The Equilibrium Energy Spectrum of Randomly Forced Two-Dimensional Turbulence

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ABSTRACT

From a statistical mechanical treatment of an ensemble of randomly forced two-dimensional flows of a viscous fluid, we derive two independent integral constraints on the form of the equilibrium energy spectrum. With a single hypothesis about the shape-preserving properties of the spectrum, those constraints determine the spectrum to within the value of a single universal dimensionless constant. In all other respects the argument is deductive and does not depend on closure approximations or hypotheses about the process of nonlinear energy transfer. The spectrum exhibits minus third-power dependence for small scales, but minus first-power dependence for large scales. It is in good agreement with the results of detailed numerical integrations of the Navier-Stokes equations.

1. Introduction

Over the past few years, the question of the predictability of the earth's atmosphere has been pursued with renewed interest, mainly because it is inextricably bound up with such eminently practical matters as the design of an economical and effective system of meteorological observation on a global scale. It is evident, for example, that one of the most serious and not totally removable uncertainties of prediction stems from the impossibility of observing the state of the atmosphere in complete detail. Thus, it has become increasingly important to determine how the atmosphere's predictability (not in an ultimate, but practical sense) depends on the frequency and spatial density of measurements or samples of the atmosphere's local state.

In the earliest attempts to deal with this question analytically, Thompson (1957) and Novikov (1960) considered the problem primarily from the standpoint of the stability (or instability) of non-equilibrium initial states to random perturbations of the initial state. Both of these treatments also contained some elements of the process of nonlinear transfer of kinetic energy between different scales of motion; neither of them, however, dealt explicitly with the eventual "transfer of uncertainty" from scales below the limit of observational resolution to much larger scales, due solely to turbulent transfer of energy from small to large scales of motion.

The latter phenomenon, first viewed in the context of the predictability problem by Robinson (1969), has been studied in greater detail and in a broader and more realistic theoretical framework by Lorenz (1969) and by Leith and Kraichnan (1972). One of the principal results of those studies is that the "transfer of uncertainty" to large scales depends crucially on the mechanism of nonlinear transfer and on the character of the kinetic energy spectrum. If, for example, the energy spectrum of the atmosphere obeyed Kolmogoroff's famous "$-5/3$ power law" for three-dimensional turbulence, the behavior of the atmosphere would probably be less "predictable in principle" than it is now actually predicted. If, on the other hand, the atmosphere followed the "-3 power law" proposed by Batchelor (1969), Kraichnan (1967) and Leith (1968), its large-scale features would be predictable over periods several times longer, with the same spatial resolution of its initial state. It has become a matter of considerable concern, therefore, to describe the statistical properties of the atmosphere's large-scale motions more exactly.

A simple and plausible model of the earth's atmosphere, viewed from the standpoint of its large- and medium-scale motions, is a homogeneous and incompressible fluid, whose motions are purely horizontal and forced by a random distribution of sources and sinks of vorticity. It has long been recognized that motions on scales of more than a few tens of kilometers are very nearly horizontal, that those motions are quasi-nondivergent, and that surfaces of constant density are almost horizontal. Moreover, the mechanism of baroclinic instability, by which available potential energy is converted into the kinetic energy of large-scale vortices, operates sporadically and at more or less randomly distributed times and places, but on a fairly narrow range of scales. Thus, one is tempted to think of the atmosphere as a fluid in two-dimensional motion, driven by random injections of vortices whose scale is near the wavelength of maximum baroclinic instability.

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It was pointed out by Fjörtoft (1953) that two-dimensional flow is peculiar in that there are *two* quadratic invariants—total kinetic energy and enstrophy—in the absence of forcing and viscous dissipation and that, in consequence, kinetic energy must be transferred simultaneously from small to larger scales and large to smaller scales through nonlinear interactions between different scales of motion. This peculiarity has led Batchelor (1969), Kraichnan (1967) and Leith (1968) to postulate the existence of two distinct "inertial" subranges of two-dimensional turbulence, above and below the scales at which energy is supplied to the system. The kinetic energy spectra in these two subranges, as inferred by dimensional arguments analogous to those advanced by Kolmogoroff (1941) and Batchelor (1946) for the case of three-dimensional turbulence, are also distinctly different. The large-scale subrange, characterized by negligible enstrophy transfer, has an energy spectrum that is determined entirely by the energy dissipation and depends on the $-5/3$ power of scalar wavenumber. The small-scale subrange, on the other hand, is distinguished by negligible energy transfer and has an energy spectrum that depends only on the enstrophy dissipation. The latter assumption implies that the kinetic energy varies as the -3 power of scalar wavenumber.

