mathbf{v}},\bar{w})$ the horizontally averaged velocity, $D(x,y,z)$ the detrained mass flux from the clouds, $l_D$ is the *liquid* water present at each level, the asterisk refers to the saturation value of the moisture, and the overbar to horizontal averages over an area large compared to cumulus scale but small compared to the environmental scale including both cloud and non-cloud areas. $(M_c-\bar{\rho}\bar{w})$ is the net environmental (non-cloud) mass flux $-\tilde{M}$. We see by the second term of (1a) that when there is net environmental subsidence, $\tilde{M}$ is negative and we obtain heating. The third term of (1a) gives heating by horizontal advection and the first is cooling by re-evaporation of detrained liquid water. [The radiative term in (1a), $Q_R$, will henceforth be ignored.] A similar interpretation can be given for the terms of (1b). We see, therefore, that the effect of the cumulus on the environment is conveniently parameterized in terms of three functions, $M_c$, $D$, $l_D$. Let us simply note that there are assumptions involved in the derivations of (1a) and (1b), the crucial one being that the fractional area covered by active cumulus convection, $\sigma_c$, be small compared to unity. This allows the identification of $\bar{q}$ and $\bar{s}$ as the environmental moisture and static energy.

Arakawa (1971b) and Arakawa and Schubert (1973) proceed to determine the mass flux function $M_c(z)$ by defining an ensemble consisting of clouds with constant fractional entrainment parameters $\lambda$:

$$M_c(z)=\int_0^{\lambda_D(z)} m_B(\lambda)\eta(z,\lambda)d\lambda, \quad (2)$$

where $\lambda_D(z)$ is the entrainment parameter of those clouds detraining at level $z$, $m_B(\lambda)$ is the mass flux at the cloud base $z_B$, and $\eta(z,\lambda)=\exp[\lambda(z-z_B)]$ below the detrainment level $z_D(\lambda)$, and zero above it. In general, clouds with a large entrainment parameter detrain lower down because the saturated cloud air becomes progressively more diluted with unsaturated environment, thereby losing its buoyancy sooner. This generally implies that above a certain level $M_c(z)$ is a decreasing function of height since it has contributions only from those clouds which have not yet detrained at level $z$.

The moist static energy in those clouds of entrainment parameter $\lambda$ is then given by

$$\eta(z,\lambda)h_c(z,\lambda)=h_B+\lambda\int_{z_B}^{z}\eta(z,\lambda)\bar{h}(z)dz, \quad (3)$$

where $h_B$ is the specific moist energy at cloud base and is a function of horizontal coordinates and time.

The buoyancy of a cloud sub-ensemble characterized by $\lambda$ is proportional to $s_c-\bar{s}$ which, assuming saturation in the cloud and small temperature differences between the cloud and environment, can be Taylor-expanded to yield

$$s_c-\bar{s}=\frac{1}{1+\gamma}(h_c-\overline{h^*}),\quad \gamma=\frac{L}{C_p}\left(\frac{\partial\overline{q^*}}{\partial T}\right)_{\bar{p}}. \quad (4)$$

Arakawa's quasi-equilibrium assumption, which is supposed to be valid for all $\lambda$, is

$$\frac{d}{dt}\int_{z_B}^{z_D(\lambda)}\eta(z,\lambda)[(1+\gamma)\bar{T}]^{-1}[h_c(z,\lambda)-\overline{h^*}(z)]dz=0. \quad (5a)$$

This fundamental equation basically states that the clouds and environment mutually adjust to each other in such a way as to maintain the net buoyancy in a cloud ensemble at a constant level. The importance of (5a) is that $\partial h_c/\partial t$ and $\partial\bar{h}^*/\partial t$ contain $m_B(\lambda)$ implicitly through Eq. (1) so that $M_c(z)$ can be obtained from this condition. $D(z)$ can also be obtained from $m_B(\lambda)$ once it is known. We see that the scheme is complete if $l_D(z)$ is also properly parameterized, i.e., the environment determines an $M_c(z)$ which in turn drives the environment.

A necessary condition can be derived by evaluating (5a) at $\lambda=0$:

$$\int_{z_B}^{z_D}[(1+\gamma)\bar{T}]^{-1}\frac{\partial}{\partial t}[h_B-\overline{h^*}(z)]dz=0, \quad (5b)$$

where $z_D$ is the maximum detrainment level. Evaluating the integral at $\lambda=0$ is *not* equivalent to assuming the clouds do not entrain. The mass flux implicit in (5b) is the fully entraining and detraining mass flux.

