On the Growth Rate of an Unstable Disturbance in a Gravitationally Stratified Shear Flow

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ABSTRACT

An improved upper bound is given for the growth rate of an unstable disturbance in a gravitationally stratified compressible shear flow, with heat conduction and viscous effects neglected. It is also shown that such a growth rate depends essentially on the characteristics of the flow in the neighborhood of the critical layer.

1. Introduction

Chimonas (1970) has shown that some of the properties of a gravitationally stratified incompressible fluid carry over to compressible fluids. In particular, he has found that the Miles-Howard theorem (1961) may be generalized to compressible fluids, so that $n^2 \geqslant \frac{1}{4} U_0'^2$ throughout the flow is a sufficient condition for stability against infinitesimal adiabatic perturbations. Here $n^2 = -g(\rho_0'/\rho_0 + g/c^2)$ is the Brunt-Väisälä frequency, $\mathbf{g}$ is the gravitational acceleration acting in the negative z direction, ρ_0 the background density, $c(z)$ the sound speed, and U_0' the vertical gradient of the flow speed which is assumed to be in the x direction. A prime represents differentiation with respect to z. In addition, for perturbations of the form

$$A(x,y,z,t) = A(z) \exp[i(\omega t - k_x x - k_y y)], \qquad (1)$$

the complex phase velocity ω/k_x for any unstable mode must lie inside the semi-circle of diameter $D = U_{0\,\max} - U_{0\,\min}$ in the upper half-plane. The semi-circle theorem due to Howard (1961), and extended to compressible fluids by Eckart (1963), limits the maximum value for the growth rate ω_i. Another upper bound on ω_i is given by Howard (1961) in the form

$$\omega_i^2 \leqslant \left(\frac{U_0'^2}{4} - n^2 \right)_{\max}, \qquad (2)$$

where the maximum of the right-hand side can take place anywhere in the flow.

Inequality (2) does not depend on the horizontal wavenumber k_x since it only involves the Brunt-Väisälä frequency and the vertical gradient of the background flow. The numerical analysis of Miles and Howard (1964) of a particular case reveals that the maximum growth rate depends on k_x also, and one expects this fact to be true in general.

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The purpose of this note is to obtain a more stringent upper bound to the growth rate ω_i and to indicate that such an upper bound depends on the flow characteristics in the neighborhood of the critical layer, where the horizontal phase speed ω_r/k_x is equal to the background velocity. Here ω_r is the real part of ω.

2. The analytical derivation

We derive here the basic result of the paper which is given by the following inequality:

$$\omega_i^2 \leqslant \left\{\left(\frac{1}{1+\Omega_r^2/\omega_i^2}\right)\left[\frac{U_0'^2/4}{1+(\Omega_r^2+\omega_i^2)/k_x^2c^2}-n^2\right]\right\}_{\max}, \quad (3)$$

where

$$\Omega=\omega-k_xU_0=\Omega_r+i\omega_i, \quad \omega=\omega_r+i\omega_i, \quad \omega_i\neq 0. \quad (4)$$

To obtain (3), we start from the second-order linear homogeneous equation (see Chimonas, 1970)

$$\left[\frac{r\Omega^2 q'}{k_x^2-(\Omega/c)^2}\right]'+r(n^2-\Omega^2)q=0, \quad (5)$$

where

$$\left.\begin{aligned} q&=\frac{w(z)}{i\Omega}\exp\left[-g\int^z c^{-2}(z')dz'\right]\\ r&=\rho_0(z)\exp\left[2g\int^z c^{-2}(z')dz'\right]\end{aligned}\right\}. \quad (6)$$

Eq. (5) is obtained from the linearized form of the equations of continuity, of momentum and of adiabatic changes of state, by eliminating the pressure, density and horizontal velocities in favor of w, the vertical velocity. Heat conduction and viscous effects are ignored. The linearization is about the background density ρ_0, pressure p_0 and horizontal velocity U_0, and all perturbation quantities are of the form given by (1).

Two forms of boundary conditions on w are considered, as discussed by Chimonas (1970). If the fluid is bounded by two horizontal rigid walls, then q must be zero on them; alternatively, the fluid may be surrounded by regions in which U_0 and c are constant, so that we can apply the outgoing radiation condition, i.e., $r^{\frac{1}{2}}q\rightarrow 0$ exponentially as $|z|\rightarrow\infty$.

