Prediction of the Monin-Obukhov Similarity Functions from an Invariant Model of Turbulence

W. S. LEWELLEN AND M. TESKE

Aeronautical Research Associates of Princeton, Inc., Princeton, N. J. 08540

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ABSTRACT

The second-order, invariant modeling technique for turbulent flows as developed by Donaldson is applied to the atmospheric surface layer. The steady, high-Reynolds number equations reduce to a universal set when the variables are scaled by the shear stress and vertical heat flux as suggested by Monin and Obukhov. Numerical integration of these equations yields results for the mean velocity gradient, mean temperature gradient, Richardson number, rms vertical velocity and temperature fluctuations, and horizontal heat flux which agree favorably with experimental observations over the complete range of stability conditions.

1. Introduction

The atmospheric layer extending a few tens of meters above the earth's surface is the most extensively studied example of a turbulent shear layer incorporating the influence of stratification on the dynamics of turbulence. Following Monin and Obukhov (1953), some success has been achieved in experimentally obtaining universal functions that describe the dependence of the mean turbulence characteristics on height, shear stress and heat flux. Thus, this shear layer should provide a good test of the invariant model (Donaldson, 1972, 1973) of the second-order turbulence equations. It provides both a means of determining the few scale parameters required and of indicating how constant these parameters are.[1]

As is customary (Monin and Yaglom, 1971), we define "surface layer" to mean that region where both the mean momentum flux and mean heat flux are sensibly constant. From the momentum and energy equations for the mean flow, this is equivalent to assuming a steady, one-dimensional shear layer within which the direct viscous contributions to shear and heat transfer are negligible. This restricts our attention to altitudes below those at which any cross flows appear due to the Coriolis force caused by the rotation of the earth, and to altitudes above the surface roughness or to altitudes above the laminar sublayer in the case of a smooth surface.

[1] We wish to thank a reviewer for pointing out Mellor's work (Mellor, 1973) which bears many similarities to the present work. The major difference is that Mellor eliminates the diffusion terms in Eqs. (4)–(10) to obtain what Donaldson (1973) terms a super-equilibrium approximation. This permits the turbulent length scale to be normalized out of the problem, eliminating any need for, or any possibility of, incorporating an influence of stratification on the scale.

2. The model equations

Within the surface layer the Reynolds equations for the second-order turbulent correlations may be written as follows:

$\overline{u_i'u_j'}$:

$$-\overline{u_j'w'}\frac{\partial u_i}{\partial z}-\overline{u_i'w'}\frac{\partial u_j}{\partial z}-\frac{\partial}{\partial z}\overline{u_i'u_j'w'}-\frac{\delta_{3i}}{\rho}\frac{\partial\overline{p'u_i'}}{\partial z}-\frac{\delta_{3j}}{\rho}\frac{\partial\overline{p'u_j'}}{\partial z}+\overline{\frac{p'}{\rho}\left(\frac{\partial u_i'}{\partial x_j}+\frac{\partial u_j'}{\partial x_i}\right)}+\delta_{3i}\frac{g}{T_0}\overline{u_j'T'}+\delta_{3j}\frac{g}{T_0}\overline{u_i'T'}-2\nu\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial u_j'}{\partial x_k}}=0 \quad (1)$$

$\overline{u_i'T'}$:

$$-\overline{u_i'w'}\frac{\partial T}{\partial z}-\overline{w'T'}\frac{\partial u_i}{\partial z}-\frac{\partial}{\partial z}\overline{u_i'w'T'}+\overline{\frac{p'}{\rho}\frac{\partial T'}{\partial x_i}}-\frac{1}{\rho}\frac{\partial}{\partial x_i}\overline{p'T'}+\delta_{3i}\frac{g}{T_0}\overline{T'^2}-(\kappa+\nu)\overline{\frac{\partial u_i'}{\partial x_k}\frac{\partial T'}{\partial x_k}}=0 \quad (2)$$

$\overline{T'T'}$:

$$-2\overline{w'T'}\frac{\partial T}{\partial z}-\frac{\partial}{\partial z}\overline{w'T'T'}-2\kappa\overline{\frac{\partial T'}{\partial x_k}\frac{\partial T'}{\partial x_k}}=0. \quad (3)$$

In the above u_i is the velocity, x_i the Cartesian coordinates, z the vertical axis, p the pressure, g gravity, T temperature, ρ density, ν viscosity, and κ the thermal conductivity coefficient. The primed quantities are the fluctuating components from their mean, and an overbar denotes the correlation time average.