If we further linearize the problem by considering small deviations from a resting, conditionally unstable, steady tropical atmosphere stratified only in the vertical, $\hat{h}(z)$, $\hat{h}^*(z)$, $\hat{s}(z)$, and assume all cloud liquid water is rained out immediately, we can write the heat equation (1a) in the form

$$\frac{\partial\bar{s}'}{\partial t}+\left[\bar{w}'-\frac{1}{\hat{\rho}}M_c(z)\right]\frac{\partial\hat{s}}{\partial z}=0, \quad (6)$$

where the primes denote the perturbation quantities. In this approximation, and under the assumption that the subcloud layer is saturated and that $\partial z_B/\partial t=0$, $\partial\overline{h^{*\prime}}/\partial t=(1+\gamma)(\partial\bar{s}'/\partial t)$ and $\partial h_B/\partial t=(1+\gamma_B)\partial s_B'/\partial t)$, the Arakawa condition (5) becomes

$$\frac{\partial}{\partial t}\int_{z_B}^{z_D}\hat{T}^{-1}[\hat{\delta}(z)\bar{s}_B'-\bar{s}']dz=0,$$

where

$$\hat{\delta}(z)=[1+\gamma(z_B)][1+\gamma(z)]^{-1}.$$

It follows that

$$\int_{z_B}^{z_D}\hat{T}^{-1}\frac{\partial\bar{s}'}{\partial t}dz=\frac{\partial\bar{s}_B'}{\partial t}\int_{z_B}^{z_D}\hat{T}^{-1}\hat{\delta}dz. \quad (7)$$

Thus, a necessary and sufficient condition for the perturbation static energy in a column to grow is that the static energy in the mixed layer grow. Under the assumption that the mixed layer responds to the environment in the simplest way possible,[3] i.e., that

$$\frac{\partial\bar{s}_B'}{\partial t}=\frac{\partial\bar{s}'}{\partial t}\bigg|_{z=z_B},$$

(7) can be rewritten, using (6) at $z=z_B$, as

$$\int_{z_B}^{z_D}\hat{T}^{-1}\frac{\partial\bar{s}'}{\partial t}dz=\left(\frac{M_B}{\hat{\rho}_B}-\bar{w}_B'\right)\frac{\partial\hat{s}(z_B)}{\partial z}\int_{z_B}^{z_D}\hat{T}^{-1}\hat{\delta}dz, \quad (7a)$$

and we see that a necessary and sufficient condition for growth of the perturbation is that the mass flux into the cloud base *exceeds* the Ekman pumping. It is to be emphasized that (7a) holds for the full entraining and detraining mass flux $M_c(z)$, where $M_B\equiv M_c(z_B)$.

The left side of Eq. (7a) can also be evaluated by using (6) yielding the integral equation for $M_c(z)$:

$$\int_{z_B}^{z_D}\hat{T}^{-1}\left[\frac{M_c(z)}{\hat{\rho}}-\bar{w}'\right]\frac{\partial\hat{s}}{\partial z}dz$$
$$=\left(\frac{M_B}{\hat{\rho}_B}-\bar{w}_B'\right)\frac{\partial\hat{s}(z_B)}{\partial z}\int_{z_B}^{z_D}\hat{T}^{-1}\hat{\delta}dz. \quad (7b)$$

In practice, solution of even the simplified integral equation (7b) is very complicated and can only be accomplished numerically. But in order to gain some insight into the method, we will use a simplified perturbation of model of *non-entraining* ($\lambda=0$) clouds. We assume the clouds form at the top of the mixed layer, $z_B$, which will be assumed to be saturated. Since there is no entrainment, $M_c(z)\equiv M_B$ will be a constant with height (but will still depend on the horizontal coordinate and the time). The clouds, starting with specific moist static energy $h_B$, rise with that constant $h_B$ and cool moist-adiabatically until they lose their buoyancy at $z_D$ so that $h_B=\bar{h}^*(z_D)$. We will assume that all the liquid water rains out upon ascent and therefore none is detrained at $z_D$—this is equivalent to ignoring the cooling effect of re-evaporation of liquid water.

[3] This is equivalent to the assumption that $\bar{h}_B=\bar{h}'(z_B)$, i.e., that the clouds have and maintain zero buoyancy at cloud base.

