The Nonlinear Quasi-Geostrophic Equation: Existence and Uniqueness of Solutions on a Bounded Domain

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ABSTRACT

The quasi-geostrophic theory leads to a single nonlinear partial differential equation for a streamfunction giving geostrophic velocity fields presumed to resemble the synoptic scales of atmospheric motion. This article is concerned with demonstrating that the quasi-geostrophic problem is well-posed mathematically, in the sense that solutions exist, and that they are continuously dependent on the initial data. The model studied is comprised of the quasi-geostrophic equation subject to the severe boundary condition that an isentrope coincides with the earth's surface.

The main technique is the use of the eigenfunctions of an elliptic operator appearing within the quasi-geostrophic equation. These eigenfunctions provide the basis for a spectral model, which can be truncated to include a finite number of scales. The convergence properties of the solutions to the truncated model allow the existence of solutions to the entire model to be inferred with the methods of functional analysis. Thus, the conclusions reached are relative to generalized solutions and to the usual norm in the Hilbert space of quadratically integrable functions.

The results have applications to the study of atmospheric predictability.

1. Introduction

The large-scale atmospheric motion of middle latitudes has the dominant characteristics of being hydrostatic, nearly geostrophic, and capable of conversions of internal and potential energy to kinetic energy. These are almost contradictory, for strictly geostrophic motions cannot convert the static forms of energy into the kinetic energy of horizontal motion. The problem may be viewed as arising from the fact that the horizontal divergence fields are a dynamically and energetically important part of the actual process, but are poorly modeled by the divergence of the geostrophic wind. The solution lies in the observation that the vorticity of the geostrophic wind is a useful approximation to the actual vorticity.

Thus, the application of the geostrophic assumption to the vorticity equation and perturbation techniques leads to the Rossby wave theory (Rossby *et al.*, 1939) that has provided both impetus and inspiration to meteorological thought, despite its failures as a realistic model of large-scale flow.

The developments following from the Rossby wave equation culminate in the modern quasi-geostrophic theory in which the assumption that the motions are nearly—but not exactly—geostrophic leaves sufficient freedom for the energy conversion processes to exist. The present quasi-geostrophic theory was developed by Charney (1947) and Eady (1949) and was given its most recent form by Charney and Stern (1962). The task of quasi-geostrophic theory is the development through systematic scaling of an approximate set of equations of motion applicable to synoptic-scale motions in middle latitudes; its signal accomplishment is that the approximate system reduces to a single nonlinear differential equation governing the evolution of a streamfunction for the motion.

The major use of this equation has been in perturbation techniques, in which it is converted into a linear form useful for studying stability criteria for disturbances superimposed on a zonal current. The results of these investigations have been summarized in a review of geostrophic motion by Phillips (1963). Integral stability theorems of the type developed in the study of small-scale shear instabilities for incompressible flow (Lin, 1955; Howard, 1961) and for compressible motion (Dutton and Fichtl, 1969) have been developed for quasi-geostrophic motion (Charney and Stern, 1962; Pedlosky, 1964).

Attention has turned to the nonlinear equation with Charney's (1971) development of a theory of geostrophic turbulence in which the conservation of a potential vorticity form couples with energy conservation to yield an atmospheric analogy to two-dimensional turbulence and produces a credible explanation of the observed kinetic energy spectra.

This article will be concerned with the mathematical properties of the nonlinear quasi-geostrophic equation, and the goal is the proof of the uniqueness and existence of solutions on a bounded domain. The development is based upon Charney's (1971) observation that the highest order terms of the quasi-geostrophic equation involve the quasi-geostrophic version of the material derivative applied to an elliptical partial differential operator. The eigenfunctions of this operator, which were described and used formally by Charney (1971), yield a spectral model which is the basis of most of the present study. The existence of solutions to truncated versions of this model is established, and then the Riesz-Fisher theorem provides the assurance that these truncated solutions converge to a generalized form of solution.

Thus, the physical phenomena under consideration are the motions modeled by the quasi-geostrophic equation; the question is whether the model is well-formulated in the sense that unique solutions exist; the mathematical techniques used are those in functional analysis which provide the completeness criteria that allow the problem to be imbedded in Hilbert spaces that possess characteristics advantageous in the study of differential equations.

2. Properties of solutions to the nonlinear equation

The major conditions sufficient for the validity of the quasi-geostrophic equation are that the flow be hydrostatic, have a small Rossby number Ro $=V/(fL)\approx 0.1$ and a large Richardson number $\mathrm{Ri}\approx 50$, and that $L/a\approx\mathrm{Ro}$ so that the β-plane approximation may be used. A more specific account of the scale requirements is given by Charney and Stern (1962) and by Phillips (1963). If ψ is a non-dimensional stream-function, then the non-dimensional geostrophic velocity is $\mathbf{v}_g=\mathbf{k}\times\nabla\psi$ and via the thermal wind equation, the non-dimensional fractional potential temperature perturbation is $\theta=\partial\psi/\partial z$. If we let subscript zero denote conditions in a hydrostatic, motionless basic state, which varies only with height, then the quasi-geostrophic equation may be written

$$\left[\frac{\partial}{\partial t}+\mathbf{v}_g\cdot\nabla_H\right]\left[\nabla_H{}^2\psi+\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\rho_0}{\sigma}\frac{\partial\psi}{\partial z}\right)+\beta y\right]=0, \quad (2.1)$$

where $\sigma=\sigma(z)$ is a stability parameter and only ρ_0 is a dimensional variable. Alternate forms are convenient. We define the quasi-geostrophic material derivative as

$$\frac{\delta}{\delta t}=\frac{\partial}{\partial t}+\mathbf{v}_g\cdot\nabla_H, \quad (2.2)$$

and the Jacobian operator as

$$J(f,g)=f_xg_y-f_yg_x, \quad (2.3)$$

where subscripts denote partial differentiation. Then with the notation L for the elliptic operator

$$L(\psi)=\nabla_H{}^2\psi+\frac{1}{\rho_0}\left(\frac{\rho_0}{\sigma}\psi_z\right)_z, \quad (2.4)$$

we have

$$\frac{\delta}{\delta t}[L(\psi)+\beta y]=\frac{\partial}{\partial t}L(\psi)+J[\psi,L(\psi)]+\beta\psi_x=0. \quad (2.5)$$

Moreover, the definition of

$$Z_g=L(\psi)+\beta y \quad (2.6)$$

as the quasi-geostrophic potential vorticity gives

$$\frac{\delta}{\delta t}Z_g=0. \quad (2.7)$$

The quasi-geostrophic equation is assumed to represent a class of flows on a domain V defined by

$$V=\left\{\mathbf{x}\,\middle|\,-\frac{L}{2}\leqslant x\leqslant\frac{L}{2},\quad -\frac{L}{2}\leqslant y\leqslant\frac{L}{2},\quad 0\leqslant z\leqslant Z_T\right\}, \quad (2.8)$$

where Z_T may approach infinity. The restriction of x and y to a specific interval is convenient but not essential. We insist on cyclic continuity on each horizontal plane (which includes the vanishing of v_g at $y=\pm L/2$ as a special case). Vertical boundary conditions follow from energy theorems that will be stated below.