Some of these conjectures have been confirmed remarkably well by two different types of results. First, the -3 law is very well substantiated by detailed numerical integrations of the two-dimensional Navier-Stokes equations (Lilly, 1969) and by numerical experiments with a considerably more realistic three-dimensional model of atmospheric motion, carried out by Wellck *et al.* (1971). Second, the kinetic energy spectra computed by Wiin-Nielsen (1967) and Kao *et al.* (1966) from actual observations of atmospheric motions also show an approximately -3 power dependence at scales smaller than the wavelength of maximum baroclinic instability ($\sim$4000 km). It would thus appear that the question is essentially resolved.

It should be pointed out, however, that all analytical or numerical approaches to a statistical theory of two-dimensional turbulence have thus far suffered from one or more of several defects. In the case of direct numerical integration of the Navier-Stokes equations, the principal drawback is that truncation error in the smallest scales of motion (particularly at high Reynolds numbers) may invalidate the calculation. On the purely theoretical side, analytic treatments of turbulence have been based either on physical hypotheses (arguments of dimensionality or "localness") or on mathematical closure approximations.

The purpose of this paper is to derive the equilibrium spectrum of randomly forced two-dimensional turbulence from the Navier-Stokes equations, subject to a single hypothesis about its universality, but without closure approximations or appeal to purely dimensional argument. From a statistical-mechanical formulation of the problem, one can extract two fundamental integral constraints that, taken together, determine a unique shape-preserving kinetic energy spectrum. This spectrum approaches the -3 power law asymptotically at small scales and a -1 power law at large scales. It agrees very well with Lilly's (1969) numerical integrations, even to details of fine structure in the neighborhood of the scale of forcing.

2. Two integral constraints

In a recent paper, Thompson (1972) has considered the statistical properties of two-dimensional flows of a homogeneous and incompressible, but viscous fluid whose motions are forced by randomly varying sources and sinks of vorticity. In essence, the approach was to represent the streamfunction as an infinite series of space-dependent (but time-independent) orthogonal functions whose coefficients or amplitudes depend only on time, and to derive from the Navier-Stokes equations a simultaneous system of ordinary differential equations governing the time evolution of the amplitudes.

The second and crucial step in the derivation was to regard the amplitudes of the orthogonal modes as the coordinates of a point (in an infinite-dimensional phase space), whose position corresponds to the state of one particular realization of the system at a single moment in time. With this representation, the time evolution of a single realization is described by the trajectory of a single point, whose parametric equations of motion in the phase space are just the evolution equations for the amplitudes.

Finally, introducing the simplest ideas of classical statistical mechanics, we require conservation of points or realizations in the phase space. In the paper referred to earlier (Thompson, 1972), it was shown that the continuity equation for the phase flow takes the form

$$\frac{\partial\rho}{\partial t}+\sum_{k=1}^{\infty}\frac{\partial}{\partial X_k}\rho(N_k-\nu\alpha_k{}^2X_k)=\sum_{k=1}^{\infty}\mu_k\frac{\partial^2\rho}{\partial X_k{}^2},\qquad(1)$$

where X_k is the "speed" amplitude of the kth mode, ρ the probability distribution in X_k space, ν the coefficient of kinematic viscosity, and α_k the eigenvalue corresponding to the kth mode. The nonlinear effects are bound up in N_k, which has the form

$$N_k=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\beta_{ijk}\alpha_j{}^2X_iX_j.\qquad(2)$$

The constant μ_k is the coefficient of diffusion of points in the X_k direction in the phase space, due solely to randomly varying speeds in the X_k direction, and is proportional to

$$\langle f_k{}^2\rangle\int_0^{\infty}\Phi_k(\tau)d\tau,$$

where $f_k(t)$ is the randomly varying forcing function for the kth mode and $\Phi_k(\tau)$ is its normalized autocorrelation function. The angle brackets denote the ensemble average.