To close this set of equations we must model the third-order correlations, the pressure correlations, and the dissipation terms appearing in these equations in terms of the lower-order correlations and their gradients. Bradshaw (1972) discusses various models proposed for these terms by various investigators. As modeled by Donaldson (1973), the following closed set is obtained for $\overline{u'u'}$, $\overline{v'v'}$, $\overline{w'w'}$, $\overline{u'w'}$, $\overline{w'T'}$, $\overline{u'T'}$, $\overline{T'^2}$:

$$-2\overline{u'w'}\frac{\partial u}{\partial z}+\frac{\partial}{\partial z}\left(\Lambda_2 q\frac{\partial\overline{u'u'}}{\partial z}\right)-\frac{q}{\Lambda}\left(\overline{u'u'}-\frac{q^2}{3}\right)-\frac{2\beta bq\overline{u'u'}}{\Lambda}-(1-\beta)\frac{2bq^3}{3\Lambda}=0 \quad (4)$$

$$\frac{\partial}{\partial z}\left(\Lambda_2 q\frac{\partial\overline{v'v'}}{\partial z}\right)-\frac{q}{\Lambda}\left(\overline{v'v'}-\frac{q^2}{3}\right)-\frac{2\beta bq\overline{v'v'}}{\Lambda}-(1-\beta)\frac{2bq^3}{3\Lambda}=0 \quad (5)$$

$$2g\frac{\overline{w'T'}}{T_0}+3\frac{\partial}{\partial z}\left(\Lambda_2 q\frac{\partial\overline{w'w'}}{\partial z}\right)+2\frac{\partial}{\partial z}\left(\Lambda_3 q\frac{\partial\overline{w'w'}}{\partial z}\right)-\frac{q}{\Lambda}\left(\overline{w'w'}-\frac{q^2}{3}\right)-2\beta bq\frac{\overline{w'w'}}{\Lambda}-(1-\beta)\frac{2bq^3}{3\Lambda}=0 \quad (6)$$

$$-\overline{w'w'}\frac{\partial u}{\partial z}+\frac{g}{T_0}\overline{u'T'}-\frac{q}{\Lambda}\overline{u'w'}(1+2\beta b)=0 \quad (7)$$

$$-\overline{u'w'}\frac{\partial T}{\partial z}-\overline{w'T'}\frac{\partial u}{\partial z}+\frac{\partial}{\partial z}\left(\Lambda_2 q\frac{\partial\overline{u'T'}}{\partial z}\right)-\frac{Aq\overline{u'T'}}{\Lambda}=0 \quad (8)$$

$$-\overline{w'w'}\frac{\partial T}{\partial z}+\frac{g}{T_0}\overline{T'^2}-\frac{Aq\overline{w'T'}}{\Lambda}=0 \quad (9)$$

$$-2\overline{w'T'}\frac{\partial T}{\partial z}+\frac{\partial}{\partial z}\left(\Lambda_2 q\frac{\partial\overline{T'^2}}{\partial z}\right)-\frac{2bsq\overline{T'^2}}{\Lambda}=0. \quad (10)$$

The first terms in Eqs. (4)–(10) not involving the Λ length scales are the turbulent production terms. The terms involving Λ_2 are the model representation for the gradients of third-order correlations. The terms with Λ_3 are the representation of the gradients of the $\overline{p'u_i'}$ correlations. The first terms with Λ in the denominator are the tendency-toward-isotropy terms, and the last terms involving b are the disssipation terms for high turbulent Reynolds numbers. When $\beta=0$, the dissipation terms are modeled as isotropic. Note, that since the third-order correlations are modeled as proportional to the gradient of the second-order terms, these correlations do not appear in Eqs. (7) and (9) for the constant shear stress $\overline{u'w'}$ and the constant heat flux $\overline{w'T'}$.