For any fixed volume V and for any cyclically continuous function ϕ whose derivative is continuous and is summable, we have

$$\frac{\partial}{\partial t}\int\rho_0\phi dV=\int\rho_0\frac{\partial\phi}{\partial t}dV=\int\rho_0\left[\frac{\delta\phi}{\delta t}-J(\psi,\phi)\right]dV$$
$$=\int\rho_0\frac{\delta\phi}{\delta t}dV, \quad (2.9)$$

which is the *quasi-geostrophic transport theorem* and follows by an integration of parts and application of the cyclic boundary conditions.

a. Constraints on the motion

With the transport theorem, it is easy to prove that quasi-geostrophic motions satisfy the energy constraint

$$2\dot{E}=\frac{\partial}{\partial t}\int\rho_0\left[|\nabla_H\psi|^2+\frac{1}{\sigma}|\psi_z|^2\right]dV$$
$$=-\int_A 2[\rho_0\psi w]_{z=0}^{Z_T}dA, \quad (2.10)$$

where w is a nondimensional vertical velocity obtained from the quasi-geostrophic isentropic condition

$$\frac{\delta}{\delta t}\psi_z+\sigma w=0. \quad (2.11)$$

As pointed out by Charney (1971), the motions also satisfy a mean-square vorticity criterion in the form

$$2\dot{F}=\frac{\partial}{\partial t}\int \rho_0|L(\psi)|^2dV=-2\beta\int_A\left[\frac{\rho_0}{\sigma}\psi_z\psi_x\right]_{z=0}^{Z_T}dA. \quad (2.12)$$

Another family of constraints follows from (2.7) and the transport theorem, for if H is a differentiable function then we shall have

$$\frac{\partial}{\partial t}\int \rho_0 H(Z_g)dV=0. \quad (2.13)$$

Now the simplest boundary problem for bounded domains is obtained by insisting that an isentrope coincide with each of the surfaces $z=0$ and $z=Z_T<\infty$, because then $\psi_z=0$ at $z=0$ and $z=Z_T$ and the boundary terms of (2.10) and (2.12) vanish. This is a severe restriction, and is not in accord with actual conditions in which isentropes intersect the earth's surface—a phenomenon that is significant in stability investigations. The necessary boundary condition (which follows from the requirement that the vertical velocity vanish at a flat lower, rigid surface) is that $\delta\psi_z/\delta t$ vanish. Relaxation of the present boundary condition to this form would be a desirable feature of any extension of the present results.

It is convenient to restrict attention to *regular flows*, those for which the integrals of ρ_0, $\rho_0\psi^2$, $\rho_0|\mathbf{v}_g|^2$, $\rho_0 w^2$, $\rho_0\theta^2=\rho_0(\psi_z)^2$ and $\rho_0|L(\psi)|^2$ are all finite. For Z_T finite and infinite, regular flows have finite mass, variance and energy in this sense. For unbounded domains ($Z_T=\infty$), regularity requires that all of these five quantities vanish more rapidly than z^{-1} as $z\rightarrow\infty$. Thus application of the Schwarz inequality at Z_T in (2.10) and (2.12) shows that the boundary terms vanish. We may retain the condition that $\psi_z=0$ at $z=0$.

DEFINITION 2.1. *Simple quasi-geostrophic flows are those regular flows that satisfy the lower condition that $\psi_z=0$ at $z=0$ and the condition that $\psi_z=0$ at Z_T when $Z_T<\infty$.*

The family of constraints (2.13) may be given further consideration now. Let F be finite, so that for $H(x)=x^2$, we may consider

$$G_2(t)=\int \rho_0 Z_g^2 dV=\int \rho_0[L(\psi)+\beta y]^2dV,$$

$$\leqslant\left\{\left[\int \rho_0|L(\psi)|^2dV\right]^{\frac{1}{2}}+\left[\int \rho_0\beta^2y^2dV\right]^{\frac{1}{2}}\right\}^2,$$

$$=\{(2F)^{\frac{1}{2}}+\beta L(M_0/12)^{\frac{1}{2}}\}^2, \quad (2.14)$$

in which we have used Minkowski's inequality and denoted the total mass by $M_0=\int\rho_0 dV$. With Hölder's inequality we have

$$G_\alpha(t)=\int_V \rho_0|Z_g|^\alpha dV=\int(\rho_0^{\alpha/2}|Z_g|^\alpha)\rho_0^{(2-\alpha)/2}dV$$

$$\leqslant\left[\int\rho_0|Z_g|^2dV\right]^{\alpha/2}\left[\int\rho_0 dV\right]^{(2-\alpha)/2}, \quad (2.15)$$

subject to the restriction that $\alpha<2$. Thus, the quantities $G_\alpha(t)$ are finite for every $\alpha\leqslant 2$, and we have proven:

THEOREM 2.1. *Simple quasi-geostrophic motion on the domain V is subject to the infinite family of constraints*

$$E(t)=E(0),\quad F(t)=F(0),\quad G_\alpha(t)=G_\alpha(0) \quad [0<\alpha\leqslant 2]. \quad (2.16)$$

b. The eigenvalue problem

The quasi-geostrophic equation can be cast in spectral form with useful consequences by employing an idea of Charney (1971). Specifically, we solve the eigenvalue problem

$$L(\psi)=-\lambda\psi, \quad (2.17)$$

and use the eigenfunctions to represent ψ. The validity of this formal procedure depends on the eigenfunctions being complete in the space of square integrable functions, and this generally requires that the domain be bounded.

By the same process that led to the energy constraint, multiplication of (2.17) by $\rho_0\psi$ and integration gives

$$\int\rho_0\left[|\nabla_H\psi|^2+\frac{1}{\sigma}|\psi_z|^2\right]dV-\lambda\int\rho_0\psi^2dV$$

$$=\int_A\left[\frac{\rho_0\psi}{\sigma}\psi_z\right]_{z=0}^{Z_T}dA. \quad (2.18)$$

If this is minimized subject to the condition that $\psi_z=0$ at $z=0$ and $z=Z_T<\infty$, the result is the first eigenfunction ψ_1 and eigenvalue λ_1. As additional constraints $\int\rho_0\psi_n^2dV=1$, $\int\rho_0\psi_n\psi_k dV=0$ for $k<n$ are added, then additional eigenvalues $\lambda_1<\lambda_2<\cdots<\lambda_n$ and associated eigenfunctions are found.

For finite domains, as shown in Courant and Hilbert (1953, p. 424), these eigenfunctions are complete in the sense that if

$$\hat{f}(n)=\int\rho_0\psi_n f dV, \quad (2.19)$$

then

$$\sum_n[\hat{f}(n)]^2=\int\rho_0 f^2dV, \quad (2.20)$$

so that we have mean convergence in the form

$$\lim_{N\to\infty}\int \rho_0|f-\sum_{n=1}^{N}\hat{f}(n)\psi_n|^2 dV=0. \qquad (2.21)$$

For unbounded domains, these results are not generally true because the eigenvalues are usually no longer discrete, giving a continuous spectrum of solutions for a range of values of λ. Whether or not this occurs with the quasi-geostrophic equation is, to my knowledge, an open question.

A simple example is instructive. Let

$$\rho_0(z)=\rho_0(0)\exp[-Dz/H],$$

where H is the scale height and D is the vertical scale of the motions. Let $\sigma=1$, and put $\psi=\phi\exp[Dz/2H]$. Then (2.17) gives

$$\nabla^2\phi+[\lambda-(D/2H)^2]\phi=0, \qquad (2.22)$$

subject to boundary conditions $\phi_z+(D/2H)\phi=0$ at $z=0$ and $z=Z_T$. Now

$$\phi=\hat{\phi}(z)\exp\left[\frac{2\pi i}{L}(mx+ny)\right] \qquad (2.23)$$

gives the standard Sturm-Liouville problem

$$\left.\begin{aligned}&\hat{\phi}_{zz}-\Lambda_{mn}^2\hat{\phi}=0\\ &\left[-\Lambda_{mn}^2=\lambda-\left(\frac{2\pi}{L}\right)^2(m^2+n^2)-(D/2H)^2\right]\end{aligned}\right\}. \qquad (2.24)$$