Since ω_i is different from zero, we can uniquely define a branch of $\Omega^{\frac{1}{2}}$ and make the substitution

$$q=\Omega^{-\frac{1}{2}}\Phi. \quad (7)$$

In this case (5) reduces to

$$\left[\frac{r\Omega\Phi'}{k_x^2\Delta}\right]'-\frac{(\Omega')^2 r}{4k_x^2\Omega\Delta}\Phi-\frac{1}{2}\left[\frac{r\Omega'}{k_x^2\Delta}\right]'\Phi+r\frac{n^2-\Omega^2}{\Omega}\Phi=0, \quad (8)$$

where

$$\Delta=1-(\Omega/k_xc)^2.$$

Multiplying (8) by Φ^* and integrating over the domain of z gives

$$\int dz\left\{\left[\frac{r\Omega\Phi^*\Phi'}{k_x^2\Delta}\right]'-\frac{1}{2}\left[\frac{r\Omega'|\Phi|^2}{k_x^2\Delta}\right]' +r(\psi_r+i\psi_i)\left[-|\Phi'|^2-\left(\frac{\Omega'}{2\Omega}\right)^2|\Phi|^2+\frac{1}{2}\frac{\Omega'}{\Omega}(\Phi\Phi^*)'\right] +\frac{r}{\Omega}(n^2-\Omega^2)|\Phi|^2\right\}=0, \quad (9)$$

where ψ_r and ψ_i are the real and imaginary parts of $\Omega/k_x^2\Delta$, respectively. The integrals of the first two terms in (9) are zero because of the boundary conditions. Taking the imaginary part of (9) gives

$$\omega_i\left[\int\frac{r}{k_x^2|\Delta|^2}Q^2dz+\int rdz|\Phi|^2\right] +\omega_i\int\frac{r}{|\Omega|^2}dz\left[n^2-\frac{(\Omega'/2k_x)^2}{\Delta_1}\right]|\Phi|^2=0, \quad (10)$$

where

$$Q^2=\Delta_1\left\{|\Phi'|^2-\frac{1}{2}\frac{\Omega'\Omega_r}{|\Omega|^2}\left(1-\frac{\Delta_2}{\Delta_1}\right)(\Phi\Phi^*)' +\left[\frac{\Omega'\Omega_r}{2|\Omega|^2}\right]^2\left[1-\frac{\Delta_2}{\Delta_1}\right]^2|\Phi|^2\right\}, \quad (11)$$

$$\Delta_1=1+|\Omega|^2/k_x^2c^2, \quad \Delta_2=1-|\Omega|^2/k_x^2c^2. \quad (12)$$

In deriving (10), use was made of the identity

$$\omega_i^2\Delta_1^2+\Omega_r^2\Delta_2^2=|\Omega|^2|\Delta|^2.$$

Since $(\Phi\Phi^*)'\leqslant 2|\Phi||\Phi'|$, Q^2 is a non-negative real number. For a statically stable fluid, n^2 and r are both real and positive; therefore, one would conclude from (10) that a sufficient condition for stability is that throughout the flow

$$n^2\geqslant\frac{U_0'^2}{4}\frac{1}{\Delta_1}. \quad (13)$$

Alternatively, one can say that a necessary condition for instability is that $n^2-(\Omega'/2k_x)^2/\Delta_1$ becomes negative somewhere in the flow. Yet, because of the $1/|\Omega|^2$ "resonant" term in the last integral, (10) strongly suggests that, for instability, $n^2-(\Omega'/2k_x)^2/\Delta_1$ should change sign not just anywhere but in the neighborhood of the critical layer. Miles's approach (1961), which no doubt can be extended to compressible fluids, would confirm this at least on the neutral boundary.

