On the Stability of a Moist Atmosphere in the Presence of a Background Wind

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ABSTRACT

Sufficient conditions for the stability of a moist atmosphere, with possible condensation occurring, in the presence of a background wind are given. The atmosphere is assumed saturated at all heights, and only frequencies below the audible range are considered. It is shown that the effect of condensation is essentially that of reducing the Brunt-Väisälä frequency and hence of decreasing the stability of the system.

1. Introduction

In a recent paper, Chimonas (1970) has shown that a sufficient condition for stability, against infinitesimal adiabatic perturbations of a gravitationally stratified compressible inviscid fluid containing a horizontal flow with only vertical shear, is that $n^2/U_0'^2 \geqslant \frac{1}{4}$ throughout the flow. Here n is the Brunt-Väisälä frequency and U_0' the vertical gradient of the flow speed. Miles (1961) and Howard (1961) had obtained an equivalent result for an incompressible fluid.

In this paper we study the stability of an atmosphere which is considered to be a compressible, inviscid fluid, when the effect of water vapor, water droplets and condensation is included. Because of the latent heat exchange associated with changes of phase, infinitesimal perturbations are not adiabatic and an energy source or sink term appears in the energy equation. Using a system of equations first derived in a companion paper by Einaudi and Lalas (1973), and hereafter referred to as I, sufficient conditions for the stability of a moist, saturated atmosphere in the presence of a background wind shear are given and discussed for frequencies below the audible range. In the internal gravity wave range it is shown that the effect of condensation is to reduce the Brunt-Väisälä frequency and hence the stability of the system.

In the limit of no background wind, it is shown that circumstances may occur in which no neutral solutions can exist.

2. Equations of motion of the system

If we assume that the water droplets are very small compared to characteristic scales of the system and numerous enough so as to treat them as a continuum, the dry air, water vapor and water droplets form a mixture of three fluids with the possibility of phase change between the last two. General equations for the system have been derived in I, by introducing the usual equations for the conservation of mass, momentum and energy, along with a set of constitutive relations that complete the description of the dynamical interactions of the mixture. A simplified form of these equations, which still preserves the most important effects of moisture, is introduced and justified here, in the linear approximation, for the case in which a horizontal background wind $U_0(z)$ is present. In particular, the equations of continuity, horizontal and vertical momentum, energy, production of water vapor per unit volume and time, the Clausius-Clapeyron equation, and equations of state, respectively are as follows:

$$\frac{d}{dt}\rho_1^{(1)}+\rho_0^{(1)}\left(\frac{\partial}{\partial x}v_x+\frac{\partial}{\partial z}v_z\right)+v_z\frac{d}{dz}\rho_0^{(1)}=0 \tag{2.1}$$

$$\frac{d}{dt}\rho_1^{(2)}+\rho_0^{(2)}\left(\frac{\partial}{\partial x}v_x+\frac{\partial}{\partial z}v_z\right)+v_z\frac{d}{dz}\rho_0^{(2)}=\Gamma_1 \tag{2.2}$$

$$\rho_0^{(1)}\frac{d}{dt}v_x+\frac{\partial}{\partial x}p_1^{(1)}+\rho_0^{(1)}v_z\frac{d}{dz}U_0=0 \tag{2.3}$$

$$\rho_0^{(1)}\frac{d}{dt}v_z+\frac{\partial}{\partial z}p_1^{(1)}+\rho_1^{(1)}g=0 \tag{2.4}$$

$$\rho_0^{(1)}c_v^{(1)}\left(\frac{d}{dt}T_1+v_z\frac{d}{dz}T_0\right)+p_0^{(1)}\left(\frac{\partial}{\partial x}v_x+\frac{\partial}{\partial z}v_z\right)=-L_v\Gamma_1 \tag{2.5}$$

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$$\Gamma_1=[\rho_{\omega1}-\rho_1^{(2)}]/\tau_m, \quad 1/\tau_m=4\pi n_p a_0 D \qquad (2.6)$$

$$\rho_{\omega1}=\rho_0^{(2)}\left[\frac{L_v}{R^{(2)}T_0}-1\right]\frac{T_1}{T_0} \qquad (2.7)$$

$$p_1^{(i)}=R^{(i)}[T_0\rho_1^{(i)}+\rho_0^{(i)}T_1], \quad i=1,2. \qquad (2.8)$$