Thus, we find upon solving this equation

$$\hat{\phi}=A\exp(\Lambda_{mn}z)+B\exp(-\Lambda_{mn}z),$$

and applying the boundary condition at $z=0$ that (for $\Lambda_{mn}\neq -D/2H$)

$$\hat{\phi}(z)=B\left[\frac{\Lambda_{mn}-(D/2H)}{\Lambda_{mn}+(D/2H)}\exp(\Lambda_{mn}z)+\exp(-\Lambda_{mn}z)\right]. \qquad (2.25)$$

Then at Z_T we require

$$\hat{\phi}_z+\frac{D}{2H}\hat{\phi}=B\{[\Lambda_{mn}-(D/2H)]\times[\exp(\Lambda_{mn}Z_T)-\exp(-\Lambda_{mn}Z_T)]\}=0. \qquad (2.26)$$

This is satisfied if $\Lambda_{mn}=ik\pi/Z_T$ for $k=0, \pm 1, \pm 2, \ldots$, or if $\Lambda_{mn}=D/2H$. Hence, we have eigenvalues

$$\left.\begin{aligned}\lambda_{m,n,k}&=\left(\frac{D}{2H}\right)^2+\left(\frac{2\pi}{L}\right)^2(m^2+n^2)+\left(\frac{\pi k}{Z_T}\right)^2\\ \lambda_{m,n}&=\left(\frac{2\pi}{L}\right)^2(m^2+n^2)\end{aligned}\right\} \qquad (2.27)$$

associated with eigenfunctions

$$\left.\begin{aligned}\psi_{m,n,k}&=(L)^{-1}\exp\left[\frac{2\pi i}{L}(mx+ny)\right]\hat{\psi}_k(z)\\ \psi_{m,n}&=(L)^{-1}\exp\left[\frac{2\pi i}{L}(mx+ny)\right]\\ &\quad\times\left\{\frac{H\rho_0(0)}{D}[1-\exp(-Z_TD/H)]\right\}^{-\frac{1}{2}}\end{aligned}\right\}, \qquad (2.28)$$

where

$$\hat{\psi}_k(z)=\left[\frac{Z_T}{2}\rho_0(0)\right]^{-\frac{1}{2}}\left(\frac{\pi^2k^2}{Z_T^2}+\frac{D^2}{4H^2}\right)^{-\frac{1}{2}}\times\left[\frac{\pi k}{Z_T}\cos\left(\frac{\pi kz}{Z_T}\right)-\frac{D}{2H}\sin\left(\frac{\pi kz}{Z_T}\right)\right]e^{zD/2H}. \qquad (2.29)$$

For these eigenvalues and eigenfunctions, we have the ranges $-\infty\leqslant m, n\leqslant\infty$ and $k=\pm 1, \pm 2, \ldots$. The eigenfunctions are orthonormal, including those functions associated with multiple eigenvalues arising from m and n being interchangeable in (2.27). Once D, H and L are specified, the eigenvalues can be arranged in an increasing sequence; it is significant that $\lambda_0=0$ is an eigenvalue and reflects the fact that the boundary conditions allow constant solutions of $L(\psi)=0$.

Returning to (2.26), we see that the only value of Λ_{mn} that will satisfy the boundary condition as $Z_T\to\infty$ is $\Lambda_{mn}=D/2H$ so that

$$\psi=B\exp\left[\frac{2\pi}{L}(mx+ny)\right]. \qquad (2.30)$$

Thus, this elliptic eigenvalue problem is characterized, as are elliptic problems in general, by having no non-trivial exponential solution (2.23) in the unbounded domain.

In the rest of this section we consider the case in which a complete set of orthonormal eigenfunctions with an unbounded set of eigenvalues exists as solutions to (2.17) and thus the results surely apply for $Z_T<\infty$; whether they are valid as $Z_T\to\infty$ remains to be determined. For convenience we assume the set is arranged with multiple real eigenfunctions accompanying the eigenvalues so that all functions and coefficients are real. Summation over the multiple eigenfunctions and multiple coefficients is understood, but not explicitly shown.

c. *Geostrophic turbulence*

The solution to the quasi-geostrophic equation may now be represented as

$$\psi(\mathbf{x},t)=\sum_{n=0}^{\infty}a_n(t)\psi_n(\mathbf{x}), \qquad (2.31)$$

where

$$L(\psi_n)=-\lambda_n\psi_n. \tag{2.32}$$

Thus, we write

$$L(\psi)=L(\sum_{n=0}^{\infty} a_n\psi_n)=-\sum_{n=1}^{\infty}\lambda_n a_n\psi_n, \tag{2.33}$$

which is justified if the terms $L(\psi_n)$ are continuous functions and if the series on the right converges uniformly (Titchmarsh, 1952, p. 37). Since this depends on the coefficients $a_n(t)$ we shall not now attempt to verify this result but use it only in a formal sense for the present.

For simple quasi-geostrophic motions, the constraints imply that

$$\left.\begin{aligned} 2E(t)&=-\int\rho_0\psi L(\psi)dV=\sum_{n=1}^{\infty}\lambda_n|a_n(t)|^2\\ &=\sum_{n=1}^{\infty}\lambda_n|a_n(0)|^2\\ 2F(t)&=\sum_{n=1}^{\infty}\lambda_n^2|a_n(t)|^2=\sum_{n=1}^{\infty}\lambda_n^2|a_n(0)|^2\\ G_\alpha(t)&=\int\rho_0[L(\psi)+\beta y]^\alpha dV\\ &=G_\alpha[\lambda_1a_1(t),\lambda_2a_2(t),\ldots]\\ &=G_\alpha[\lambda_1a_1(0),\lambda_2a_2(0),\ldots] \end{aligned}\right\}. \tag{2.34}$$

The first two constraints are responsible for the resemblance of quasi-geostrophic motions to two-dimensional turbulence. We define

$$2E_{M,N}=\sum_{n=M}^{N}\lambda_n a_n^2,\quad 2F_{M,N}=\sum_{n=M}^{N}\lambda_n^2 a_n^2. \tag{2.35}$$

Then using Charney's approach we find

$$\left.\begin{aligned} 2E_{M,\infty}&\leqslant\frac{1}{\lambda_M}\sum_{n=M}^{\infty}\lambda_n^2a_n^2=\frac{2F_{M,\infty}}{\lambda_M}\leqslant\frac{2F_{1,\infty}}{\lambda_M}(0)\\ 2F_{1,M-1}&=\sum_{1}^{M-1}\lambda_n^2a_n^2\leqslant\lambda_{M-1}\sum_{1}^{M-1}\lambda_n a_n^2\\ &=2\lambda_{M-1}E_{1,M-1} \end{aligned}\right\}. \tag{2.36}$$

The quantity on the right in the first inequality vanishes as $M\rightarrow\infty$ giving Charney's result that energy cannot accumulate at high "wavenumbers" M.

The two inequalities taken together give the stronger result

$$\frac{E_{M,\infty}}{E_{1,M-1}}\leqslant\frac{\lambda_{M-1}}{\lambda_M}\frac{F_{M,\infty}}{F_{1,M-1}}. \tag{2.37}$$

Thus, at every M, continual simultaneous fluxes $E_{1,M-1}\rightarrow E_{M,\infty}$ and $F_{M,\infty}\rightarrow F_{1,M-1}$ are impossible, and as a result the only permitted modes of response in which the flux of energy E and enstrophy F (mean-square vorticity) are always of the same sign at any M must have $E_{M,\infty}\rightarrow E_{1,M-1}$ and $F_{1,M-1}\rightarrow F_{M,\infty}$. The minus-three law for spectra $E(\kappa)$ of this mode of quasi-geostrophic motion now follows from the assumption that $E(\kappa)=C\eta^\alpha\kappa^\beta$, where η is the flux of enstrophy across wavenumber κ, because dimensional analysis gives $\alpha=\frac{2}{3}$ and $\beta=-3$.