The nonlinear interaction coefficients β_{ijk} have several distinctive properties, which follow directly from the particular kind of nonlinearity displayed by the Navier-Stokes equations. The most important features are that β_{ijk} vanishes if any two indices are equal, is invariant under cyclic permutation of indices, and changes sign under non-cyclic permutation of indices. The consequences of these properties will be noted later in this section.

Although it was shown that there are exact and unique equilibrium solutions of Eq. (1) for an infinite family of types of random forcing, it is clear that those cases are highly special, in that there is no *net* nonlinear transfer of energy into or out of any mode. Thus those solutions are of limited interest, considered from the standpoint of turbulent transfer of energy.

It is also apparent, however, that some interesting and more general statistical properties of two-dimensional turbulence can be extracted from Eq. (1), without solving it explicitly for the probability distribution. For example, let us multiply Eq. (1) by $F(X_p)$ and integrate over the entire phase space. Integrating by parts and noting that ρ vanishes as X_k approaches infinity, we may write the condition for statistical equilibrium as

$$\int_V \rho\left[\frac{\partial F}{\partial X_p}(N_p-\nu\alpha_p{}^2X_p)+\mu_p\frac{\partial^2 F}{\partial X_p{}^2}\right]dV=0, \qquad (3)$$

in which the integration over V is taken over the whole of the phase space and dV is a "volume" element of the phase space.

Specializing the general equilibrium condition stated in Eq. (3), we now let $F(X_p)=\frac{1}{2}X_p{}^2$, with the result that

$$\int_V \rho[X_pN_p-\nu\alpha_p{}^2X_p{}^2+\mu_p]dV=0,$$

or, since ρdV is just the probability that any particular realization is located within the "volume" element dV in the neighborhood of X_k,

$$\langle X_pN_p\rangle-\nu\alpha_p{}^2\langle X_p{}^2\rangle+\mu_p=0.$$

Thus, summing over all p, we have

$$\langle\sum_{p=1}^{\infty} X_pN_p\rangle-\nu\sum_{p=1}^{\infty}\alpha_p{}^2\langle X_p{}^2\rangle+\sum_{p=1}^{\infty}\mu_p=0. \qquad (4)$$

Examining the first term of the equation above, we note that Eq. (2) implies that

$$\sum_{p=1}^{\infty} X_pN_p=\sum_{j=1}^{\infty}\alpha_j{}^2X_j\sum_{i=1}^{\infty}\sum_{p=1}^{\infty}\beta_{ijp}X_iX_p.$$

This summation clearly vanishes, since the summand reverses sign under the exchange (a noncyclic permutation) of the indices i and p and because $\beta_{iji}=\beta_{pjp}=0$. Accordingly, Eq. (4) reduces to

$$\nu\sum_{p=1}^{\infty}\alpha_p{}^2\langle X_p{}^2\rangle=\sum_{i=1}^{\infty}\mu_p. \qquad (5)$$

This is one of two exact constraints on two-dimensional turbulence.

Letting the arbitrary function $F(X_p)=\frac{1}{2}\alpha_p{}^2X_p{}^2$, we find that a similar calculation leads to

$$\langle\sum_{p=1}^{\infty}\alpha_p{}^2X_pN_p\rangle-\nu\sum_{p=1}^{\infty}\alpha_p{}^4\langle X_p{}^2\rangle+\sum_{p=1}^{\infty}\mu_p\alpha_p{}^2=0. \qquad (6)$$

Again, from Eq. (2) we have

$$\sum_{p=1}^{\infty}\alpha_p{}^2X_pN_p=\sum_{i=1}^{\infty}X_i\sum_{j=1}^{\infty}\sum_{p=1}^{\infty}\beta_{ijp}\alpha_j{}^2\alpha_p{}^2X_jX_p.$$

But this summation also vanishes, simply because the summand reverses sign under exchange of j and p and $\beta_{ijj}=\beta_{ipp}=0$. Eq. (6) thus reduces to

$$\nu\sum_{p=1}^{\infty}\alpha_p{}^4\langle X_p{}^2\rangle=\sum_{p=1}^{\infty}\mu_p\alpha_p{}^2. \qquad (7)$$

This is an independent but also exact constraint on two-dimensional turbulence. Eqs. (5) and (7), which are completely general and were derived without approximation, will be recognized as the conditions for balance between the kinetic energy and enstrophy generated by random sources of vorticity and the molecular dissipation of energy and enstrophy in statistical equilibrium.