Now Eq. (7b) can be solved algebraically to yield:

$$M_B=\frac{\displaystyle\int_{z_B}^{z_D}\hat{T}^{-1}(\bar{w}'-\hat{\delta}\bar{w}_B')dz}{\displaystyle\int_{z_B}^{z_D}\hat{T}^{-1}(\hat{\rho}^{-1}-\hat{\delta}\hat{\rho}_B^{-1})dz}, \quad (8)$$

where we have assumed $\partial\hat{s}/\partial z=c_p(\partial\hat{T}/\partial z)+g$ is a constant. $M_B>0$ is the condition for cumulus convection in a region: if the integrals turn out negative, $M_B$ is defined to be zero and there is no convection at that point. The value of $M_B$ calculated from (8) can be inserted back into (6) to give an equation that no longer explicitly refers to the clouds; this, indeed, was the purpose of cumulus parameterization in terms of the large-scale motions. Note finally that $M_B$ is a first-order perturbation quantity. There is cumulus convection always going on to maintain the mean state, $\hat{h}(z)$ and $\hat{h}^*(z)$, of the atmosphere (Yanai, 1971b), but it is *not* included in the $M_B$ given by (8).

## 3. Two-level models of CISK

A recent work of Charney (1971) considered the intertropical convergence zone (ITCZ) as a zonally symmetric linear disturbance driven by the latent heat of condensation released by the Ekman pumping produced by frictional convergence of moisture in the boundary layer. His heating equation was taken to be

$$g\frac{\partial}{\partial t}\ln(\bar{\theta}+\theta')+\hat{N}^2(\bar{w}'-F\eta\bar{w}_B')=0, \quad (9)$$

where $F$ is a vertical form function which is assumed constant and equal to unity in regions of clouds (cycylonic vorticity) and 0 otherwise; and

$$\eta=(Lqs/c_pT)(\partial\ln\bar{\theta}/\partial z)^{-1}$$

has the numerical value 2.14. Assuming an isothermal atmosphere, and searching for an exponential growth $e^{\sigma t}$, Charney found the condition for an ITCZ to exist, of width $2a$ (i.e., rising motion for $y<|a|$, sinking for $y>|a|$) to be[4] $\eta\geqslant 2$. The resulting $a$ vs $\sigma$ curve is shown in Fig. 1. Note that the maximum growth rate is at

[4] The correct condition (5.27) in Charney (1971) should read

$$\tan\frac{a}{\lambda_+}=\frac{\lambda_+}{\lambda_-},$$

where

$$\lambda_+=[(\eta-2-\sigma)/(2+2\sigma)]^{\frac{1}{2}}\lambda,\quad \lambda_-=[(2+\sigma)/(2+2\sigma)]^{\frac{1}{2}}\lambda.$$

Here $\lambda$ is the nondimensional radius of deformation and should not be confused with the entrainment parameter.

zero scale and does *not* provide evidence that the CISK process allows the cumulus to organize motion on the larger ITCZ scale of a few hundred kilometers.

The heating equation (6) can be rewritten to facilitate comparison with (9):

$$g\frac{\partial}{\partial t}\ln(\bar{\theta}+\theta')+\hat{N}^2\left(\bar{w}'-\frac{M_B}{\hat{\rho}}\right)=0. \qquad (10)$$

If $M_B$ is constant and simply chosen to be given by the Ekman pumping, and (10) is evaluated at mid-level in the atmosphere as it is in Charney's model, the expression (10) becomes identical to (9) but with $\eta=\rho_B/\rho_{\frac{3}{2}}\leqslant 2$ which is *not* sufficient for growth of the ITCZ in this model. We saw that $M_B>\hat{\rho}_B\bar{w}_B'$ is a *necessary* condition for an ITCZ, so that Ekman pumping by itself does not provide enough mass flux to give enough subsidence heating to give an ITCZ when we use the "correct" heat equation (10).

If, however, we use the Arakawa condition (8) to evaluate $M_B$, we *can* get an ITCZ and one that has maximum growth rate at some scale *larger* then the cumulus scale. If, consistent with the two-level model, we evaluate $M_B$ as given by (8) by the trapezoid rule, we obtain

$$M_B=\hat{\rho}_{\frac{3}{2}}(\alpha\bar{w}_B'+\beta\bar{w}_{\frac{3}{2}}'),$$

where the subscript $\frac{3}{2}$ denotes the mid-level by weight of the atmosphere, and $\alpha$, $\beta$ are computed from the integrals in (8). The calculation proceeds as in Charney (1971) and the final relation between $a$ and $\sigma$ is

$$a=\lambda_+ \arctan(\lambda_+/\lambda_-),$$

where

$$\left.\begin{aligned}\lambda_-&=[(2+\sigma)/(2+2\sigma)]^{\frac{1}{2}}\lambda\\ \lambda_+&=\left\{\left[\left(\frac{\beta}{2}-1\right)(\sigma+2)+\frac{\alpha}{2}\right]\Big/(2+2\sigma)\right\}^{\frac{1}{2}}\lambda\end{aligned}\right\}. \qquad (11)$$