Here the model used by Donaldson (1973) has been slightly generalized. The equations reduce to the same if we set $\beta=1$, $A=1+2b$, and $s=1$. Only b, Λ_2/Λ and Λ_3/Λ have been obtained from previous computer parameter searches. The numerical results given here are for isotropic dissipation where $\beta=0$. Results for $\beta=1$ are given in Lewellen and Teske (1972). The value of the parameter A will be chosen to give the desired turbulent Prandtl number and s will be chosen to give the correct scale to $\overline{T'^2}$.

Since $\overline{u'u'}$ and $\overline{v'v'}$ enter the remaining equations only in terms of $q=(\overline{u'u'}+\overline{v'v'}+\overline{w'w'})^{\frac{1}{2}}$, Eqs. (4) and (5) may be eliminated in favor of one equation for q^2 obtained by adding Eqs. (4), (5) and (6). Eqs. (7) and (10) are algebraic expressions for $\partial u/\partial z$ and $\partial T/\partial z$ which may be used to eliminate these derivatives. This results in a set of four ordinary differential equations for q^2, $\overline{w'w'}$, $\overline{u'T'}$ and $\overline{T'^2}$.

Normalization of the variables in terms of the Monin-Obukhov scaling permits this set to reduce to a universal set, i.e., to eliminate the explicit dependence on $\overline{u'w'}$ and $\overline{w'T'}$. To accomplish this task we introduce the following dimensionless variables:

$$\left.\begin{aligned} &\xi=-\frac{g\overline{w'T'}}{T_0|\overline{u'w'}|^{\frac{3}{2}}}z, && l=-\frac{g\overline{w'T'}}{T_0|\overline{u'w'}|^{\frac{3}{2}}}\Lambda \\ &Q=q/|\overline{u'w'}|^{\frac{1}{2}}, && WW=\overline{w'w'}/|\overline{u'w'}| \\ &UT=\overline{u'T'}/\overline{w'T'}, && TT=\frac{\overline{T'^2}|\overline{u'w'}|}{(\overline{w'T'})^2} \end{aligned}\right\}. \quad (11)$$

We have chosen not to follow the usual procedure of using the von Kármán constant k in the scale length:

$$\text{Monin-Obukhov length}=-\frac{T_0|\overline{u'w'}|^{\frac{3}{2}}}{kg\overline{w'T'}}\equiv L. \quad (12)$$

This would unnecessarily introduce several k's into our dimensionless equations. We also restrict our attention to the case where $\overline{u'w'}=-|\overline{u'w'}|$, which corresponds to a positive gradient of mean velocity with respect to z. The resulting equations are

$$\frac{2}{WW}\left[\frac{Q(1+2\beta b)}{l}-UT\right]-2+\frac{\partial}{\partial\xi}\left(l_2Q\frac{\partial Q^2}{\partial\xi}\right)+2\frac{\partial}{\partial\xi}\left(l_2Q\frac{\partial WW}{\partial\xi}\right)+2\frac{\partial}{\partial\xi}\left(l_3Q\frac{\partial WW}{\partial\xi}\right)-\frac{2bQ^3}{l}=0, \quad (13)$$

$$-2+3\frac{\partial}{\partial\xi}\left(l_2Q\frac{\partial WW}{\partial\xi}\right)+2\frac{\partial}{\partial\xi}\left(l_3Q\frac{\partial WW}{\partial\xi}\right)$$
$$-\frac{Q}{l}(1+2\beta b)WW+\frac{Q^3}{3l}[1-2b(1-\beta)]=0, \quad (14)$$

$$-\frac{1}{WW}\left[TT+\frac{Q}{l}(1+2\beta b+A)-UT\right]$$
$$+\frac{\partial}{\partial\xi}\left(l_2Q\frac{\partial UT}{\partial\xi}\right)-\frac{AQUT}{l}=0, \quad (15)$$

$$\frac{2}{WW}\left(TT+\frac{AQ}{l}\right)+\frac{\partial}{\partial\xi}\left(l_2Q\frac{\partial TT}{\partial\xi}\right)-\frac{2bsTTQ}{l}=0. \quad (16)$$