Let us next suppose that the spatial domain occupied by the flow is very large and very nearly square in shape, so that the distribution of the modes in wavenumber space is uniformly dense. Further, we also suppose that the flow is isotropically forced in a narrow band of wavenumbers of width Δk centered on wavenumber k_0. In the limit, the sums that appear in Eqs. (5) and (7) become integrals over wavenumber space, i.e.,

$$\nu\int_\lambda^\infty k^2E(k)dk=\mu k_0\Delta k, \qquad (8)$$

$$\nu\int_\lambda^\infty k^4E(k)dk=\mu k_0{}^3\Delta k, \qquad (9)$$

where k is the scalar wavenumber and $E(k)$ is the kinetic energy contained in a ring of unit width and radius k in wavenumber space. The wavenumber λ corresponds to the lowest mode and is independent of k_0. The pair of equations (8) and (9) is the fundamental theoretical basis for the remaining development.

3. The equilibrium energy spectrum

We next proceed to show that, with an hypothesis about the universality of the energy spectrum, the form of the equilibrium spectrum is determined by Eqs. (8) and (9). Let us first note that, if $\Delta k \ll k_0$, the natural unit for scaling k (in the low-wavenumber domain over which viscous effects are relatively unimportant) is just k_0. Thus, it is not unreasonable to suppose that the energy spectrum below the dissipation range has the "universal" form

$$E(k)=E_0 f(r), \tag{10}$$

where $r=k/k_0$, $f(r)$ is a universal "shape" function that depends only on r, and E_0 depends only on the externally fixed parameters μ, ν, Δk and k_0. This supposition will later be justified by verifying its consequences.

Introducing the hypothesis stated above, we differentiate Eq. (8) with respect to k_0. We then have

$$\frac{dE_0}{dk_0}\int_\lambda^\infty k^2 f dk - E_0\int_\lambda^\infty \frac{k^{3\prime} f}{k_0^2}dk=\frac{\mu\Delta k}{\nu},$$

in which the prime denotes differentiation with respect to r. Thus, eliminating $\mu\Delta k/\nu$ between Eq. (8) and the equation above, we have

$$\frac{dE_0}{dk_0}\int_\lambda^\infty k^2 f dk - E_0\int_\lambda^\infty \frac{k^3 f'}{k_0^2}dk=E_0\int_\lambda^\infty \frac{k^2 f}{k_0}dk. \tag{11}$$

Similarly, turning to a completely independent constraint, we differentiate Eq. (9) with respect to k_0 and resubstitute from Eq. (9) with the result that

$$\frac{dE_0}{dk_0}\int_\lambda^\infty k^4 f dk - E_0\int_\lambda^\infty \frac{k^5 f'}{k_0^2}dk=3E_0\int_\lambda^\infty \frac{k^4 f}{k_0}dk. \tag{12}$$

Thus, regarding Eqs. (11) and (12) as a simultaneous system of two linear homogeneous algebraic equations in which the variables are E_0 and dE_0/dk_0, the condition for the existence of a nontrivial solution of this system is

$$\frac{\int_\lambda^\infty \left(\frac{k^5 f'}{k_0^2}+\frac{3k^4 f}{k_0}\right)dk}{\int_\lambda^\infty \left(\frac{k^3 f'}{k_0^2}+\frac{k^2 f}{k_0}\right)dk}=\frac{\int_\lambda^\infty k^4 f dk}{\int_\lambda^\infty k^2 f dk}. \tag{13}$$

Eqs. (8) and (9) imply, however, that

$$\frac{\int_\lambda^\infty k^4 f dk}{\int_\lambda^\infty k^2 f dk}=k_0^2,$$

so that the preceding equation may be written as

$$\int_{\lambda/k_0}^\infty \phi(r)dr=0, \tag{14}$$

where

$$\phi(r)=r^3\left[(1-r^2)f'-\left(3r-\frac{1}{r}\right)f\right]. \tag{15}$$

Taken together, Eqs. (14) and (15) determine the shape of the energy spectrum.