We see from (11) that at $\sigma=\infty$, if $a$ is small,

$$\frac{a}{\lambda}\approx 2^{-\frac{1}{2}}\left(\frac{\beta}{2}-1\right),$$

and there is a well-defined preferred scale for CISK in this model. Since $\sigma=\infty$ clearly cannot be realized, the actual preferred CISK scale will be chosen at some $\sigma_{\max}$ for which the model ceases to be valid.

To be more explicit, a calculation using this two-level model and the reasonable parameterization

$$\hat{\delta}=\begin{cases}3.2(3p+0.2)^{-1}, & p>0.2\\ 4, & p<0.2\end{cases}$$

(all $p$'s are here normalized to $p_B=900$ mb) yields the value $a=0.11\lambda$ at $\sigma=\infty$ for a detrainment level $p_D=140$ mb. Since the Rossby radius of deformation is $\sim$2500 km in the tropics, this gives a preferred scale of growth of about 250 km. This result is very sensitive to choice of the detrainment level $p_D$ so it cannot be considered a calculation of the ITCZ; it indicates, however, that a reasonable solution for the ITCZ does exist when the heating is given by the "correct" heat equation (1) and the cloud mass flux $M_B$ is given by Arakawa's quasi-equilibrium condition.

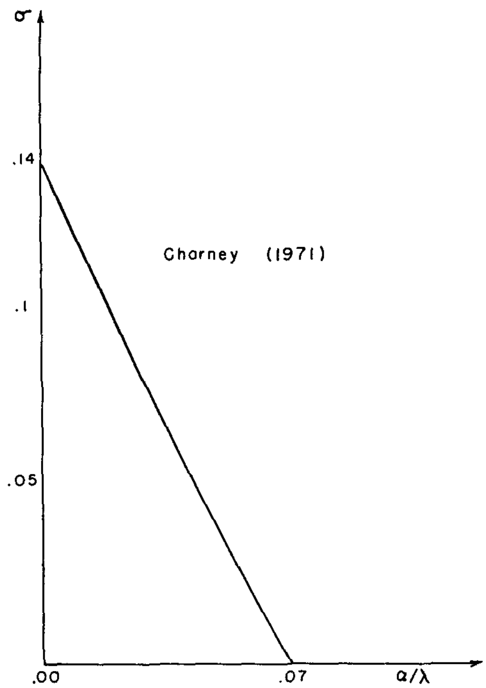


FIG. 1. Growth rate vs width of ITCZ for Charney's (1971) two-level model of CISK.

A similar calculation for circular geometry, analagous to Charney and Eliassen (1964), but using heating at only the middle level, yields the condition

$$\frac{J_1(a/\lambda_+)}{J_0(a/\lambda_+)}=\frac{\lambda_+}{\lambda_-}\frac{K_1(a/\lambda_-)}{K_0(a/\lambda_-)},$$

where the $J$'s and $K$'s are the usual and modified Bessel functions and the $\lambda_\pm$ are the same as those for the linear geometry. Using the same detrainment level $p_D=140$ mb, we find $\dot{a}=0.34\lambda$ at $\sigma=\infty$.

It should be pointed out that while the curves for the two-level model look like Fig. 2, the growth rate at $a=0$ may already be so large that the model is invalid. A recent work of Charney (1973), however, indicates that the Ekman pumping at these scales is really quite inefficient (in fact, zero at zero scale) so that a more realistic treatment of the Ekman layer would have the growth rate curve start out at $\sigma=0$ for $a=0$.

## 4. Semi-analytic model of CISK

Since the simple two-level model presented in the last section was very sensitive to the position of the detrainment level, we consider a model that eliminates the following sources of error:

1. Discretization error in the continuity equation.
2. Discretization error in the heat equation.
3. Discretization error of the integrals in the evaluation of $M_B$.
4. Evaluation of the boundary condition at 750 mb instead of at the "top of the Ekman layer."

This is achieved by parameterizing the vertical velocity, and allows us to draw some conclusions about the possible effect of increased vertical resolution and the release of heat at different elevations.