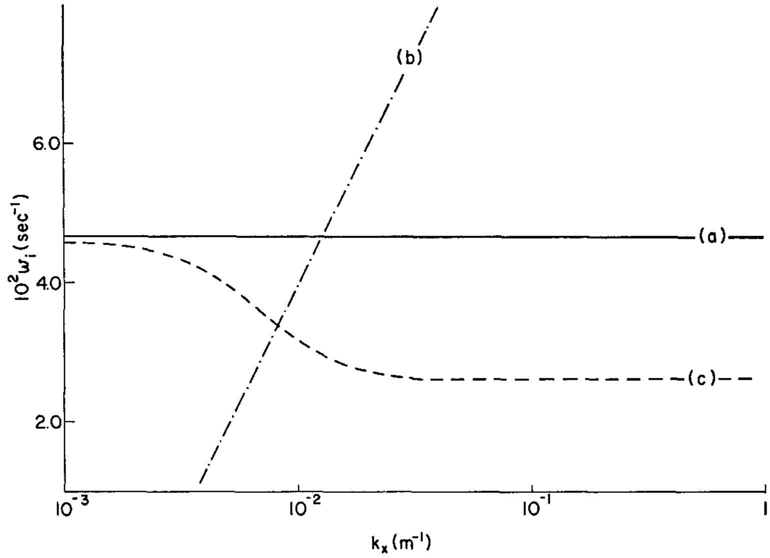

FIG. 1. The growth rate ω_i (sec^{-1}) vs k_x the horizontal wavenumber (m^{-1}). Line (a) is the Chimonas (1970) maximum growth rate, line (b) the Eckart (1963) maximum growth rate imposed by the semi-circle theorem, and line (c) the present Eq. (3), for the particular case of

$$U(z)=10.0\tanh\left(\frac{z}{H_1}\right),$$

$$\rho_0(z)=\rho_{0_0}\exp\left(-\frac{z}{H_2}\right),$$

where $U(z)$ is in m sec^{-1}, with $H_1=100$ m, $H_2=RT_0/g$, $T_0=300$K, and ρ_{0_0} a reference density at $z=0$ level. The above correspond to an isothermal atmosphere, with $U'_{\max}=0.1$ sec^{-1} and constant Brunt-Väisälä frequency$=1.76\times10^{-2}$ sec^{-1}. The critical level is taken at $z=70.0$ m where $U=6.04$ m sec^{-1} and Ri (the Richardson number)$=0.08$. Ri is less than 0.25 between $-90<z<90$ m.

The interesting point to be emphasized here is that the integral approach followed by Howard (1961), if properly modified, is capable of providing a good deal of information concerning the specific role of the critical layers in the flow.

For ω_i non-zero, we can write from (10)

$$\int rdz|\Phi|^2\leqslant\int\frac{r}{|\Omega|^2}dz\left[\frac{(\Omega'/2k_x)^2}{\Delta_1}-n^2\right]|\Phi|^2$$

$$\leqslant\left[\left(\frac{1}{1+\Omega_r^2/\omega_i^2}\right)\left(\frac{(\Omega'/2k_x)^2}{\Delta_1}-n^2\right)\right]_{\max}$$

$$\times\frac{1}{\omega_i^2}\int rdz|\Phi|^2,\quad(14)$$

from which (3) follows.

Inequality (3) limits the maximum value that the growth rate can have to values smaller than that imposed by (2); its form is somewhat more complicated since ω_i appears in both sides and it may require a numerical analysis. More important, perhaps, is the presence in (3) of the resonant terms $1/(1+\Omega_r^2/\omega_i^2)$ and $1/(1+|\Omega|^2/k_xc^2)$ which result in ω_i being essentially limited by the values of n^2, U_0 and U_0' in the neighborhood of the critical level, $\Omega_r=0$.

An illustrative example of the comparison between Howard's (1961) maximum growth rate given by Eq. (2), the new maximum growth rate given by Eq. (3), and that given by the semi-circle theorem as generalized by Eckart (1963), is shown in Fig. 1. In conjunction with the semi-circle theorem, it strongly indicates that the most unstable wave will have $k_x\approx9.3\times10^{-3}$ m^{-1} with maximum growth rate 25% less than the Howard rate which is independent of wavenumber.

It should be pointed out, of course, that the difference between the maximum growth rates due to (2) and (3) would disappear, except for a compressibility effect, if the maximum of $[(U_0'^2/4)-n^2]$ takes place at the critical level where $\Omega_r=0$.

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