The above is a system of nine equations for the nine perturbation unknowns: v_x and v_z, the horizontal and vertical velocities; $\rho_1^{(1)}$ and $\rho_1^{(2)}$, the partial densities for air and water vapor; $\rho_{\omega1}$, the partial density of vapor at the droplet surface; $p_1^{(1)}$ and $p_1^{(2)}$, the partial pressures for air and water vapor; T_1, the temperature of the system; and Γ_1 the production of water vapor per unit volume and unit time. The quantity $c_v^{(1)}$ is the specific heat at constant volume for dry air; $R^{(1)}$ and $R^{(2)}$ are the gas constants for dry air and water vapor, respectively; L_v is the latent heat of evaporation; n_p is the number of droplets, all assumed the same, per unit volume of the mixture, and is an externally specified quantity depending on the number of nuclei around which each droplet can grow; a_0 is the average radius of the droplets; D is the diffusivity of water vapor; g is the gravitational acceleration acting in the negative z direction; and d/dt is given by

$$\frac{d}{dt}=\frac{\partial}{\partial t}+U_0\frac{\partial}{\partial x}.$$

The background pressures $p_0^{(i)}$, densities $\rho_0^{(i)}$ and temperature T_0, denoted by the suffix 0, depend on z only and are governed by the usual equations of state, the hydrostatic equation and the Clausius-Clapeyron equation

$$\frac{p_0^{(2)}}{p_g^{(2)}}=\exp\left[\frac{L_v}{R^{(2)}T_g}\left(1-\frac{T_g}{T_0}\right)\right], \qquad (2.9)$$

where the suffix g denotes background quantities at the reference level $z=0$.

Eqs. (2.1), (2.3), (2.4) and (2.8) for $i=1$ are the same as those governing the propagation of acoustic gravity waves in a dry atmosphere (see Hines, 1960). The energy equation, on the other hand, contains a source term which is equal to minus the product of the latent heat times the rate of production of water vapor. The latter depends on the microstructure surrounding each droplet. This is a region, around each droplet, extending over a few droplet diameters only, within which all the detailed phenomena take place that affect the mass, momentum and energy exchanges between constituents. In this region the local average values of $p_1^{(2)}$, $\rho_1^{(2)}$ and T_1 differ, in general, from the vapor pressure, density and temperature at the droplet surface ($p_{\omega1}, \rho_{\omega1}, T_{\omega1}$). Thus, if we assume that the controlling mechanism of mass exchange between droplet and ambient vapor is diffusion, then we can write (following Byers, 1965) for $\rho_0^{(2)}=\rho_{\omega0}$

$$\frac{d}{dt}m_p=4\pi Da_0[\rho_1^{(2)}-\rho_{\omega1}], \qquad (2.10)$$

where m_p is the mass of the droplet. Disregarding nuclei generation, the production of water vapor per unit volume and time Γ_1 is equal to the production of water vapor at each droplet multiplied by the number of droplets, and (2.6) follows immediately. Eq. (2.7) is the linearized form of the Clausius-Clapeyron equation relating saturation density and temperature. The linearization assumes that the ratio

$$\frac{L_v}{R^{(2)}T_0}\frac{T_1}{T_0}\ll 1.$$

Since $L_v/[R^{(2)}T_0]\approx 20$ in the lower atmosphere, it follows that the above condition could be important in limiting the maximum value that T_1/T_0 can reach for the linearization to hold. Finally, we point out that we assume (2.7) to hold at all heights and times, which is equivalent to saying that the atmosphere is saturated.