Again consider zonally symmetric line perturbations, anti-symmetric about $y=0$, in an isothermal atmosphere (Charney, 1971). The scaled perturbation equations of motion are

$$\left.\begin{aligned}&\bar{u}=-\frac{\partial\bar{\varphi}}{\partial y},\quad \bar{v}=\frac{\partial\bar{u}}{\partial t}\\ &\frac{\partial\bar{v}}{\partial y}+\frac{1}{\rho}\frac{\partial}{\partial z}(\rho\bar{w})=0\end{aligned}\right\}, \tag{12}$$

and the heat equation is, in our non-entraining cloud model,

$$\frac{\partial^2\bar{\varphi}}{\partial t\partial z}+\lambda^2\left(\bar{w}-\frac{M_B}{\rho}\right)=0, \tag{13}$$

where $\varphi$ is the scaled pressure and $\lambda$ the nondimensional Rossby radius of deformation. Primes and carets have been dropped.

We choose as our parameterization

$$\bar{w}=A_y p^{m_1}+B_y p^{m_2}, \tag{14}$$

where $A$ and $B$ are functions of $y$ only, $p$ is defined as the ratio of pressure to cloud base pressure, and the subscript $y$ denotes the partial derivative. Since we require no mass flux out of the atmosphere, we must demand $pw\rightarrow 0$ as $p\rightarrow 0$ so that both $m_1$ and $m_2$ must be greater than $-1$.

The meridional velocity can be found by integrating the continuity equation in (12) subject to the anti-symmetry condition at $y=0$, that $v\rightarrow 0$ as $y\rightarrow\pm\infty$ and that $v(0,p)=0$ for all $p$. This yields

$$\bar{v}=(m_1+1)Ap^{m_1}+(m_2+1)Bp^{m_2}. \tag{15}$$

We take the cloud base to be at the top of the Ekman layer where the geostrophic Ekman pumping is assumed to be proportional to the vorticity:

$$\bar{w}_B=e\frac{\partial^2\bar{\varphi}}{\partial y^2}\quad\text{at}\quad p=p_B, \tag{16}$$

where $e$, the proportionality constant, is $2^{-\frac{1}{2}}D_E$ in the classical Ekman sense, but is much smaller in general.[5] Assuming we look for a solution that grows as $e^{\sigma t}$ everywhere, the condition (16) can be expressed as

$$\frac{\partial\bar{v}}{\partial y}=-\frac{\sigma}{e}\bar{w}_B\quad\text{at}\quad p=p_B. \tag{17}$$

This now provides a relation between (14) and (15) which can be expressed as

$$B_y=-A_y g_1 g_2^{-1},\quad\text{where}\quad g_i=p_B^{m_i}\left(\frac{\sigma}{e}+1+m_i\right).$$

If we now choose $A_y$ to be positive in the rising cloud region, $\bar{w}=A_y[p^{m_1}-(g_1/g_2)p^{m_2}]$ can be seen to be positive if $m_2>m_1$.

If we differentiate the heat equation (13) with respect to $y$, change to $p$ coordinates (recalling that for an isothermal atmosphere, $p=\rho gH$),

$$\frac{\partial\bar{v}}{\partial p}+\lambda^2\left(\frac{\bar{w}_y}{p}-\frac{gHM_{By}}{p^2}\right)=0,$$

and integrate from $p_B$ to $p_D$, we have

$$\bar{v}(p_B)-\bar{v}(p_D)+\lambda^2\left[\int_{p_B}^{p_D}\frac{\bar{w}_y}{p}dp-gHM_{By}\int_{p_B}^{p_D}\frac{dp}{p^2}\right], \tag{18}$$

where $M_{By}$ was taken out of the integral because we are using the non-entraining model of $M_B$ given by (8).

Let us *define* an $\eta$ to satisfy the relation

$$\int_{p_B}^{p_D}\left[\frac{\bar{w}_y}{p}-\frac{gHM_{By}}{p^2}\right]dp=(1-\eta)\int_{p_B}^{p_D}\frac{\bar{w}_y}{p}dp, \tag{19}$$

so that the heat equation reads

$$\bar{v}(p_D)-\bar{v}(p_B)+\lambda^2(1-\eta)\int_{p_B}^{p_D}\frac{\bar{w}_y}{p}dp. \tag{20}$$