It is noteworthy that this development has used only the energy and enstrophy constraints. What effect inclusion of some or all of the other constraints may have appears to be an open question.

d. Orbits in phase space

The expansions (2.31) provide a spectral model of quasi-geostrophic motion which is obtained by substituting (2.31) into (2.5), and then multiplying by $\rho_0\psi_n$ and integrating to isolate $\partial a_n(t)/\partial t=\dot{a}_n$. This provides an infinite nonlinear system of equations for the quantities $\dot{a}_n$ with interaction coefficients formed by integrals of products of the eigenfunctions. Such models, truncated at the Nth coefficient, are an important part of the later developments.

For now, we observe that the space $\mathfrak{A}=\{a_1,a_2,\ldots\}$ is an infinite dimensional phase space in which the trajectories $\{a_1(t),a_2(t),\cdots\}$ describe quasi-geostrophic motion. In fact, we have the relations

$$\left.\begin{aligned} \tfrac{1}{2}\sum_{n=1}^{\infty}\lambda_n a_n^2&=E(0)\\ \tfrac{1}{2}\sum_{n=1}^{\infty}\lambda_n^2 a_n^2&=F(0)\\ G_\alpha(\lambda_1a_1,\lambda_2a_2,\ldots)&=G_\alpha(0),\quad 0<\alpha<2 \end{aligned}\right\}, \tag{2.38}$$

so that the trajectories are confined to the common intersection in $\mathfrak{A}$ of the hypersurfaces specified by the constraints (2.38).

3. Uniqueness of classical solutions

The mathematical question of whether solutions to the quasi-geostrophic equation are unique has implications for the important and difficult questions associated with determination of the degree of predictability of atmospheric motions. Moreover, since solutions to the Navier-Stokes equations are known to be unique (Serrin, 1959), any successful model of quasi-geostrophic motion should share this property.

In this section, we assume that simple quasi-geostrophic flows exist as solutions of (2.5) in the interval $0\leqslant t\leqslant T$. We use $\Phi(\mathbf{x},t)$ to denote $L(\psi)$. We suppose that (2.5) has two solutions ψ and $\tilde{\psi}$ for the same initial data and boundary conditions.

Hence we have

$$\frac{\delta}{\delta t}(\Phi+\beta y)=0,\quad \frac{\tilde{\delta}}{\delta t}(\tilde{\Phi}+\beta y)=0. \tag{3.1}$$

The calculations are much simplified by the following:

LEMMA 3.1. *For cyclically continuous functions with continuous derivatives the Jacobian operator of (2.3) has the properties that*

(i) $J(f,g)=-J(g,f)$

(ii) $J(f+g,h)=J(f,h)+J(g,h)$

(iii) $\int \rho_0 hJ(f,g)dV=\int \rho_0 fJ(g,h)dV$

PROOF. The first two are immediate consequences of the definition; the third follows from integration by parts and use of cyclic continuity.

LEMMA 3.2 (Serrin, 1959). *Let* $\phi'=\tilde{\phi}-\phi$. *Then*

$$\frac{\tilde{\delta}\tilde{\phi}}{\delta t}-\frac{\delta\phi}{\delta t}=\frac{\delta\phi'}{\delta t}+J(\psi',\tilde{\phi}).$$

PROOF. The result follows immediately from Lemma 3.1 because

$$\begin{aligned}J(\tilde{\psi},\tilde{\phi})-J(\psi,\phi)&=J(\psi,\phi')+J(\psi',\tilde{\phi}),\\&=\mathbf{v}_g\cdot\nabla\phi'+J(\psi',\tilde{\phi}).\end{aligned}$$

Finally, we observe that if ψ is a classical solution, then all the terms that appear in the equation must be finite, including $\nabla_H\Phi$.

With these preliminaries complete, we find easily from (3.1) that

$$\frac{\delta\Phi'}{\delta t}+J(\psi',\tilde{\Phi})+\beta v_g'=0. \quad (3.2)$$

This equation controls the development of difference fields in the study of quasi-geostrophic predictability. From it we calculate with the transport theorem that

$$\begin{aligned}\frac{\partial}{\partial t}\int\rho_0\frac{|\Phi'|^2}{2}dV&=\int\rho_0\Phi'\frac{\delta\Phi'}{\delta t}dV,\\&=-\int\rho_0\Phi'[J(\psi',\tilde{\Phi})+\beta v_g']dV. \quad (3.3)\end{aligned}$$

The β term here vanishes upon integration, as it did in (2.12) with the present boundary conditions. For the remaining term we have

$$\begin{aligned}\left|\int\rho_0\Phi'J(\psi',\tilde{\Phi})dV\right|&\leqslant\int\rho_0|\Phi'||\mathbf{v}_g'||\nabla_H\tilde{\Phi}|dV,\\&\leqslant M\left\{\int\rho_0|\Phi'|^2dV\int\rho_0|\mathbf{v}_g'|^2dV\right\}^{\frac{1}{2}},\\&\leqslant 2M\{F'E'\}^{\frac{1}{2}}. \quad (3.4)\end{aligned}$$

Here we have defined $M=\mathrm{Max}|\nabla_H\tilde{\Phi}|$.

The functions ψ' satisfy homogeneous boundary conditions as well as (2.17). For homogeneous boundary conditions and for bounded domains V, the smallest eigenvalue of $L(\psi)=-\mu\psi$ is easily seen from (2.17) to be $\mu_1>0$ since now the only constant solution is the trivial one that vanishes. Therefore, Rayleigh's principle [obtained from (2.18) with the argument that μ will be greater than μ_1 unless ψ is the minimizing function] applies, and so

$$\begin{aligned}\mu_1&\leqslant-\int\rho_0\psi'\Phi'dV\bigg/\int\rho_0|\psi'|^2dV\\&\leqslant\left\{\int\rho_0|\Phi'|^2dV\bigg/\int\rho_0|\psi'|^2dV\right\}^{\frac{1}{2}}=\left\{\frac{2F'}{D'}\right\}^{\frac{1}{2}}. \quad (3.5)\end{aligned}$$

Another implication of (3.5) is that

$$E'=-\frac{1}{2}\int\rho_0\psi'\Phi'dV\leqslant\tfrac{1}{2}\{2F'D'\}^{\frac{1}{2}}. \quad (3.6)$$

In both these inequalities we are using

$$D=\int\rho_0|\psi'|^2dV.$$

The combination of these results gives

$$\frac{\partial F'}{\partial t}\leqslant 2MF'/\sqrt{\mu_1}. \quad (3.7)$$

This differential inequality implies that

$$F'\leqslant F'(0)\exp\left[\frac{2M}{\sqrt{\mu_1}}t\right],\quad 0\leqslant t\leqslant T. \quad (3.8)$$

This proves that the vorticity fields are unique, for $F'(0)=0$ implies that F' remains zero in every finite interval of time in which the solution $\tilde{\psi}$ exists. Moreover, it now follows from (3.6) that the velocity fields are unique when $F'(0)$ vanishes, and finally (3.5) shows that the streamfunctions are unique.

Therefore we have proven:

THEOREM 3.1. *If simple quasi-geostrophic flows exist as classical solutions of (2.5) in the interval* $0\leqslant t\leqslant T$, *then they are unique.*

COROLLARY 3.1.1. *The coefficients* $\{a_n(t),n\geqslant 1\}$ *for simple quasi-geostrophic flows are unique in those cases in which a complete set of eigenfunctions satisfying (2.33) exists.*

PROOF. We have

$$\int\rho_0|\Phi'|^2dV=\sum_{n=1}^{\infty}\lambda_n^2|a_n'|^2=0. \quad (3.9)$$

COROLLARY 3.1.2. *Let a complete set of eigenfunctions satisfying (2.33) exist. Then if* $A(t)=\sum \lambda_n^2 |a_n'|^2$, *we have for simple quasi-geostrophic flows*

$$A(t)\leqslant A(0)\exp\left[\frac{2M}{\sqrt{\mu_1}}t\right]. \tag{3.10}$$

PROOF. The result follows from (3.8).

