On the Short-Wave Cutoff of CISK

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18 June 1973 and 13 September 1973

1. Introduction

The theory of Conditional Instability of the Second Kind (CISK) for tropical storm development was proposed about ten years ago. The main problem associated with the original CISK models (Charney and Eliassen, 1964; Ooyama, 1964; and others) is that the growth rate increases as the horizontal scale of the convective area decreases. Among the efforts to resolve this problem, the inclusion of lateral diffusion is very unsatisfactory, since lateral diffusion can hardly be expected to have any significant effect before the storm matures. On the other hand, the "movable CISK" mechanism proposed by Charney (1973) requires the presence of a basic current. In addition, Holton (1974) has indicated that Charney's analysis is, in effect, an analogy to the "critical latitude" mechanism due to temporal acceleration of the boundary layer flow (Holton *et al.*, 1971). Thus, the scaling of the boundary layer depth in Charney's model may require more careful consideration.

The purpose of this note is to point out that there is an important mechanism related to Ekman pumping which may be responsible for the short-wave cutoff of growth rate in a CISK model. The mechanism involves the large Ekman damping of small-scale motions in a stratified atmosphere. This effect, included in Holton's (1965) stratified spin-up problem and Kuo's (1973) study of the stratified Ekman layer, is explicitly contained in an earlier work by Chang (1971).

2. Simplified solutions

In the study by Chang (1971) [hereafter referred to as (C)], the effect of Ekman damping and heating may be easily separated for the case when $\beta=0$. The solutions in (C) for the disturbance geopotential may be written[1]

$$\Phi(x,y,z,t)=\sin\lambda y\,\phi(z)e^{i(kx+\omega t)+z/2}. \tag{1}$$

When this solution is substituted into the potential vorticity equation, we obtain Eq. (5) of (C):

$$i\omega\left[\frac{d^2\phi}{dz^2}-\left(\frac{k_h^2}{\epsilon}+\frac{1}{4}\right)\phi\right]=\left(\frac{\partial}{\partial z}-\frac{1}{2}\right)q. \tag{2}$$

Here we have set $\bar{u}=\beta=0$, q is the vertical heating function, $\epsilon=(f^2L^2/gH^2)/[\partial(\ln\theta_s)/\partial z]$ is the "internal rotational Froude number," and $k_h=(k^2+\lambda^2)^{\frac{1}{2}}$ is the total horizontal wavenumber. The lower boundary condition which is applied at the top of the Ekman layer (TE) may be written

$$i\omega\left(\frac{\partial}{\partial z}+\frac{1}{2}\right)\phi+\frac{w_{\mathrm{TE}}}{\epsilon}=q. \tag{3}$$

As in (C), the upper boundary condition requires the geopotential and temperature fields be continuous at the tropopause. The vertical velocity at the top of the Ekman layer takes the form

$$w_{\mathrm{TE}}=-\alpha k_h^2\phi_{\mathrm{TE}}, \tag{4}$$

where α is a proportionality constant determined by the depth of the Ekman layer and the ageostrophic deviation angle at the surface.

We first examine the adiabatic solution for $q=0$ which may be written

$$\phi=a_1\exp(-k_h/\epsilon^{\frac{1}{2}})z+a_2\exp(k_h/\epsilon^{\frac{1}{2}}),\quad \text{for } z\leqslant 2. \tag{5}$$

The coefficients a_1, a_2 are determined by the boundary conditions and in our case $a_2=0$. The frequency is given by

$$i\omega=-\alpha k_h\epsilon^{\frac{1}{2}}\bigg/\left(1-\frac{\epsilon^{\frac{1}{2}}}{2k_h}\right). \tag{6}$$

When the horizontal scale is small, the ϕ field decays rapidly above the boundary layer. The decay rate becomes infinite when the horizontal scale goes to zero. [The denominator on the right-hand side of (6) approaches 1 for very small scales, and it is identically 1 if the Boussinesq approximation is used.] This behavior results from the requirement [Eq. (2)] that the potential vorticity must vanish in the interior. Note that (6) also shows that the damping rate is proportional to the square root of dry static stability.

This large damping for small horizontal scales may be interpreted from the boundary condition (3) with $q=0$ and the solution (5). In a region of cyclonic horizontal shear (low pressure) the air rising out of the Ekman layer will cause adiabatic cooling just above the boundary layer. Through hydrostatic pressure changes this process tends to eliminate the surface low pressure area, but the pressure patterns at higher levels will not be changed. The damping rate can be found directly from (3) if we write $d\phi/dz=-k_z\phi_{\mathrm{TE}}$, where k_z is the scale of vertical structure, to obtain the expression

$$i\omega=-\frac{\alpha k_h^2}{\epsilon(k_z-\frac{1}{2})}. \tag{7}$$

Eq. (7) becomes identical to (6) with the use of (5) where $k_z=k_h\epsilon^{\frac{1}{2}}$.