Integration of these four equations with the model parameters specified over the interval $-\infty<\xi<+\infty$ should produce universal functions of ξ which may then be compared with those obtained experimentally. Positive ξ corresponds to negative $\overline{w'T'}$ and thus to stable stratification conditions, while negative ξ represents unstable stratification. The Richardson number, defined as

$$\mathrm{Ri}=\frac{g}{T_0}\frac{\partial T}{\partial z}\Big/(\partial u/\partial z)^2, \quad (17)$$

is a universal function of ξ given in our dimensionless variables as

$$\mathrm{Ri}=\frac{WWl(lTT+QA)}{[(1+2\beta b)Q-lUT]^2}. \quad (18)$$

The scale parameter l has the same sign as ξ, so that Ri has the same sign as ξ.

For boundary conditions, we require that: (i) as $\xi\rightarrow\infty$, a critical Richardson number is approached and the variables WW, UT, TT and Q all approach constant values; (ii) as $\xi\rightarrow-\infty$, free convection exists so that the velocity gradient approaches zero; and (iii) at $\xi=0$, no singularities exist in the turbulence characteristics or their derivatives.

To complete the specification of the model, the scales l, l_2, l_3 must be determined. Near $z=0$, we expect the principal eddy size to be proportional to z. Therefore we expect to have

$$l=\alpha\xi, \quad \text{as } \xi\rightarrow 0, \quad (19)$$

with α a constant. This relationship is probably correct for all $\xi<0$, since as $\xi\rightarrow-\infty$, the physical problem corresponds to local free convection where we still expect the eddy size to scale with distance from the surface. However, for $\xi>0$, we expect to reach an upper bound on Ri at which the turbulence should be damped. This implies an upper bound on l. We will designate this upper bound as $\alpha\xi_1$ so that

$$\left.\begin{array}{ll} l=\alpha\xi, & \xi\leqslant\xi_1 \\ l=\alpha\xi_1, & \xi>\xi_1 \end{array}\right\}. \quad (20)$$

Both l_2 and l_3 will be taken as everywhere proportional to l.

As previously mentioned, b, l_2/l and l_3/l have been obtained by fitting the invariant model to other flows ($b=0.125$, $l_2/l=0.3$, $l_3/l=-0.3$). The scale length slope α is chosen as 0.6 to yield a von Kármán constant compatible with the data by Businger *et al.* (1971), i.e.,

$$k=|\overline{u'w'}|^{\frac{1}{2}}\Big/\left(\xi\frac{\partial u}{\partial\xi}\right)_{\xi=0}=0.36, \quad (21)$$

which is somewhat less than the generally accepted laboratory value of 0.4. The value of A is chosen as 0.75 so that the ratio of the temperature gradient to the velocity gradient under neutral conditions agrees with the experimental data. The value of s is chosen as 1.8 to fix the magnitude of $\overline{T'^2}$. The upper bound on the scale length $\alpha\xi_1$ is chosen as 0.55 to yield a critical Richardson number of 0.20. This corresponds to requiring that the eddy length scale not exceed more than $\sim$20% of the Monin-Obukhov length.

The resulting curves are relatively insensitive to values of l_2/l and l_3/l, most sensitive to the value of b, and moderately sensitive to the other constants.

With the values of the variables determined at $\xi=-\infty$, 0 and $+\infty$ by the boundary conditions, it is possible to numerically integrate Eqs. (13)–(16) across the complete stability range by applying a forward-time, centered-space, implicit finite-difference technique to the unsteady equation set. This set is constructed by adding $\partial Q^2/\partial t$, $\partial WW/\partial t$, $\partial UT/\partial t$ and $\partial TT/\partial t$ to the right-hand sides of Eqs. (13)–(16), respectively. The pseudo-time t is then incremented from $t=0$ until a steady solution results.

3. Model results compared with data

The numerical results are compared with the data of Businger *et al.* (1971) in Figs. 1–3 and with the complementary data of Wyngaard *et al.* (1971) in Figs. 4–6. In order to facilitate comparison with the data, the coordinate z has been normalized with the Monin-Obukhov length defined in Eq. (12).