As an alternative to use of Rayleigh's principle, we note that when the eigenfunction expansion is valid then in (2.36) we may set $\lambda_1=\mu_1$ with the result that

$$E'\leqslant F'/\mu_1, \tag{3.11}$$

which gives the same result when substituted in (3.4).

4. Generalized solutions

To prove the existence of solutions to the quasi-geostrophic equation, we shall adopt a wider view of what constitutes a solution than the classical one. The technique of posing the relevant questions with respect to norms in function spaces has proven successful in the study of differential equations, and is typified by Ladyzhenskaya's (1963) book on incompressible flow of viscous fluids.

The problem considered here is the quasi-geostrophic equation (2.5) subject to the boundary conditions that ψ_z vanish at $z=0$ and at $z=Z_T<\infty$. The initial data are assumed to be regular, so that in particular, the integrals $E(0)$, $F(0)$, and $\int\rho_0\psi^2dV$ are all finite at time $t=0$. For this situation, the eigenvalue problem discussed in Section 2b yields a complete, orthonormal set $\{\psi_n(\mathbf{x})\}$ of eigenfunctions and an associated set $\{\lambda_n\}$ of discrete eigenvalues with $\lambda_n\rightarrow\infty$ as $n\rightarrow\infty$. Any continuous function square integrable with respect to ρ_0 has an expansion that converges in mean-square.

a. The truncated spectral equation

We begin with truncated versions of the eigenfunction expansions for the streamfunction ψ. That is, we define the functions

$$\psi^N=a_0(0)\psi_0+\sum_{n=1}^{N}a_n(N,t)\psi_n(\mathbf{x}), \tag{4.1}$$

and because the number of terms is finite, we have now without difficulty that

$$L(\psi^N)=-\sum_{n=1}^{N}\lambda_n a_n\psi_n. \tag{4.2}$$

Substitution of (4.1) and (4.2) in (2.5) gives the equation

$$-\sum_{m=1}^{N}\frac{\partial a_m}{\partial t}\lambda_m\psi_m+\beta\sum_{j=1}^{N}a_j\overset{j}{\psi_x} +J\left[\sum_{k=1}^{N}a_k\psi_k,\,-\sum_{m=1}^{N}\lambda_m a_m\psi_m\right]=0, \tag{4.3}$$

in which we have adopted the notation $\partial\psi_n/\partial x=\overset{n}{\psi_x}$. When this equation is multiplied by $\rho_0\psi_n$ and integrated, the orthonormality of the eigenfunctions produces

$$\lambda_n\frac{\partial a_n}{\partial t}-\beta\sum_{j=1}^{N}a_jC_{jn}+\sum_{k=1}^{N}\sum_{m=1}^{N}a_ka_m\lambda_mD_{kmn}=0,\quad n=1,2,\ldots,N, \tag{4.4}$$

where

$$\left.\begin{aligned}C_{jn}&=\int\rho_0\psi_n\overset{j}{\psi_x}dV\\ D_{kmn}&=\int\rho_0\psi_nJ(\psi_k,\psi_m)dV\end{aligned}\right\}. \tag{4.5}$$

Thus, the function ψ^N will be a solution provided that the coefficients $\{a_n\}$ satisfy the nonlinear system (4.4). Before establishing the existence of solutions to this equation, we prove that the coefficients (4.5) are bounded.

For C_{jn} we have by virtue of the Schwarz inequality and normality of the eigenfunctions

$$|C_{jn}|\leqslant\left[\int\rho_0(\psi_n)^2dV\int\rho_0(\overset{j}{\psi_x})^2dV\right]^{\frac{1}{2}},$$

$$\leqslant\left[\int\rho_0\left\{|\nabla_H\psi_j|^2+\frac{1}{\sigma}|\overset{j}{\psi_z}|^2\right\}dV\right]^{\frac{1}{2}}=(\lambda_j)^{\frac{1}{2}}. \tag{4.6}$$

For D_{kmn}, we observe first that ψ_n is differentiable throughout the closed and bounded domain V so that it is continuous and hence bounded. Let $\mathrm{Max}_V|\psi_n|=\overset{n}{\psi_M}$. Then

$$|D_{kmn}|\leqslant\overset{n}{\psi_M}\int\rho_0|\overset{k}{\psi_x}\overset{m}{\psi_y}-\overset{k}{\psi_y}\overset{m}{\psi_x}|dV,$$

$$\leqslant\overset{n}{\psi_M}\left\{\left[\int\rho_0(\overset{k}{\psi_x})^2dV\int\rho_0(\overset{m}{\psi_y})^2\right]^{\frac{1}{2}}+\left[\int\rho_0(\overset{k}{\psi_y})^2dV\int\rho_0(\overset{m}{\psi_x})^2\right]^{\frac{1}{2}}\right\},$$

$$\leqslant 2\overset{n}{\psi_M}(\lambda_k\lambda_m)^{\frac{1}{2}}. \tag{4.7}$$

The system (4.4) is analytic in the variables $\{a_n\}$ and admits of a power series solution in the form

$$a_n=a_n(0)+\sum_{K=1}^{\infty}A_{Kn}t^K,\quad n=1,\ldots,N, \tag{4.8}$$

which gives the relationship

$$\sum_{K=1}^{\infty} KA_{Kn}t^{K-1}=\lambda_n^{-1}\{\beta\sum_{j=1}^{N} C_{jn}[a_j(0)+\sum_{K=1}^{\infty} A_{Kj}t^K]$$
$$-\sum_{k=1}^{N}\sum_{m=1}^{N}\lambda_m D_{kmn}[a_k(0)+\sum_{K=1}^{\infty} A_{Kk}t^K]$$
$$\times[a_m(0)+\sum_{K=1}^{\infty} A_{Km}t^K]\}, \quad (4.9)$$

provided the series (4.8) is convergent on the interval $[0,T]$. If it is, then it is uniformly convergent in $[0,T-\epsilon]$ for $\epsilon>0$ and may be differentiated as many times as we choose in the latter interval. The relations (4.9) yield explicit formulas for the coefficients:

$$\left.\begin{aligned} A_{1n}&=\lambda_n^{-1}[\beta\sum_{j=1}^{N} C_{jn}a_j(0) \\ &\quad -\sum_{k=1}^{N}\sum_{m=1}^{N}\lambda_m D_{kmn}a_k(0)a_m(0)] \\ A_{2n}&=(2\lambda_n)^{-1}\{\beta\sum_{j=1}^{N} C_{jn}A_{1j}-\sum_{k=1}^{N}\sum_{m=1}^{N}\lambda_m D_{kmn} \\ &\quad \times[a_k(0)A_{1m}+a_m(0)A_{1k}]\} \\ &\quad n=1,2,\ldots,N \end{aligned}\right\}, \quad (4.10)$$

which obviously are uniquely related to the initial values.