A number of assumptions already used limit the frequency range and the maximum amplitude of the perturbation quantities for which (2.1)–(2.8) give meaningful results. The limitations on the frequency range are due to having neglected the velocity and temperature differences among the three components of the mixture; this implies that the period of oscillation must be much larger than the two time scales

$$\tau_M=\frac{9\mu^{(1)}}{2a_0^2\rho_w}, \quad \tau_T=\frac{\rho_w a_0^2 c_v^{(3)}}{3\kappa^{(1)}}. \qquad (2.11)$$

Here $\mu^{(1)}\approx 1.82\times10^{-4}$ gm sec^{-1} cm^{-1} and $\kappa^{(1)}\approx 6.2\times10^{-6}$ kcal m^{-1} sec^{-1} (°K)$^{-1}$ are the viscosity coefficient and the thermal conductivity of dry air, respectively[2]; and $c_v^{(3)}=c_p^{(3)}=1$ cal gm^{-1} and $\rho_w=1$ gm cm^{-3} are the specific heat and density of liquid water, respectively. The quantity τ_M is the characteristic time scale with which the droplet velocity approaches that of the gas, while τ_T is the characteristic time for the heat transfer via conduction. For an average radius a_0 of the order of 10^{-3} cm both time scales are less than 10^{-3} sec. Therefore, if we limit ourselves to periods much greater than 10^{-3} sec, i.e., frequencies below the audible range, we are justified in neglecting differences in velocities and temperatures among the three components of the mixture.

Three smallness parameters (ϵ, δ, Δ) appear naturally in the problem. The first, ϵ, describes the amplitude of the disturbance and therefore is such that $|\rho_1^{(1)}/\rho_0^{(1)}| \sim |\rho_1^{(2)}/\rho_0^{(2)}| \sim |v_z/c_r| \sim O(\epsilon)$, etc., where c_r is a characteristic velocity of the system; and δ and Δ are

[2] In I, the factor 10^{-6} was inadvertently omitted in the expression for $\kappa^{(1)}$ following Eq. (4.16); the last term of the same equation should read $5.3\times10^{-5}T_\omega/(T-T_\omega)$.

the ratios of water vapor and water droplet densities to the total density of the mixture, respectively. The quantities δ and Δ are never larger than 0.03–0.04 in the atmosphere (Tverskoi, 1965), and rather often they are of the order of 10^{-3}. Only terms $O(\epsilon)$ are retained in the governing equations. All terms of $O(\epsilon\delta)$ and of $O(\epsilon\Delta)$ are neglected except in the right-hand side of the energy equation where δ is multiplied by $\{L_v/[R^{(2)}T_0]\}^2$ which is large compared to unity. The best way to justify this point is to rewrite the energy equation in terms of T_1, using Eqs. (2.2) and (2.7), so that

$$\rho_0^{(1)}c_v^{(1)}\left[i\Omega T_1(z)+v_z(z)\frac{d}{dz}T_0\right]+p_0^{(1)}\left[-ikv_x(z)+\frac{d}{dz}v_z(z)\right]=-\left[\frac{L_v}{R^{(2)}T_0}\right]^2\frac{R^{(2)}T_0\rho_0^{(2)}}{1+i\Omega\tau_m}\left[i\Omega\frac{T_1}{T_0}+\frac{v_z(z)}{T_0}\frac{dT_0}{dz}\right],\quad(2.12)$$

where

$$\Omega=\omega-kU_0(z),\quad k\text{ real and }\omega=\omega_r+i\omega_i,\quad(2.13)$$

and the perturbation amplitudes are assumed of the form

$$A(x,z,t)=A(z)\exp[i(\omega t-kx)].\quad(2.14)$$

The ratio of the right-hand side of (2.12) to the first term on its left hand side reads

$$\frac{\rho_0^{(2)}}{\rho_0^{(1)}}\frac{R^{(2)}}{c_v^{(1)}}\left|\frac{1}{1+i\Omega\tau_m}\right|\left[\frac{L_v}{R^{(2)}T_0}\right]^2\approx400\frac{\rho_0^{(2)}}{\rho_0^{(1)}},$$

so that for $\delta\sim\rho_0^{(2)}/\rho_0^{(1)}\approx10^{-3}$, the ratio is of the order of 0.4 and the right-hand side cannot be neglected. In deriving the right-hand side of (2.12), the implicit assumption was made that all the energy released by condensation is added to the dry air component of the mixture [compare (2.5) of this paper with (4.20) in I].

Finally, we point out that the terminal velocity u_{z0} of the droplets has been neglected, since it is much smaller (<1.0 cm sec^{-1}) than either the speed of sound, $C_0\approx3.2\times10^4$ cm sec^{-1}, or a typical gravity wave speed of a few meters per second.