The dimensionless velocity gradient in Fig. 1, the dimensionless temperature gradient in Fig. 2, and the vertical velocity fluctuations in Fig. 4 agree remarkably well with the experimental data. The temperature fluctuations in Fig. 6 agree reasonably well although the predictions are low on the unstable side and high on the stable side. The horizontal heat flux in Fig. 5 appears to agree with the experiments at

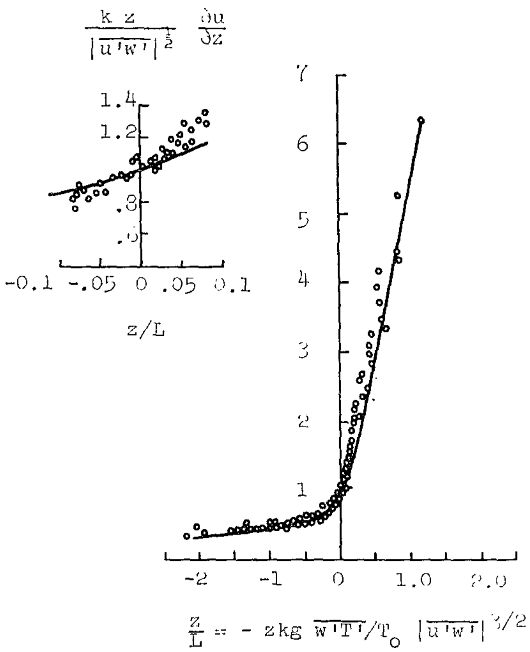

FIG. 1. Normalized velocity gradient as a function of the Monin-Obukhov similarity variable: model predictions, solid lines; observations (circles) [from Businger *et al.* (1972)].

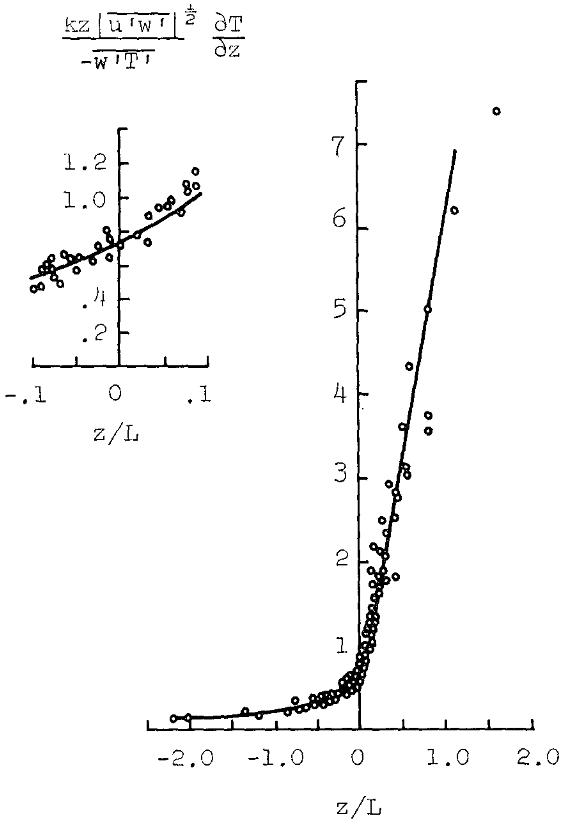

FIG. 2. Normalized temperature gradient as a function of the Monin-Obukhov similarity variable: model predictions, solid lines; observations (circles) [from Businger *et al.* (1972)].

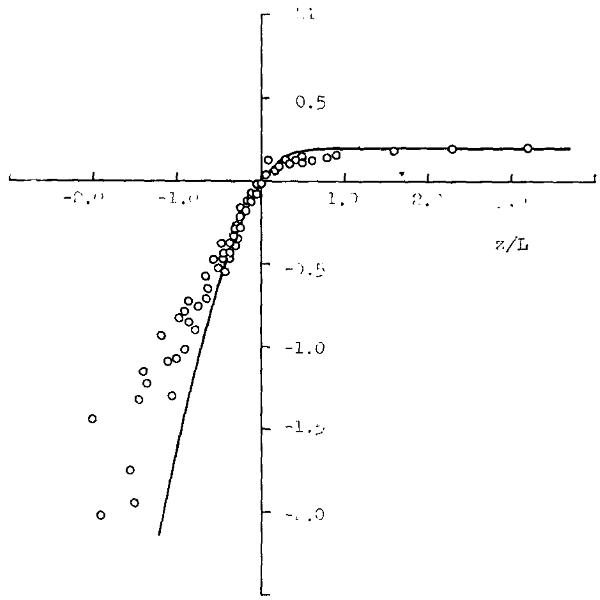

FIG. 3. Richardson number as a function of the Monin-Obukhov similarity variable: model prediction, solid line; observations (circles) [from Businger *et al.* (1972)].