To prove that the series is convergent, it is sufficient to show that $a_n(t)$ is bounded, and this follows immediately from the system of equations itself. It is easy to see that $C_{nj}=-C_{jn}$ through use of an integration by parts and cyclic continuity, and Lemma 3.1 (iii) shows that in addition to the antisymmetry of D_{kmn} in k and m, it is also antisymmetric in k and n and m and n. If (4.4) is multiplied by a_n and summed over $n=1,\ldots,N$, then these observations yield immediately that

$$\frac{\partial}{\partial t}\sum_{n=1}^{N}\lambda_n a_n^2(t)=0, \quad (4.11)$$

so that

$$\sum_{n=1}^{N}\lambda_n a_n^2(t)=\sum_{n=1}^{N}\lambda_n a_n^2(0). \quad (4.12)$$

Clearly then $a_n(t)$ is bounded for all time when the initial energy is finite; thus, we have proven existence of solutions for the truncated spectral model. The same operation with $\lambda_n a_n$ on (4.4) gives

$$\frac{\partial}{\partial t}\sum_{n=1}^{N}\lambda_n^2 a_n^2(t)=\beta\sum_{n=1}^{N}\sum_{j=1}^{N} a_n a_j\lambda_n\int_V \rho_0\psi_n\psi_x^j dV,$$
$$=-\beta\int_V \rho_0 L(\psi^N)\psi_x^N dV,$$
$$=-\beta\left[\int\frac{\rho_0}{\sigma}\psi_x^N\psi_z^N\right]_0^{Z_T} dA=0, \quad (4.13)$$

so that we have

$$\sum_{n=1}^{N}\lambda_n^2 a_n^2(t)=\sum_{n=1}^{N}\lambda_n^2 a_n^2(0). \quad (4.14)$$

The point here is that the truncated solutions conserve the energy $E^N(0)$ and mean-square vorticity $F^N(0)$ of their initial conditions. Thus, for flows with finite initial energy and vorticity moment

$$2E_0=\sum_{n=1}^{\infty}\lambda_n a_n^2(0), \quad 2F_0=\sum_{n=1}^{\infty}\lambda_n^2 a_n^2(0), \quad (4.15)$$

the series

$$2E^N=\sum_{n=1}^{N}\lambda_n[a_n(N,t)]^2, \quad 2F^N=\sum_{n=1}^{N}\lambda_n^2[a_n(N,t)]^2 \quad (4.16)$$

converge as $N\to\infty$. To see this, we observe that for $M<N$,

$$|E^M-E^N|=|E^M(0)-E^N(0)|=\tfrac{1}{2}|\sum_{M+1}^{N}\lambda_n a_n^2(0)|, \quad (4.17)$$

so the fact that $E^N(0)$ is a Cauchy sequence implies that $\lim_{M,N}|E^M-E^N|\to 0$ as M, $N\to\infty$. The same is true for F^N, and the fact that F^N is convergent shows that $a_n^2\lambda_n^2\to 0$ as $n\to\infty$. Finally, we know that there is an integer P such that $\lambda_n>1$ for all $n\geqslant P$. Therefore, the comparison test assures us that the sequence

$$D^N=\sum_{n=1}^{N}[a_n(N,t)]^2 \quad (4.18)$$

converges since both E^N and F^N are convergent. Thus, D^N, E^N and F^N are all Cauchy sequences.

b. *Existence and properties of generalized quasi-geostrophic solutions*

The developments of the previous subsection allow us to prove that functions ψ, $\mathbf{v}_g$, and Φ exist as limits in quadratic mean of the truncated solutions. The assertion that these limit functions are solutions for the simple quasi-geostrophic problem depends upon the concepts of generalized derivatives and generalized solutions. Although we do not emphasize it later, the proofs utilize the Hilbert spaces (containing those quadratically integrable functions whose generalized derivatives are also quadratically integrable) now often referred to as Sobolev spaces and discussed in detail in Sobolev (1963).

We shall use L_2 here to denote the Lebesgue space of equivalence classes of measurable functions quadratically integrable with respect to ρ_0 and we use

$$\|f\|=\left[\int_V \rho_0|f^2|dV\right]^{\frac{1}{2}} \quad (4.19)$$

to denote the norm in L_2; we allow vector-valued functions in L_2 with the understanding that $|f|^2=\mathbf{f}\cdot\mathbf{f}$.

The essential concept here is the Riesz-Fisher theorem (for example, see Riesz and Sz.-Nagy, 1955, pp. 59 and 70) which asserts that if $\{\phi_n\}$ is an orthonormal sequence of functions in L_2 and if c_n is an arbitrary sequence of numbers such that the series $\sum_{n=1}^{\infty} c_n^2$ converges, then there exists a function f in L_2 such that

$$\lim_{N\to\infty}\|\sum_{n=1}^{N} c_n\phi_n-f\|=0, \quad (4.20)$$

with the property that

$$\int f\phi_n dV=c_n.$$

The theorem can be stated in another way:

$$\lim_{M,N\to\infty}\|f^M-f^N\|=0,$$

if and only if there exists $f\in L_2$ such that

$$\lim_{N\to\infty}\|f^N-f\|=0.$$

This asserts that the Banach space L_2 is complete in norm and hence is a Hilbert space. We note that all theorems about L_2 with the usual unweighted norm apply with norm (4.19) through the device of putting $g=\sqrt{\rho_0}f$.

THEOREM 4.1. *The functions ψ^N, $\mathbf{v}_g^N$ and Φ^N obtained from the truncated solutions converge in L_2 norm to functions ψ, $\mathbf{v}_g$ and Φ in L_2.*

PROOF. Because F^N is a Cauchy sequence, it converges as $N\to\infty$; thus, the Riesz-Fisher theorem assures us of the existence of a function Φ such that

$$\lim_{N\to\infty}\|\sum_{n=1}^{N}\lambda_n a_n\psi_n-\Phi\|=0. \quad (4.21)$$

Similarly, because D^N is a convergent sequence, there is a function χ such that

$$\lim_{N\to\infty}\|\sum_{n=1}^{N} a_n\psi_n-\chi\|=0. \quad (4.22)$$

But an arbitrary constant is unimportant in the definition of the quasi-geostrophic streamfunction, so we may put $\psi=a_0(0)\psi_0+\chi$, which is still in L_2 since V is a bounded domain.

Now we use the Riesz-Fisher theorem in the alternate form. Because the limits-in-mean ψ and Φ of ψ^N and Φ^N exist, it is true that

$$\lim_{M,N\to\infty}\|\psi^M-\psi^N\|=0, \quad \lim_{M,N\to\infty}\|\Phi^M-\Phi^N\|=0. \quad (4.23)$$

Thus we obtain

$$\begin{aligned}\|\mathbf{v}_g{}^M-\mathbf{v}_g{}^N\|^2 &\leqslant \int\rho_0\left[|\nabla_H(\psi^M-\psi^N)|^2+\frac{1}{\sigma}|(\psi^M-\psi^N)_z|^2\right]dV,\\ &=-\int\rho_0(\psi^M-\psi^N)L(\psi^M-\psi^N)dV,\\ &\leqslant\|\psi^M-\psi^N\|\,\|\Phi^M-\Phi^N\|\to 0, \quad (4.24)\end{aligned}$$

which shows there exists a function $\mathbf{v}_g$ in L_2 which is the limit-in-mean of $\mathbf{v}_g^N$, completing the proof.

The next question is whether these functions obtained as limits have a relation resembling that of solutions to the equation. We turn now to the notion of generalized derivatives and generalized solutions.

Let ϕ be an element of $C^\ell(V)$ for $\ell\geqslant 1$, where $C^\ell(V)$ is the space of cyclically continuous functions on V whose ℓth derivatives are continuous. Then if f is cyclically continuous and if $\nabla_H f$ is continuous, the divergence theorem applies and

$$\int\phi\nabla_H f dV=\int\nabla_H(\phi f)dV-\int f\nabla_H\phi dV = -\int f\nabla_H\phi dV, \quad (4.25)$$

because the boundary integral vanishes by cyclic continuity. Regardless of whether $\nabla_H f$ exists, we use this notion in:

DEFINITION 4.1. *Let $\phi\in C^\ell(V)$ for $\ell\geqslant 1$. If $F\in L_2$ exists such that*

$$\int\rho_0 F\phi dV=-\int\rho_0 f\nabla_H\phi dV, \quad (4.26)$$

then F is a generalized gradient of f and may be denoted $F=\nabla_{GH}f$.