Having briefly discussed some of the features and assumptions of Eqs. (2.1)–(2.8), we proceed in the next section to utilize them in studying the stability of the system. Again, for more detailed discussion of the governing equations, the reader is referred to I.

3. The stability analysis

To reduce the initial system of equations to a second-order differential equation for the vertical component of the velocity, $v_z(z)$, we write the energy equation (2.12), with the use of (2.1) and (2.8) as

$$i\Omega p_1^{(1)}(z)=[i\Omega\rho_1^{(1)}(z)-v_z(z)\rho_0^{(1)}\tilde{n}^2/g]\tilde{c}^2,\quad(3.1)$$

where

$$\tilde{c}^2=R^{(1)}T_0\left\{1+\frac{R^{(1)}/c_v^{(1)}}{1+\tilde{L}_v^2B\dfrac{R^{(2)}}{c_v^{(1)}}\dfrac{\rho_0^{(2)}}{\rho_0^{(1)}}}\right\},\quad(3.2)$$

$$\tilde{n}^2=-g\left[\frac{g}{\tilde{c}^2}+\frac{1}{\rho_0^{(1)}}\frac{d\rho_0^{(1)}}{dz}\right],\quad(3.3)$$

$$\tilde{L}_v=\frac{L_v}{R^{(2)}T_0},\quad B=1/(1+i\Omega\tau_m).\quad(3.4)$$

Both $\tilde{c}$ and $\tilde{n}$ go to the values of the speed of sound c and of the Brunt-Väisälä frequency n as the ratio $\rho_0^{(2)}/\rho_0^{(1)}$ goes to zero. Eqs. (2.1), (2.3), (2.4), (3.1) and (2.8), for $i=1$, form a system of linear equations from which we can eliminate the various amplitudes in terms of $v_z(z)$, obtaining

$$\left\{\frac{\Omega^2rq'}{k^2[1-(\Omega/k\tilde{c})^2]}\right\}'+rq(\tilde{n}^2-\Omega^2)+\frac{rq\Omega^2}{k^2[1-(\Omega/k\tilde{c})^2]}\left(\frac{\tilde{\epsilon}'}{\tilde{\epsilon}}-\frac{E'}{E}\right)\left(\frac{\tilde{\epsilon}'}{\tilde{\epsilon}}-\frac{E^{*\prime}}{E^*}\right)+q\left\{\frac{\Omega^2r}{k^2[1-(\Omega/k\tilde{c})^2]}\left(\frac{\tilde{\epsilon}'}{\tilde{\epsilon}}-\frac{E'}{E}\right)\right\}'=0,\quad(3.5)$$

where

$$E=\exp\left[g\int^z\frac{dz}{\tilde{c}^2}\right],\quad\tilde{\epsilon}^2=EE^*,\quad(3.6)$$

$$q=\frac{v_z(z)}{i\tilde{\epsilon}\Omega},\qquad r=\rho_0^{(1)}EE^*.\quad(3.7)$$

In the above equations, primes represent differentiation with respect to z and stars denote the complex conjugate. In the limit of $\delta\to0$, (3.5) reduces to (2.8) of Chimonas (1970).

We assume the fluid to be bounded by horizontal rigid walls at $z=z_1$ and $z=z_2$, which in limiting cases can be extended to $z=\pm\infty$; it follows that v_z must vanish for these values of z. We can also consider the fluid bounded by some height above which $\tilde{c}^2$ and U_0 do not depend on z any more and $r^{\frac{1}{2}}q$ goes exponentially to zero for large enough z as discussed by Chimonas (1970).

Let us first discuss the physical significance of the new quantities $\tilde{c}^2$ and $\tilde{n}^2$ defined by (3.2) and (3.3), respectively. These are most readily understood by considering first the oscillation of a parcel of moist air when displaced vertically from its initial position. No background velocity is now included. Following

Eckart (1960), let us imagine the parcel to be enclosed by a flaccid balloon-like membrane which allows for changes of density and such that pressures inside and outside are always the same.