both large positive ξ and negative ξ but does not show the peak at the neutral value of $\xi=0$ as evident in the data. The Richardson number variation with height (Fig. 3) shows rather good agreement across the full range. Note that the small discrepancies in velocity gradient on the unstable side (Fig. 1) are responsible for the large discrepancies in Ri in the same region. The overall agreement in the six figures is quite favorable. The most significant discrepancies between the model and the data are *TT* and *UT* at $\xi=0$. However, as explained by Wyngaard *et al.*,

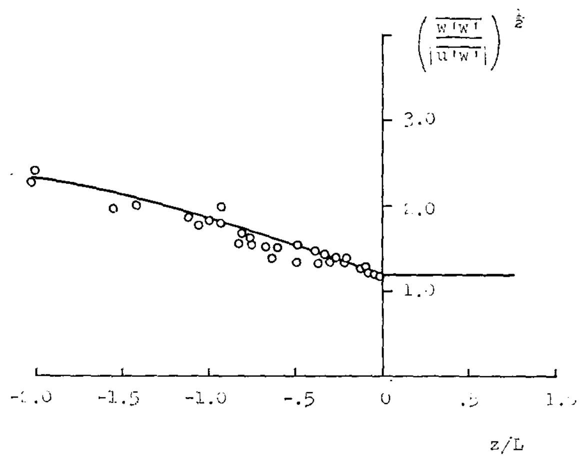

FIG. 4. Dimensionless rms vertical velocity fluctuations as a function of the Monin-Obukhov similarity variable: model prediction, solid line; observations (circles) [from Wyngaard *et al.* (1971)].

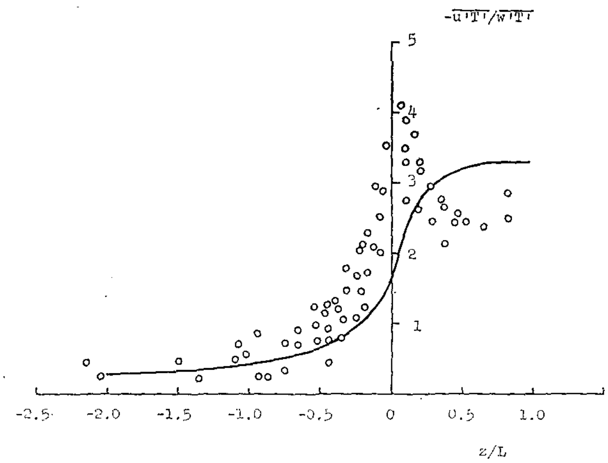

FIG. 5. Ratio of horizontal to vertical heat flux as a function of the Monin-Obukhov similarity variable: model prediction, solid line; observations (circles) [from Wyngaard *et al.* (1971)].

the cusps near $\xi=0$ in their data may reflect the effect of measuring heat fluxes under conditions near transition rather than at steady, near-neutral conditions.

It may be argued that we have forced the good agreement between model and data by the choice of our model constants. This is certainly true to some extent, since this much-studied flow was chosen for the purpose of determining the constants. However, all of the constants except $\alpha\xi_1$ are determined by conditions at $\xi=0$ so the faithful reproduction of the general shape of the curves is significant.