THEOREM 4.2. *The function $\mathbf{v}_g\in L_2$ obtained as the limit-in-mean of $\mathbf{k}\times\nabla_H\psi^N$ is $\mathbf{k}\times\nabla_{GH}\psi$, where ψ is the L_2 limit of ψ^N.*

PROOF. We have

$$\left|\int \rho_0[(\mathbf{k}\times\mathbf{v}_g)\phi-\psi\nabla_H\phi]dV\right|$$

$$=\left|\int \rho_0[\mathbf{k}\times(\mathbf{v}_g-\overset{N}{\mathbf{v}_g})\phi-(\psi-\psi^N)\nabla_H\phi+\mathbf{k}\times\overset{N}{\mathbf{v}_g}\phi-\psi^N\nabla_H\phi]dV\right|$$

$$\leqslant\|\mathbf{v}_g-\overset{N}{\mathbf{v}_g}\|\,\|\phi\|+\|\psi-\psi^N\|\,\|\nabla_H\phi\|\rightarrow 0, \quad (4.27)$$

because the norms of both ϕ and $\nabla_H\phi$ are finite for $\ell\geqslant 1$ on bounded domains V, and we may choose N to make the other two norms arbitrarily small. The last two terms of the integral on the right vanish because ψ^N is continuously differentiable.

COROLLARY 4.2.1. *The generalized geostrophic velocity* $\mathbf{v}_g$ *is horizontal and solenoidal in the generalized sense.*

PROOF. We have

$$\left|\int \rho_0\mathbf{k}\cdot\mathbf{v}_g dV\right|=\left|\int \rho_0\mathbf{k}\cdot[\overset{N}{\mathbf{v}_g}+\mathbf{v}_g-\overset{N}{\mathbf{v}_g}]dV\right|,$$

$$\leqslant M_0^{\frac{1}{2}}\|\mathbf{v}_g-\overset{N}{\mathbf{v}_g}\|\rightarrow 0, \quad (4.28)$$

where M_0 is the mass $\int\rho_0 dV$. Since $\int\rho_0\mathbf{v}_g\cdot\nabla_H\phi dV=0$ if $\mathbf{v}_g$ is solenoidal in the classical sense, we use

$$\left|\int \rho_0\mathbf{v}_g\cdot\nabla_H\phi dV\right|=\left|\int \rho_0[\overset{N}{\mathbf{v}_g}+\mathbf{v}_g-\overset{N}{\mathbf{v}_g}]\cdot\nabla_H\phi dV\right|$$

$$\leqslant\|\nabla_H\phi\|\,\|\mathbf{v}_g-\overset{N}{\mathbf{v}_g}\|\rightarrow 0 \quad (4.29)$$

to show that $\mathbf{v}_g$ is solenoidal in the generalized sense, completing the proof.

The same notion is used to define a generalized solution. Let ϕ now be in $C^\ell(\mathbf{x},t)$ for $\mathbf{x}\in V$ and let ϕ vanish for $t=0$ and for $t\geqslant T$. Then if ψ, $\mathbf{v}_g$ and Φ are solutions in the classical sense, integrations by parts show that

$$0=\int_0^T\int_V\rho_0\phi\left[\frac{\partial\Phi}{\partial t}+\mathbf{v}_g\cdot\nabla_H\Phi+\beta\psi_x\right]dVdt,$$

$$=-\int_0^T\int_V\rho_0\left[\Phi\frac{\partial\phi}{\partial t}+\Phi\mathbf{v}_g\cdot\nabla_H\phi+\beta\psi\phi_x\right]dVdt. \quad (4.30)$$

DEFINITION 4.2. *Let* $\phi\in C^\ell(\mathbf{x},t)$ *for* $\mathbf{x}\in V$, $\ell\geqslant 1$, *and let* ϕ *have compact support for time in* $(0,\infty)$, *that is,* ϕ *is non-zero only on a closed and bounded set in* $(0,\infty)$. *A generalized solution of the quasi-geostrophic equation is any triple of functions which gives*

$$\int_0^\infty\int\rho_0\left[\Phi\left(\frac{\partial\phi}{\partial t}+\mathbf{v}_g\cdot\nabla_H\phi\right)+\beta\psi\phi_x\right]dVdt=0 \quad (4.31)$$

for all such ϕ.

THEOREM 4.3. *The triple* $(\psi,\mathbf{v}_g,\Phi)$ *obtained as limits in* L_2 *of the truncated solutions* $(\psi^N,\mathbf{v}_g^N,\Phi)$ *is a generalized solution of the quasi-geostrophic equation valid for bounded domains* V.

PROOF. We set

$$\epsilon=\int_0^\infty\int_V\rho_0\left[\Phi\left(\frac{\partial\phi}{\partial t}+\mathbf{v}_g\cdot\nabla_H\phi\right)+\beta\psi\phi_x\right]dVdt, \quad (4.32)$$

and we have

$$0=\int_0^\infty\int_V\rho_0\left[\Phi^N\left(\frac{\partial\phi}{\partial t}+\overset{N}{\mathbf{v}_g}\cdot\nabla_H\phi\right)+\beta\psi^N\phi_x\right]dVdt, \quad (4.33)$$

so that

$$\epsilon=\int_0^\infty\int_V\rho_0\left[(\Phi-\Phi^N)\frac{\partial\phi}{\partial t}+\Phi\mathbf{v}_g\cdot\nabla_H\phi-\Phi^N\overset{N}{\mathbf{v}_g}\cdot\nabla_H\phi+\beta(\psi-\psi^N)\phi_x\right]dVdt. \quad (4.34)$$

For the first term (T_1) of this expression the Schwarz inequality gives

$$|T_1|\leqslant\int_0^\infty\left\|\frac{\partial\phi}{\partial t}\right\|\,\|\Phi-\Phi^N\|dt\rightarrow 0, \quad (4.35)$$

and for the last (T_3)

$$|T_3|\leqslant\int_0^\infty\beta\|\phi_x\|\,\|\psi-\psi^N\|dt\rightarrow 0. \quad (4.36)$$

The middle term (T_2) yields

$$|T_2|=\left|\int_0^\infty\int_V\rho_0[(\Phi^N+\Phi-\Phi^N)(\overset{N}{\mathbf{v}_g}+\mathbf{v}_g-\overset{N}{\mathbf{v}_g})-\Phi^N\overset{N}{\mathbf{v}_g}]\cdot\nabla_H\phi dVdt\right|,$$

$$=\left|\int_0^\infty\int_V\rho_0[\Phi^N(\mathbf{v}_g-\overset{N}{\mathbf{v}_g})+(\Phi-\Phi^N)\overset{N}{\mathbf{v}_g}+(\Phi-\Phi^N)(\mathbf{v}_g-\overset{N}{\mathbf{v}_g})]\cdot\nabla_H\phi dVdt\right|,$$

$$\leqslant\int_0^\infty\{\|\Phi^N\|\,\|\mathbf{v}_g-\overset{N}{\mathbf{v}_g}\|+\|\Phi-\Phi^N\|\,\|\overset{N}{\mathbf{v}_g}\|+\|\Phi-\Phi^N\|\,\|\mathbf{v}_g-\overset{N}{\mathbf{v}_g}\|\}\max_V|\nabla_H\phi|dt\rightarrow 0. \quad (4.37)$$

In each of the integrals (4.35)–(4.37) we use the argument that for any η there is an integer $P(\eta,t)$ such that $\|\psi-\psi^N\|<\eta$, $\|\mathbf{v}_g-\mathbf{v}_g^N\|<\eta$ and $\|\Phi-\Phi^N\|<\eta$ whenever $N>P$. The choice of $N>P_M$ where P_M is the maximum value of P over the support of ϕ allows use of the mean value theorem to reach the conclusion that the integrals vanish for each admissible function ϕ. Hence $|\epsilon| \leqslant 0$, completing the proof.