Let us displace the balloon from the position z to $z+\xi$ in the time interval Δt. With the degree of approximation of this discussion, the pressure changes by

$$\Delta p_0 = -\rho_0^{(1)} g\xi, \tag{3.8}$$

while the change in the density outside the balloon is

$$(\Delta\rho_0)_{\text{outside}} = \frac{d\rho_0^{(1)}}{dz}\xi. \tag{3.9}$$

To calculate the changes in the density inside the balloon we have to include the effect of the latent heat which may be released or absorbed in the process of moving the parcel of air. The amount of condensation produced in the time Δt is given by

$$\Delta\Gamma = \frac{\Delta\rho_0^{(2)}}{\Delta t} = \frac{1}{R^{(2)}}\frac{p_0^{(2)}}{T_0^2}(\tilde{L}_v - 1)\frac{\Delta T_0}{\Delta t} \approx \frac{\rho_0^{(2)}\tilde{L}_v}{T_0}\frac{\Delta T_0}{\Delta t}.$$

Using (2.1) and (2.5), we can express the variation of energy of the parcel as

$$\rho_0^{(1)} c_v^{(1)} \Delta T_0 - \frac{p_0^{(1)}}{\rho_0^{(1)}}(\Delta\rho_0)_{\text{inside}} = -L_v \tilde{L}_v \frac{\rho_0^{(2)}}{T_0}\Delta T_0, \tag{3.10}$$

where ΔT_0 is the change of temperature of the parcel of air and can be readily expressed in terms of Δp_0 and $(\Delta\rho_0)_{\text{inside}}$ via the equation of state. The change in density inside the balloon can then be written as

$$(\Delta\rho_0)_{\text{inside}} = \Delta p_0/\alpha_0^2, \tag{3.11}$$

where

$$\alpha_0^2 = R^{(1)} T_0 \left[1 + \frac{R^{(1)}/c_v^{(1)}}{1 + \tilde{L}_v^2 \dfrac{R^{(2)}}{c_v^{(1)}}\dfrac{\rho_0^{(2)}}{\rho_0^{(1)}}}\right]. \tag{3.12}$$

The balloon experiences a buoyancy force given by

$$g[(\Delta\rho_0)_{\text{outside}} - (\Delta\rho_0)_{\text{inside}}] = g\xi\left[\frac{\Delta\rho_0^{(1)}}{\Delta z} + g\rho_0^{(1)}/\alpha_0^2\right]$$
$$= -\rho_0^{(1)} n_0^2 \xi, \tag{3.13}$$

where

$$n_0^2 = -g\left[\frac{1}{\rho_0^{(1)}}\frac{d\rho_0^{(1)}}{dz} + g/\alpha_0^2\right], \tag{3.14}$$

and its equation of motion can be written as

$$\frac{d^2\xi}{dt^2} + n_0^2 \xi = 0. \tag{3.15}$$

The quantity n_0 has now a clear physical meaning: it represents the frequency of oscillation of the parcel of moist air. It could be obtained from the standard definition of the Brunt-Väisälä frequency by replacing the dry adiabatic lapse rate by the now appropriate moist one calculated in a manner consistent with the linearization scheme of the basic equations. For the oscillations to be stable, n_0 must satisfy the inequality

$$n_0^2 > 0. \tag{3.16}$$

In this order of approximation, α_0^2 represents the coefficient of proportionality between changes of pressure and density within the parcel of moist air. A number of assumptions have been tacitly made in deriving (3.15). The use of the hydrostatic equation implies that the displacement of the parcel should be small and its movement slow; in fact, α_0^2 and n_0^2 can be derived from (3.2), (3.3) and (3.4), i.e., from $\tilde{c}^2$ and $\tilde{n}^2$, respectively, by taking the limit of $\omega \to 0$ and remembering that no background wind is present in these considerations. We now return to the investigation of (3.5), for three cases of special interest.

a. *Stability analysis for* $\Omega\tau_m \ll 1$

First we consider the case in which

$$\tau_m |\omega - kU_0| \ll 1. \tag{3.17}$$

For $a_0 \approx 10^{-3}$ cm, $D \approx 2.58\ 10^{-1}$ cm² sec⁻¹ and $n_p \approx 300$ particles cm^{-3}, $\tau_m \approx 1$ sec; hence, (3.17) implies that we restrict ourselves to frequencies in the internal gravity wave range and, for $U_0 \approx 10$–100 m sec^{-1}, to horizontal wavelengths λ_x of at least 1 km. For smaller values of U_0, correspondingly smaller values of λ_x can be considered. Thus, (3.17) is still satisfied for most internal gravity wave propagation conditions of practical interest.