If we now insert (14) and (15) into (20), and recall that outside the region of clouds, $y>|a|$, $M_B$ is absent, we find that

$$A_{yy}+\lambda_+^{-2}A=0,\quad\text{for}\quad y<|a|, \tag{21a}$$

$$A_{yy}-\lambda_-^{-2}A=0,\quad\text{for}\quad y>|a|, \tag{21b}$$

where

$$\left.\begin{aligned}&\lambda_+^2=(\eta-1)ND^{-1}\lambda^2,\quad \lambda_-^2=ND^{-1}\lambda^2\\ &N=-m_1^{-1}(p_D^{m_1}-p_B^{m_1})+g_1g_2^{-1}m_2^{-1}(p_D^{m_2}-p_B^{m_2})\\ &D=(m_1+1)(p_D^{m_1}-p_B^{m_1})-g_1g_2^{-1}(p_D^{m_2}-p_B^{m_2})\end{aligned}\right\}.$$

[5] The constant $e$ can be evaluated from the relation given by Charney and Eliassen (1949), i.e., $w_B=\frac{1}{2}D_E\zeta_g\sin 2\alpha$, where $D_E$ is the Ekman depth, $\zeta_g$ is the vorticity of the geostrophic wind, and $\alpha$ is the angle between the surface geostrophic wind and the surface isobars. The inverse $e^{-1}$ has the interpretation of the (scaled) spin-up time scale.

We will get an ITCZ-like solution corresponding to Charney (1971), where $\lambda_-^2$ and $\lambda_+^2$ are positive and $\lambda_-^2$ and $a$ are related by

$$\tan\frac{a}{\lambda_+}=\frac{\lambda_+}{\lambda_-}. \tag{22}$$

Although it is not obvious, it can be checked that $m_2>m_1$ implies that both $N$ and $D$ are positive for all $p_D<p_B$. The necessary condition for CISK then becomes

$$\eta>1. \tag{23}$$

The explicit form for $\eta$ can be evaluated directly from the definition of $\eta$ in (19) and the $y$-differentiated form of the explicit expression for $M_B$ given by (8). The result is

$$\eta=\frac{gHM_{B_y}\int_{p_B}^{p_D}p^{-2}dp}{\int_{p_B}^{p_D}\bar{w}_yp^{-1}dp}=\frac{p_D^{-1}-p_B^{-1}}{I}\left[1-\frac{I_1}{N}\left(p_B^{m_1}-\frac{g_1}{g_2}p_B^{m_2}\right)\right], \tag{24}$$

where

$$I=p_D^{-1}-p_B^{-1}-I_1>0,$$

$$I_1=-\int_{p_B}^{p_D}\hat{\delta}(p)p^{-1}dp>0.$$

As

$$\sigma\rightarrow\infty,$$

$$\eta\rightarrow\frac{p_D^{-1}-p_B^{-1}}{p_D^{-1}-p_B^{-1}-I_1}>1,$$

so that we can always find a strong CISK solution.

The restriction to an isothermal atmosphere and constant cloud mass flux was made for the sake of explicitness and clarity. A more general treatment can be given as follows:

Begin with the heat equation

$$\frac{\partial^2\bar{\varphi}'}{\partial t\partial z}+\lambda^2\left[\bar{w}'-\frac{M_c(z)}{\hat{\rho}(z)}\right]=0,$$

where we look for a CISK solution with rising motion for $y<|a|$ and sinking motion for $y>|a|$.

Integrating the heat equation over

$$d\bar{z}\equiv(dz/\hat{T})(\partial\hat{s}/\partial z),$$

where the limits will be understood to be $\bar{z}_B$ to $\bar{z}_D$, yields

$$\int\frac{\partial^2\bar{\varphi}'}{\partial t\partial z}d\bar{z}-\lambda^2(\eta-1)\int\bar{w}'d\bar{z}=0, \tag{25}$$

where we have defined

$$\eta=\left[\int\bar{w}'d\bar{z}\right]^{-1}\int\hat{\rho}^{-1}(z)M_c(z)d\bar{z}.$$

We look for separable solutions of the form

$$\bar{\varphi}'=A(y)\Phi(z)e^{\sigma t}, \tag{26a}$$

which, by the continuity equation, implies

$$\bar{w}'=-\sigma\hat{\rho}^{-1}A_{yy}\langle\hat{\rho}\Phi\rangle e^{\sigma t}, \tag{26b}$$

where the angle brackets mean $z$ integration from $z_B$ to $z$.

Inserting (26) into (25) gives

$$A+\lambda_-^2(\eta-1)A_{yy}=0, \tag{27}$$

where

$$\frac{\lambda_-^2}{\lambda^2}=\left[\int\Phi_zd\bar{z}\right]^{-1}\int\langle\hat{\rho}\Phi\rangle\hat{\rho}^{-1}d\bar{z}.$$

Eq. (27) is the generalization of (21) and it is clear that the requirement for a CISK solution, when $\lambda_-^2$ is positive, is that $\eta>1$.