To prove the latter point, let us next differentiate Eq. (14) with respect to k_0. Since k_0 now enters explicitly only in the limits of integration, Eq. (14) implies that

$$\frac{\lambda}{k_0^2}\phi\left(\frac{\lambda}{k_0}\right)=0,$$

whence ϕ vanishes for all k_0 or all r. Accordingly, Eq. (15) implies that the shape function $f(r)$ is determined by the ordinary differential equation[2]

$$(1-r^2)f'=\left(3r-\frac{1}{r}\right)f. \tag{16}$$

The general solution of this equation is easily found by quadratures. It is just

$$f(r)=\frac{C}{r(r-1)(r+1)}, \tag{17}$$

where C is a dimensionless constant, either positive or negative.

The reader will undoubtedly have perceived that there is an apparent inconsistency between the form of $f(r)$ and the integral constraints from which it was derived: $f(r)$ has a singularity at the wavenumber at which the system is being forced.[3] Lacking any rigorous analytical method of treating the time-evolution of the probability distribution, we conjecture that the "equilibrium" spectrum corresponds to an asymptotic state which is never exactly achieved in a finite period of time. We do not, therefore, speculate about the exact behavior of $E(k)$ near $k=k_0$. Recognizing, however, that $E(k_0)$ is finite in actuality, it is reasonable to impose a continuity

[2] It should be noted that the shape function $f(r)$ is determined (except for its "amplitude") by the homogeneous Eq. (16), and that Eq. (16) implies that the integrals $\int_\lambda^\infty k^4 f dk$ and $\int_\lambda^\infty k^2 f dk$ are both divergent. Thus, although it is legitimate to require that their *ratio* be k_0^2, it is not admissible to impose the stronger condition $\int_\lambda^\infty k^2(k^2-k_0^2)fdk=0$ as an independent constraint. Following the argument above, the latter constraint would lead only to the trivial and physically unrealistic solution $f(r)=0$.

[3] The formal mathematical difficulty can be removed by leaving $f(r)$ undefined in a very narrow but finite range $1-\epsilon<r<1+\epsilon$. We require only that $f(r)$ is finite in that range and satisfies a "continuity" condition $f(1+\epsilon)=f(1-\epsilon)$. The development proceeds in the same way, but the results are approximations that depend on the smallness of ϵ. The main conclusions are unaffected.

condition, namely, that $f'(1+\epsilon)=-f'(1-\epsilon)$. The latter condition implies that

$$f(r)\approx\begin{cases}\dfrac{C}{r(r-1)(r+1)}, & r>1\\[2ex] \dfrac{C}{r(1-r)(1+r)}, & r<1\end{cases}$$

Finally, we observe that Eqs. (8) and (9) may be rewritten as

$$\int_{\lambda}^{\infty} r^2E_0f(r)dr=\frac{\mu\Delta k}{\nu k_0^2},$$

$$\int_{\lambda}^{\infty} r^4E_0f(r)dr=\frac{\mu\Delta k}{\nu k_0^2}.$$

These relationships suggest that E_0 is proportional to $\mu\Delta k/(\nu k_0^2)$ by a dimensionless factor, in which case the energy spectrum is given by

$$E(k)=\begin{cases}\dfrac{C\mu k_0\Delta k}{\nu k(k-k_0)(k+k_0)}, & k>k_0\\[2ex] \dfrac{C\mu k_0\Delta k}{\nu k(k_0-k)(k_0+k)}, & k<k_0\end{cases}\tag{18}$$

in which C is presumably a universal dimensionless constant, independent of k_0 or any other external conditions of the system.

4. Conclusions and comparison with independent results

The most distinctive features of the energy spectrum given by Eq. (18) are as follows:

1) The kinetic energy per unit scalar wavenumber rapidly approaches the -3 power of wavenumber above the wavenumber at which energy is injected into the system. This result, of course, is in accord with the dimensional arguments of Batchelor, Kraichnan and Leith.