If (3.17) is satisfied, we have that $B \to 1$, $\tilde{c}^2 \to \alpha_0^2$, $\tilde{n}^2 \to n_0^2$, $\tilde{\epsilon} = E = E^*$; and Eq. (3.5) reduces to Chimonas' Eq. (2.8), provided we use in the latter α_0^2 and n_0^2 instead of Chimonas' C^2 and n^2. Because of (3.16), n_0^2, α_0^2 and r are all real and positive and the stability analysis of (3.5) can be carried out exactly as done by Chimonas (1970).

The effect of condensation in a moist, saturated atmosphere is to replace the usual Brunt-Väisälä frequency by n_0 so that a sufficient condition for the stability of the system is that throughout the flow

$$\frac{n^2}{U_0'^2} \geq \frac{1}{4}\left(\frac{n}{n_0}\right)^2. \tag{3.18}$$

The quantitative significance of the effect of condensation can be assessed by calculating the ratio

$$S = \frac{n^2 - n_0^2}{n^2}. \tag{3.19}$$

For the same density stratification, and $R^{(2)}=1.6R^{(1)}$, $c=3.2\times10^4$ cm sec^{-1}, $\gamma=1.4$, $\tilde{L}_v=20$, $g=981$ cm sec^{-2}, $\rho_0^{(2)}/\rho_0^{(1)}=10^{-2}$, and n measured in sec^{-1}, (3.19) becomes

$$S=2.5\times10^{-4}/n^2. \tag{3.20}$$

Taking the rather high value for n of $2\pi/n=4$ min, for example, in (3.20), one obtains $S=0.37$ which corresponds to a decrease of the maximum allowable shear of 20%, a decrease of rather important practical significance. Equivalently, the usual critical Richardson number, $n^2/U_0'^2$, increases from 0.25 to 0.39. Higher values of $\rho_0^{(2)}/\rho_0^{(1)}$ or lower values of n would give even lower allowable shear rates for stability.

b. *Stability analysis in the general case*

The procedure followed by Chimonas (1970) requires here some modifications due to the nature of $\tilde{c}^2$ and $\tilde{n}^2$ which are now complex numbers. Making the substitution

$$q=\Omega^{-\frac{1}{2}}\Phi, \tag{3.21}$$

Eq. (3.5) can be rewritten as

$$\left[\frac{r\Omega}{k^2d}\Phi'\right]'-\frac{1}{2}\left[\frac{r\Omega'}{k^2d}\right]'\Phi-\frac{1}{4}\frac{r\Omega'^2}{k^2d\Omega}\Phi+\frac{r(n^2-\Omega^2)}{\Omega}\Phi$$
$$+\frac{r\Omega}{k^2d}\left(\frac{\tilde{\epsilon}'}{\tilde{\epsilon}}-\frac{E'}{E}\right)\left(\frac{\tilde{\epsilon}'}{\tilde{\epsilon}}-\frac{E^{*\prime}}{E^*}\right)\Phi$$
$$+\left[\frac{r\Omega^2}{k^2d}\left(\frac{\tilde{\epsilon}'}{\tilde{\epsilon}}-\frac{E'}{E}\right)\right]'\frac{\Phi}{\Omega}=0, \tag{3.22}$$

where a branch of $\Omega^{\frac{1}{2}}$ can be uniquely defined and d is given by

$$d=1-(\Omega/k\tilde{c})^2. \tag{3.23}$$

Multiplying (3.22) by Φ^* and integrating over the z domain, we have

$$\int dz\left\{\left[\frac{r\Omega}{k^2d}\Phi'\Phi^*\right]'-\frac{1}{2}\left[\frac{r\Omega'}{k^2d}\Phi\Phi^*\right]'+\left[\frac{r\Omega}{k^2d}\left(\frac{\tilde{\epsilon}'}{\tilde{\epsilon}}-\frac{E'}{E}\right)\Phi\Phi^*\right]'\right.$$
$$-\frac{r\Omega}{k^2d}\left[|\Phi'|^2+\left(\frac{1}{2}\frac{\Omega'}{\Omega}+ig\,\mathrm{Im}(1/\tilde{c}^2)\right)^2|\Phi|^2\right.$$
$$\left.-\left(\frac{1}{2}\frac{\Omega'}{\Omega}+ig\,\mathrm{Im}(1/\tilde{c}^2)\right)(\Phi\Phi^*)'\right]$$
$$\left.+r\frac{(\tilde{n}^2-\Omega^2)}{\Omega}|\Phi|^2\right\}=0, \tag{3.24}$$