Using (7b), we can explicitly evaluate

$$\eta-1=\left[\int\bar{w}'d\bar{z}\right]^{-1}\left[\int\frac{M_c(z)}{\hat{\rho}(z)}-\bar{w}'\right]d\bar{z},$$

$$=\left(\frac{M_B}{\hat{\rho}_B}-\bar{w}_B'\right)\frac{\partial\hat{s}_B}{\partial z}\left[\int\bar{w}'d\bar{z}\right]^{-1}\int_{z_B}^{z_D}\hat{\rho}\hat{T}^{-1}dz, \tag{28}$$

and again we see that a necessary condition for our CISK solution is that

$$M_c(z_B)>\hat{\rho}_B\bar{w}_B'.$$

Numerical calculations have been carried out using the parameterization (14) for the vertical dependence but including $\hat{\delta}(z)$ and $\hat{T}(z)$ consistent with Yanai's (1971b) ITCZ atmosphere. The relation between $a$ and $\sigma/e$, as given by (22), is plotted in Fig. 2.

Again $\sigma/e$ *increases* with $a$ and becomes infinite at a finite value of $a$. A solution does not exist for larger values of $a$. The position of curves depends on the parameters $m_1$ and $m_2$ which, in turn, determine the vertical distribution of adiabatic heating (as $m_1$ and $m_2$ decrease, the bulk of the adiabatic heating moves to the upper parts of the atmosphere). It follows (see Section 5) that quantitative results are unreliable and this must be true of all two-level modes.

## 5. Discussion

1) The present (and first) application of Arakawa's new parameterization scheme to a model of CISK shows that the scheme can be readily implemented in dynamical models.

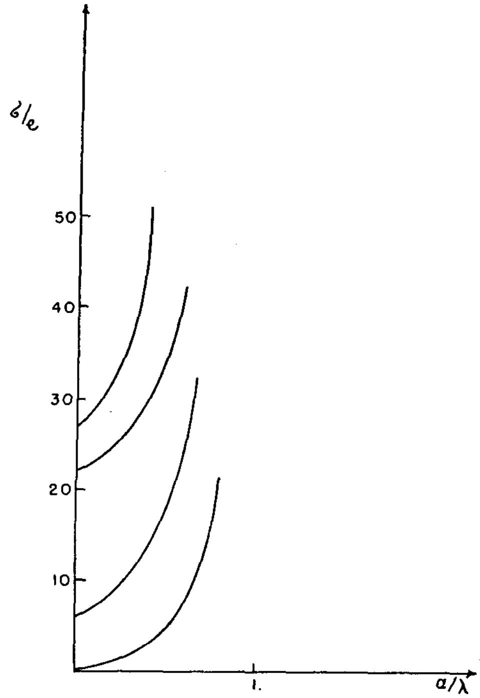


FIG. 2. Growth rate vs width of ITCZ for semi-analytic model of CISK using Arakawa's parameterization in the non-entraining cloud approximation.

2) Since the parameterization involves integration of functions that increase rapidly with height, the two-level model that involves crude integrations must be very sensitive to the position of the correct detrainment level.

3) A semi-analytic model is introduced that involves the parameterization of the vertical velocity $\bar{w}$; it satisfies all the boundary conditions and equations of the model exactly except the heat equation, which is satisfied in an integrated form. The resulting curves of $\sigma/e$ vs $a$ depend on the parameters that characterize the form of $\bar{w}$. This shows that any two-level model cannot give quantitative results because the particular choice of levels and discretization scheme will be equivalent to a choice of $\bar{w}$ (i.e., $m_1$ and $m_2$) in our model.

4) Both our two-level model and the semi-analytic (vertically-averaged) model predict an increase of the growth rate when the size of the active region ($2a$) is increased. This indicates the existence of a non-zero preferred scale of convection and is in contradistinction to all present models that use a less sophisticated parameterization. Those models all give a maximal growth rate at $a=0$ (but see Charney, 1973).

5) The magnitude of the growth rate in our model can be between 0 and $\infty$. Clearly the maximum growth rate will be limited by nonlinear and relaxation processes neglected in our model. For example, the convection relaxation time cannot be approached without invalidating the convection parameterization scheme. However, the important point is that our models allow growth time scales of the order of a few days; this should be compared with Charney (1971), for example, where the maximum growth rate $\sigma_{\max}$ corresponds to a period of a month and is too slow to account for the observed variability in the ITCZ.