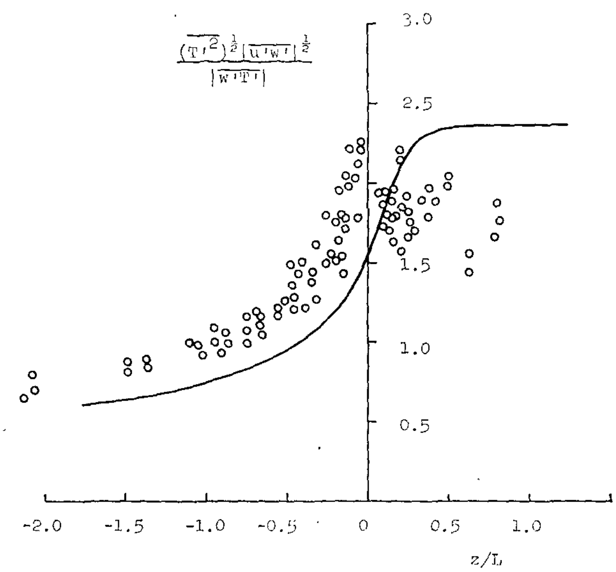

FIG. 6. Dimensionless rms temperature fluctuations as a function of the Monin-Obukhov similarity variable: model prediction, solid line; observations (circles) [from Wyngaard *et al.* (1971)].

The constant $\alpha\xi_1$ may best be interpreted as a limit on the Richardson number based on eddy scale length and turbulent energy, i.e.,

$$-\frac{\Lambda^2 g}{q^2 T_0}\frac{\partial T}{\partial z}\leqslant 0.25. \tag{22}$$

For a fixed energy this sets a limit on the size of the eddy which can overcome a fixed gravitational force.

If we were to compare the model values of the third-order correlations with the limited data available, the comparison would not be too good. With our gradient diffusion model we certainly cannot predict the counter-gradient diffusion sometimes reported under unstable conditions. This would require a model with many more empirical constants such as Lumley's (Lumley and Khajeh-Nouri, 1973). The intent of Donaldson's model is not to predict all the details of higher-order correlations, but to obtain reasonable predictions for the second-order correlations over a wide variety of conditions.

4. Conclusions

The agreement between model results and experimental observations demonstrates that Donaldson's invariant model of the second-order turbulence equations can provide reasonable results across the complete range of stratification conditions from highly stable to highly unstable. The closest agreement is obtained when the coefficient of the tendency-toward-isotropy term in the heat flux equation is about 25% less than the corresponding term in the shear stress equation, the coefficient of the dissipation in the temperature variance equation is about twice the dissipation in the turbulent energy equation, and the turbulent length scale is limited to less than 20% of the Monin-Obukhov length under stable conditions.

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REFERENCES

Bradshaw, P., 1972: The understanding and prediction of turbulent flow. *Aeronaut. J.*, **76**, 403–418.

Businger, J. A., J. C. Wyngaard, Y. Izumi and E. F. Bradley, 1971: Flux-profile relationships in the atmospheric surface layer. *J. Atmos. Sci.*, **28**, 181–189.

Donaldson, Coleman duP., 1972: Calculation of turbulent shear flows for atmospheric and vortex motions. *AIAA J.*, **10**, 4–12.

——, 1973: Construction of a dynamic model of the production of atmospheric turbulence and the dispersal of atmospheric pollutants. *Workshop on Micrometeorology*, Boston, Amer. Meteor. Soc., 313–390.

Lewellen, W. S., and M. Teske, 1972: Prediction of the Monin-

Obukhov similarity functions from an invariant model of turbulence. A.R.A.P. Rept. No. 185.

Lumley, J. L., and B. Khajeh-Nouri, 1973: Computational modeling of turbulent transport. *Proc. Second IUTAM–IUGG Symp. Turbulent Diffusion in Environmental Pollution* (in press).

Mellor, G. L., 1973: Analytic prediction of the properties of stratified planetary surface layers. *J. Atmos. Sci.*, **30**, 1061–1069.

Monin, A. S., and A. M. Obukhov, 1953: Dimensionless characteristics of turbulence in the atmospheric surface layer. *Dokl. Akad. Nauk SSSR*, **93**, 223–226.

——, and A. M. Yaglom, 1971: *Statistical Fluid Mechanics.* Cambridge, The MIT Press, 417–526 pp.

Wyngaard, J. C., O. R. Coté and Y. Izumi, 1971: Local free convection, similarity, and the budgets of shear stress and heat flux. *J. Atmos. Sci.*, **28**, 1171–1182.