2) The kinetic energy per unit scalar wavenumber approaches the -1 power of wavenumber below the wavenumber of concentrated forcing. This result does not agree with the $-5/3$ power law based on purely dimensional argument.

3) The energy spectrum has a sharp peak or "spike" at the narrow wavenumber band in which the energy input is concentrated.

These features are illustrated in Fig. 1, on which the solid curve is a plot of the theoretical value of $\log E(r)$ against $\log r$. For comparison, the spectrum obtained from Lilly's (1969) detailed numerical integrations is shown by the dashed curve, plotted on the same scale. The generally good agreement between the analytic and numerical results deserves little comment, except to emphasize that there is a 500-fold decrease of energy density between the largest and smallest scales of motion shown in the figure. Beyond that, it is worth pointing out that, despite the fact that Lilly's numerical model had not reached statistical equilibrium, his computed spectrum shows a sharp "spike" at the wavenumber of concentrated forcing. Moreover, the numerically computed spectrum at low wavenumbers is not inconsistent with the -1 power law predicted by our analytic result.

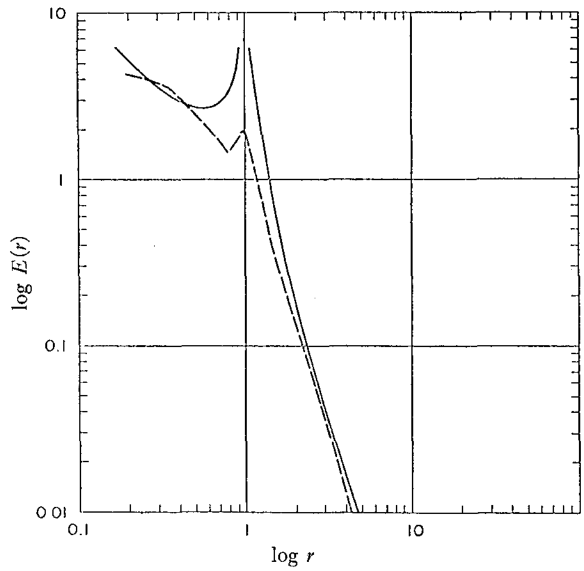

FIG. 1. Curves of $\log E(r)$ as a function of $\log r$. The theoretical result is shown by the heavy solid line, and the result based on Lilly's detailed numerical integrations is shown by the dashed line. Successive decades of the log scale are shown by the light solid lines.

As a further check, one can test the conclusion that C is a universal constant that is independent of k_0. Lilly's calculations for two cases (which differed only in that the wavenumber of concentrated random forcing was increased from 2 to 8) indicated that the energy density at wavenumber 20 was increased by a factor of about 4.5. The theoretical estimate, based on Eq. (18), predicts an increase by a factor of 4.7. The discrepancy is probably within the limits of transient departures from equilibrium in Lilly's numerical calculation. For comparisons with spectra computed from actual observations of the atmosphere, we again refer to the papers of Wiin-Nielsen (1967) and Kao *et al.* (1966), both of which show a minus third-power dependence at scales smaller than the wavelength of maximum baroclinic instability.

The principal conclusion is that an ensemble of two-dimensional randomly forced flows of a viscous fluid approach an asymptotic state of statistical equilibrium, but probably do not reach it rapidly and certainly not in a finite period of time. The "equilibrium" spectrum is evidently determined by two fundamental integral constraints, which are derivable from a statistical mechanical formulation of the problem and whose forms depend explicitly on the type of nonlinearity exhibited by two-dimensional flows. The results do not depend on closure approximations or hypotheses about the physical nature of the processes of nonlinear energy transfer.[4]

[4] It is natural to inquire about the propriety of making use of only the two quadratic invariants of unforced inviscid two-dimensional flow—*viz.* total kinetic energy and enstrophy—and ignoring the invariance of integrals of other powers or functions of vorticity. In the author's view, this type of flow is unique in the respect that the mere statement of invariance of both kinetic energy and enstrophy (together with mass conservation) implies the existence of all other known invariants. This result I proved to my own satisfaction several years ago, but assumed it was common knowledge. I will try to reconstruct the proof.

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