6) Since the semi-analytic model that depends on free parameters is not conclusive, work is now in progress on an analytic model utilizing a differentiated form of the heat equation [as in (6)], and a $z$-dependent $M_c$. Preliminary results indicate that $\sigma$ is independent of $a$ for the range of interest 0.1 to 0.5; on the other hand, $\sigma$ itself depends strongly on the form of $M_c(z)$ and is, in fact, almost zero for a constant $M_c$. For more realistic distributions $\sigma/e$ is between 0.7 and 0.5.

7) The condition $M_B > \rho_B w_B$ was derived as a very general property of the Arakawa parameterization in CISK. This condition is independent of the assumption about the nature of the boundary layer and does not depend on the pumping being Ekman. It does assume $w_B$ is coupled to the system in such a way that all quantities grow as $e^{\sigma t}$. The meaning of this condition is that the environmental subsidence at cloud base, $-\tilde{M}=M_B-\rho_B w_B$, must be positive. Thus, there must be heating by subsidence at cloud base in the disturbed region in order for the disturbance to grow. The inclusion of the two neglected cooling effects, radiation and re-evaporation, can only strengthen this conclusion.

*Acknowledgments.* We would like to thank Prof. Jule Charney for his continued advice and encouragement and for many helpful discussions; and Prof. Akio Arakawa for critically reading the manuscript and for suggesting several improvements and corrections. This work has been supported by the National Science Foundation under Contract G A28724X. One of the authors (E.S.S.) is also indebted to the National Science Foundation for his appointment as a post doctoral fellow.

REFERENCES

Arakawa, A., 1971a: Parameterization of cumulus convection (in lecture notes distributed at Workshop of the UCLA General Circulation Model, 14–22 June 1971. Also to be published as *Design of the UCLA General Circulation Model. Numerical Simulation of Weather and Climate*, Tech. Rept. No. 7, Dept. of Meteor., University of California, Los Angeles).

——, 1971b: A parameterization of cumulus convection and its application to numerical simulation of the tropical general circulation. Paper presented at the Seventh Technical Conference on Hurricanes and Tropical Meteorology, Barbados, W. I.

——, and W. Schubert, 1973: The interaction of a cumulus cloud ensemble with the large scale environment. To be published.

Bates, J. R., 1972: Tropical disturbances and the general circulation. *Quart. J. Roy. Meteor. Soc.*, **98**, 1–16.

Charney, J. G., 1971: Tropical cyclogenesis and the formation of Intertropical Convergence Zone. *Lectures in Applied Mathematics*, Vol. 13, Providence, R. I., Amer. Math. Soc., 355–368.

——, 1973: Movable CISK. *J. Atmos. Sci.*, **30**, 50–52.

——, and A. E. Eliassen, 1964: On the growth of the hurricane depression. *J. Atmos. Sci.*, **21**, 68–75.

——, and ——, 1949: A numerical method for predicting the perturbations of the middle-latitude westerlies. *Tellus*, **1**, 38–54.

Gray, W. M., 1970: Fundamental role of cumulus convection for kinetic energy transformation in the tropics and general circulation. *Preprints Symp. Tropical Meteorology* University of Hawaii, Amer. Meteor. Soc., DV 1-6.

Hayashi, Y., 1971: Instability of large-scale equatorial waves under the radiation condition. *J. Meteor. Soc. Japan*, **49**, 458–466.

Ooyama, K., 1964: A dynamical model for the study of tropical cyclone development. *Geofis. Intern.*, **4**, 187–198.

——, 1971: A theory on parameterization of cumulus convection. *J. Meteor. Soc. Japan*, **49**, 744–756.

Ogura, Y., 1964: Frictionally controlled, thermally driven circulations in a circular vortex with applications to tropical cyclones. *J. Atmos. Sci.*, **21**, 610–621.

Yamasaki, M., 1969: Large-scale disturbances in a conditionally unstable atmosphere in low latitudes. *Papers Meteor. Geophys.*, **20**, 289–336.

——, 1971: A further study of wave disturbances in the conditionally unstable model tropics. *J. Meteor. Soc. Japan*, **49**, 391–415.

Yanai, M., 1971a: A review of recent studies of tropical meteorology relevant to the planning of GATE. Report prepared for the Interim Scientific and Management Group for GATE.

——, 1971b: The mass, heat and moisture budgets and the convective heating within tropical cloud clusters. Paper presented at the Seventh Technical Conference on Hurricanes and Tropical Meteorology, Barbados, W. I.