Let us now examine a purely growing solution which is obtained with the lower boundary condition

$$i\omega\left(\frac{\partial}{\partial z}+\frac{1}{2}\right)\phi=0,\quad \text{at}\quad z=0. \tag{8}$$

This implies that the heating balances adiabatic cooling at the lower boundary. If we use the same heating function as in (C), the growth rate is given by the frequency equation

$$i\omega=\pi m\alpha k_h^2\bigg/\left(\frac{\pi^2}{2}+\frac{2k_h^2}{\epsilon}\right). \tag{9}$$

This growth rate reaches a maximum finite value when the horizontal scale is decreased to zero. The sum of (6) and (9) gives the growth/decay rate when both heating and frictional effects are included [which is just Eq. (17) in (C)]. For any arbitrary vertical heating profile, as long as there is no heating at the top of the Ekman layer and the tropopause, Fourier transforms can be applied to obtain the particular solution of ϕ, and the resultant frequency equation would always contain the frictional decay term expressed by (6). Thus, the frictional effect will always dominate for small scales because of the unbounded Ekman damping rate at zero scale.

It can be inferred from (3) that, as long as there is some temperature fluctuation at the top of the Ekman layer [using Ooyama's (1969a) notation, $\eta<1$], this damping effect will be present.[2] Recently, Arakawa and Chao (1973) applied Arakawa's new cumulus parameterization scheme to damp out small-scale motion in a

[1] All notations follow Chang (1971) unless otherwise stated.

[2] After the completion of this note the authors became aware that similar results have been reported by Ooyama in previous public lectures (1969b). Ooyama has also shown that this short wave cutoff can also result at the Ekman layer top if $\eta=1$ but $d\eta/dz<1$.

tropical model, which seems to include this effect because in Arakawa's scheme the cloud base is slightly above the top of the boundary layer.

3. Ekman damping in two-level models

We will now examine the Ekman damping in a simple two-level model. The atmosphere between $z=0$ and $z=2$ is divided into four layers of thickness $\Delta z=\frac{1}{2}$; the levels between the layers are numbered from 0 to 4 from the top down. A rigid lid is placed at $z=2$ and the Boussinesq approximation is used. If the vorticity equation is applied at levels 1 and 3 we obtain

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial \Phi_1}{\partial t}+w_2=0, \tag{10}$$

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{\partial \Phi_3}{\partial t}-w_2+w_4=0. \tag{11}$$

The first law of thermodynamics applied at level 2 is

$$\frac{\partial}{\partial t}(\Phi_1-\Phi_3)+\frac{w_2}{\epsilon}=0, \tag{12}$$

where the heating is zero. In this notation the boundary condition (4) becomes

$$w_4=\alpha\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Phi_4. \tag{13}$$

We now write

$$\Phi_4=\Phi_3, \tag{14}$$

which corresponds to the boundary condition (8). When Eqs. (10)–(14) are combined and the solution (1) is employed, we obtain

$$i\omega=-\alpha(\epsilon+k_h^2)/(2\epsilon+k_h^2). \tag{15}$$

This equation shows that the damping rate, which is just the "spin-down" rate in the two-level model, is nearly independent of scale and has a value of α for small scales. Thus, the short-wave cutoff mechanism of the continuous equation is not present in the equations for the two-level model.

The cutoff mechanism can be included in the two-level model if the first law of thermodynamics is applied at the top of the frictional layer; this gives

$$\frac{\partial}{\partial t}(\Phi_3-\Phi_4)+2\frac{w_4}{\epsilon}=0. \tag{16}$$

When this equation is used in place of (14) the frequency becomes

$$i\omega=\frac{-2\alpha k_h^2}{\epsilon}-\frac{(\epsilon+k_h^2)\alpha}{2\epsilon+k_h^2}. \tag{17}$$

This solution shows strong damping for small scales, but the damping rate is an overestimate since the continuous solution (6) gives damping proportional to the first power of k_h.

4. Remarks

The consequences of the above discussion on the CISK mechanism is obvious. For short scales, the adiabatic cooling due to Ekman pumping would damp out the flow near the lower boundary and thus diminish the Ekman layer and turn off CISK. One may argue about the validity of hydrostatic and geostrophic balances for the very small scales, but we think that for the large-scale motion system, which is more or less quasi-geostrophic, the tendency does indicate the short-wave cutoff.

An important point noted is that the basic solution may not be reproduced when the continuous vertical structure is replaced by a number of layers. The two-level model of Charney and Eliassen (1964), as can be seen in Section 3, does not contain the strong damping at small scales because they used (8) as their lower boundary condition. If the two-level model is modified so that temperature fluctuation is allowed at the lower boundary, a damping rate proportional to k_h^2 is obtained. This damping rate, which is even greater than that for the continuous case (where it is proportional to k_h), can also be seen from (7), where k_z is replaced by a constant which is determined by the vertical spacing between the two levels. As a consequence, proper treatment of this cutoff mechanism requires adequate vertical resolution in the lower troposphere.

Acknowledgments. We wish to express our appreciation to Prof. J. R. Holton for several valuable discussions and to Prof. G. J. Haltiner for reading the manuscript. We would also like to thank Prof. K. Ooyama for making several valuable comments on the manuscript. This research was supported by the Environmental Prediction Research Facility.

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