COROLLARY 4.3.1. *The generalized solutions are unique up to an equivalence class in* L_2 (*that is, unique in* L_2 *norm*).

PROOF. Let $(\psi, \mathbf{v}_g, \Phi)$ and $(\tilde{\psi}, \tilde{\mathbf{v}}_g, \tilde{\Phi})$ be solutions for the same initial data. Since $(\psi^N, \mathbf{v}_g^N, \Phi^N)$ are unique, we have, for example, that

$$\|\psi-\tilde{\psi}\| = \|\psi-\psi^N+\psi^N-\tilde{\psi}\| \leqslant \|\psi-\psi^N\| + \|\tilde{\psi}-\psi^N\| \rightarrow 0. \quad (4.38)$$

The same argument applies to $\mathbf{v}_g$ and Φ, completing the proof.

COROLLARY 4.3.2. *Within any bounded interval* $[0,t]$, *the generalized solutions (and hence classical solutions) depend continuously in norm on the initial data.*

PROOF. Let $(\psi, \mathbf{v}_g, \Phi)$ be associated via $(\psi^N, \mathbf{v}_g^N, \Phi^N)$ with initial fields $(\psi^0, \mathbf{v}_g^0, \Phi^0)$. Denote another triple, associated with initial data $(\tilde{\psi}^0, \tilde{\mathbf{v}}_g^0, \tilde{\Phi}^0)$ with a tilde. Then

$$\|\psi-\tilde{\psi}\| = \|\psi-\psi^N+\tilde{\psi}^N-\tilde{\psi}+\psi^N-\tilde{\psi}^N\| \leqslant \|\psi-\psi^N\| + \|\tilde{\psi}-\tilde{\psi}^N\| + \|\psi^N-\tilde{\psi}^N\|. \quad (4.39)$$

For any value of η, we may find an N large enough that we have $\|\psi-\psi^N\| < \eta$ and $\|\tilde{\psi}-\tilde{\psi}^N\| < \eta$. Now the difference $\psi^N-\tilde{\psi}^N$ in the last term obviously has an expansion of form $\sum_{n=0}^{N} (a_n-\tilde{a}_n)\psi_n$, in which the coefficients are given by (4.1), (4.8) and (4.10). Hence, we find

$$\|\psi^N-\tilde{\psi}^N\| \leqslant [\sum_{n=0}^{N} |a_n-\tilde{a}_n|^2]^{\frac{1}{2}}. \quad (4.40)$$

But for every $n \leqslant N$, the coefficients a_n are given by (4.1) or (4.10) as polynomials of finite order in the initial coefficients a_n^0, so that we may apply the mean value theorem to obtain

$$\begin{aligned}\tilde{a}_n - a_n &= a_n(\overset{0}{\tilde{a}}_0, \ldots, \overset{0}{\tilde{a}}_N) - a_n(\overset{0}{a}_0, \ldots, \overset{0}{a}_N), \\ &= \sum_{k=0}^{N} (\overset{0}{\tilde{a}}_k - \overset{0}{a}_k)[\partial a_n/\partial \overset{0}{a}_k]_{\bar{\mathbf{a}}^0}, \end{aligned} \quad (4.41)$$

where $(\bar{a}_0^0, \ldots, \bar{a}_N^0)$ lies on the line between $(a_0^0, \ldots, a_N^0)$, and $(\tilde{a}_0^0, \cdots, \tilde{a}_N^0)$. The derivatives $\partial a_n/\partial a_k^0$ are polynomials in $\{a_n^0\}$ and are bounded for all families of bounded coefficients a_k^0 if $t \leqslant T < \infty$.

Upon choosing a range for each $a_n^0 [|a_n^0| \leqslant (E_0/N\lambda_n)^{\frac{1}{2}}$, for $n \geqslant 1$, for example], we may determine bounds $B_{nk} = B_{nk}(T,N) < \infty$ such that $|\partial a_n/\partial a_k^0| \leqslant B_{nk}$ so that

$$|\tilde{a}_n - a_n| \leqslant \sum_{k=0}^{N} B_{nk} |\overset{0}{\tilde{a}}_k - \overset{0}{a}_k|.$$

Thus, we now have

$$\|\psi-\tilde{\psi}\| \leqslant 2\eta + [\sum_{n=0}^{N} (\sum_{k=0}^{N} B_{nk} |\overset{0}{\tilde{a}}_k - \overset{0}{a}_k|)^2]^{\frac{1}{2}}. \quad (4.42)$$

For any ϵ we may set $\eta = \epsilon/4$, say, which determines a lower bound for N. Then we may select a set of numbers $\delta_k = \delta_k(N)$ such that

$$[\sum_{n=0}^{N} (\sum_{k=0}^{N} B_{nk}\delta_k)^2]^{\frac{1}{2}} < \epsilon/2.$$

Thus, the norm in (4.42) will be less than ϵ whenever $|\tilde{a}_k^0 - a_k^0| < \delta_k$, and hence the functions ψ depend continuously on the initial data.

Both $\tilde{\mathbf{v}}_g - \mathbf{v}_g$ and $\tilde{\Phi} - \Phi$ have eigenfunction expansions with differences of coefficients [although λ_n and λ_n^2 now appear in the equivalents of (4.40)], so the same argument applies and the proof is complete.

5. Conclusion

A considerable amount of useful information about the nonlinear quasi-geostrophic equation has been developed in this article. The main accomplishment is the demonstration that generalized solutions exist for the nonlinear equation on bounded domains, and the corollary result that the solutions depend continuously on the initial data. Although the existence of solutions hardly provides assurance that the quasi-geostrophic motions are truly relevant to atmospheric problems, at least it is established that the quasi-geostrophic equation represents a well-posed mathematical problem in the generalized sense when it is applied to the class of simple quasi-geostrophic flows—those in which an isentrope coincides with the earth's surface. Extension of the results to the case in which $\delta\psi_z/\delta t$ vanishes at the surface would widen the applicability of the present theory to more realistic cases.

The continuous dependence of solutions on initial data for any finite interval of time is of interest in predictability studies. To the extent that quasi-geostrophic motions provide a useful model of large-scale flow, the study of the characteristics and rate of growth of the difference fields governed by (3.2) should be of interest. The eigenfunctions provide a convenient means of studying the spectral properties of the difference fields which result from slightly different initial conditions. In this regard, it is worth noting that the truncated spectral quasi-geostrophic equation (4.4) is of the form used by Epstein (1969) to develop the theory of stochastic dynamic prediction. A slightly simpler equation has been employed by Lorenz (1969) in a study of the predictability of two-dimensional flow.

It appears, then, that the nonlinear quasi-geostrophic equation, presumably an adequate model of large-scale atmospheric flow and surely a more tractable one than the complete Navier-Stokes equations, together with the techniques of functional analysis in suitable Hilbert or Sobolev spaces, offers an opportunity for analytical study of some of the most significant problems of modern atmospheric dynamics.

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APPENDIX

Symbols and Operators[1]

a	earth's radius
f	Coriolis parameter [$=2\Omega\sin\phi$, where ϕ is the latitude and Ω the earth's angular velocity]
β	latitudinal derivative of f [$=\partial f/a\partial\phi$, and assumed constant in the β-plane approximation]
V	typical magnitude of the horizontal velocity
L	typical horizontal scale of the motions (L is approximately a quarter wavelength) or the non-dimensional length and width of V
∇	the gradient operator
∇_H	the gradient operator restricted to the horizontal plane
$\int(\ )dV$	integral over the domain V
$\int(\ )dA$	surface integral over a horizontal plane bounding V.

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[1] Not defined in the text.