where we have used the fact that

$$\tilde{\epsilon}'/\tilde{\epsilon}-E'/E=-(\tilde{\epsilon}'/\tilde{\epsilon}-E^{*\prime}/E^*)=-ig\,\mathrm{Im}(1/\tilde{c}^2). \tag{3.25}$$

The first three terms can be integrated immediately and are identically zero because of the boundary conditions. Taking the imaginary part of (3.24), we obtain, with some rearrangement of terms,

$$\omega_i\left\{\int\frac{rdz}{k^2|d|^2}\left[k^2|d|^2|\Phi|^2+\left(1+\frac{|\Omega|^2}{k^2}\mathrm{Re}(1/\tilde{c}^2)\right)P_1\right]\right.$$
$$+\int\frac{rdz}{|\Omega|^2}\left[\mathrm{Re}(\tilde{n}^2)-\frac{|\Omega|^2}{k^2|d|^2}P_2\right]|\Phi|^2\Big\}$$
$$+\int\frac{rdz}{k^2|d|^2}\Omega_r\,\mathrm{Im}(1/\tilde{c}^2)$$
$$\times\left[\frac{|\Omega|^2}{k^2}P_1+\frac{g^2k^2|d|^2}{|\Omega|^2}|\Phi|^2-P_3|\Phi|^2\right]=0, \tag{3.26}$$

where $\Omega_r=\omega_r-kU_0$ and

$$P_1=|\Phi'|^2+\left[\frac{1}{2}\frac{\Omega'\Omega_r}{|\Omega|^2}+\frac{\Psi_r}{\Psi_i}P_0\right]^2|\Phi|^2$$
$$-(\Phi\Phi^*)'\left[\frac{1}{2}\frac{\Omega'\Omega_r}{|\Omega|^2}+\frac{\Psi_r}{\Psi_i}P_0\right], \tag{3.27}$$

$$P_2=\left(1+\frac{\Psi_r^2}{\Psi_i^2}\right)\left[1+\frac{|\Omega|^2}{k^2}\mathrm{Re}(1/\tilde{c}^2)\right]P_0^2, \tag{3.28}$$

$$P_3=\frac{|\Omega|^2}{k^2}\left(1+\frac{\Psi_r^2}{\Psi_i^2}\right)P_0^2, \tag{3.29}$$

$$P_0=g\,\mathrm{Im}(1/\tilde{c}^2)-\frac{1}{2}\frac{\Omega'\omega_i}{|\Omega|^2}, \tag{3.30}$$

$$\Psi_i=\frac{1}{k^2}\mathrm{Im}\left[\frac{\Omega}{1-(\Omega/k\tilde{c})^2}\right],\quad \Psi_r=\frac{1}{k^2}\mathrm{Re}\left[\frac{\Omega}{1-(\Omega/k\tilde{c})^2}\right]. \tag{3.31}$$

Using the fact that

$$(\Phi\Phi^*)'\leqslant 2|\Phi||\Phi'|, \tag{3.32}$$

and assuming that

$$\mathrm{Im}\left(\frac{1}{\tilde{c}^2}\right)<\mathrm{Re}\left(\frac{1}{\tilde{c}^2}\right), \tag{3.33}$$

it can be readily shown that P_1, P_2 and P_3 are non-negative real numbers. In addition, $\Omega_r\,\mathrm{Im}(1/\tilde{c}^2)=-\Omega_r^2N^2$, where N^2 is a non-negative real number. Because of (3.16) and (3.33), we make the further assumption that

$$\mathrm{Re}(\tilde{n}^2)>0. \tag{3.34}$$

It follows from the above considerations that a sufficient condition for stability of the system, i.e., $\omega_i > 0$, is that throughout the flow

$$\mathrm{Re}(\tilde{n}^2) - \frac{|\Omega|^2}{k^2|d|^2} P_2 \geqslant 0, \tag{3.35}$$

$$g^2 k^2 \frac{|d|^2}{|\Omega|^2} - P_3 \geqslant 0. \tag{3.36}$$

In the limit of $\mathrm{Im}(1/\tilde{c}^2) \rightarrow 0$, the second condition disappears since the last integral in (3.26) tends to zero. Furthermore, in this limit, (3.35) reduces to

$$n_0^2 \geqslant \frac{\Omega'^2}{4k^2} \frac{1}{(1+|\Omega|^2/k^2\alpha_0^2)}. \tag{3.37}$$

The above inequality is certainly verified if the condition $n_0^2 \geqslant U_0'^2/4$ is satisfied, and it will differ significantly from the latter only when $\omega_i^2/(k^2\alpha_0^2)$ is large. The stringency of (3.35) compared to (3.37) will depend on whether or not $\mathrm{Im}(1/\tilde{c}^2)$ is much less than $\mathrm{Re}(1/\tilde{c}^2)$, i.e., on the magnitude of $|\Omega\tau_m|$.

Inequality (3.36) does not appear to impose strong conditions on the flow parameters and is verified in most cases when (3.35) is satisfied. Expressing P_3 in terms of P_2 and using (3.35), it can be shown that (3.36) is certainly satisfied if the much more stringent inequality

$$-\frac{\rho_0'}{\rho_0} < 2g\,\mathrm{Re}(1/\tilde{c}^2) + \frac{k^2}{|\Omega|^2} g \tag{3.38}$$

is true. The above equation is likely to be verified except for particularly large gradients of density.

c. *The stability of the atmosphere at rest*

Let us now consider the case in which $U_0 \equiv 0$. The main point that can be inferred from (3.26) is that, in contrast with the case for a dry atmosphere, the last integral is different from zero, in general. But in this case of $U_0 = 0$, (3.36) is satisfied. If then the inequality (3.35) with $U_0 = 0$ is verified, we can conclude that ω_i must be greater than zero for (3.26) to hold and that the system does not allow neutral solutions ($\omega_i = 0$) to exist. In the neutral gravity wave range, however, where $\omega\tau_m \ll 1$, the imaginary part of ω can indeed become zero, so that neutral solutions can occur.

4. Conclusions

The stability of the system of equations governing the propagation of acoustic gravity waves in a moist, saturated atmosphere with a horizontal background wind is discussed. Sufficient conditions, i.e., (3.35) and (3.36), are given for the stability of the system; in the internal gravity wave range and for horizontal wavelengths sufficiently large one simply has to replace the Brunt-Väisälä frequency by a new characteristic frequency n_0. Since $n_0 < n$, a moist saturated atmosphere is less stable than a dry one. It should be pointed out that, very recently, Dudis (1972) has reached similar conclusions for the case $\Omega\tau_m \ll 1$, by an entirely different method. His treatment, though, is only valid in this range, because of his use of the Boussinesq approximation and the neglect of the microstructure of the droplet-vapor mass exchange.

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REFERENCES

Byers, H. R., 1965: *Elements of Cloud Physics.* The University of Chicago Press, Chap. 5.

Chimonas, G., 1970: The extension of the Miles-Howard theorem to compressible fluids. *J. Fluid Mech.*, **43**, 833–836.

Dudis, J. J., 1972: The stability of a saturated, stably-stratified shear layer. *J. Atmos. Sci.*, **29**, 774–778.

Eckart, C., 1960: *Hydrodynamics of Oceans and Atmospheres.* New York, Pergamon Press, 59–60.

Einaudi, F., and D. P. Lalas, 1973: The propagation of acoustic-gravity waves in a moist atmosphere. *J. Atmos. Sci.*, **30**, 365–376.

Hines, C. O., 1960: Internal atmospheric gravity waves at ionospheric heights. *Can. J. Phys.*, **38**, 1441–1481.

Howard, L. N., 1961: Note on a paper of John W. Miles. *J. Fluid. Mech.*, **10**, 509–512.

Miles, J. W., 1961: On the stability of heterogeneous shear flows. *J. Fluid. Mech.*, **10**, 496–508.

Tverskoi, P. N., 1965: *Physics of the Atmosphere.* Jerusalem, Israel Program of Scientific Translations, p. 107.