Wave-Induced Instabilities in an Atmosphere Near Saturation

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ABSTRACT

In this paper the stability and the propagation characteristics of internal gravity waves that propagate in an atmosphere near saturation and, over some height range, create the appropriate thermodynamic conditions for condensation to occur for a fraction of the wave cycle, are investigated. It is shown that if the atmosphere, over some height range, is close enough to saturation, a linear stability analysis is possible and results in a modified Richardson criterion, based on a new Brunt-Väisälä frequency n_{av} smaller than the corresponding n_u with condensation effects neglected. Since n_{av}^2 can become negative, even though the atmosphere is originally statically stable, the ability of gravity waves to trigger convective processes in a moist atmosphere is demonstrated. Finally, a numerical example is presented to illustrate the alterations of the characteristics of propagation.

1. Introduction

The importance of gravity waves in influencing, modifying and coupling different elements of mesoscale systems has lately received much attention (Matsumoto and Ninomiya, 1969; Uccellini, 1973; Bosart and Cussen, 1973). A reason for this interest is the unique ability of gravity waves to span the scale range from fronts (500 km), through squall lines (100 km) and jet-associated vertical shear maxima (50 km) down to convective cells (5 km), and to effectively transport momentum, as well as organize and trigger other phenomena.

If gravity waves are to play an important role in the dynamical processes of mesoscale systems, they should be able to interact strongly with the environment. A measure of this interaction between gravity waves and such systems is the time evolution of the waves themselves, i.e., the growth rates or decay times of waves, if any, since unstable waves grow and alter the initial background in which they propagate. Thus stability analyses of the atmosphere against disturbances of the gravity wave type are important and form the first step in theoretically investigating the underlying dynamical processes.

Of particular interest are the situations in which the atmosphere contains moisture, so that condensation effects enter or even dominate the dynamical processes. For an atmosphere saturated throughout and with liquid phase present, the propagation characteristics of internal gravity waves (with period larger than the Brunt-Väisälä period) were investigated by Einaudi and Lalas (1973) and were found to be substantially altered by release of latent heat. Lalas and Einaudi (1973, 1974) carried out stability analyses of such an atmosphere against infinitesimal perturbations using both a rigorous analytical technique in the manner of Howard (1961) and parcel method considerations (see Sutton, 1953; Brunt, 1952; Holton, 1972) and showed that the stability of a saturated atmosphere depends on a suitably defined Brunt-Väisälä frequency which is considerably smaller than the equivalent unsaturated one, for the same temperature profile.

The case of a fully saturated atmosphere provides a bound for condensation effects, since, qualitatively, latent heat exchange is maximal. There still remains, however, the more frequently encountered, and more interesting, case of an atmosphere which is close to, but not at saturation, over some height interval, and which may be brought to saturation by the wave itself, over some portion of its cycle. Examples of this situation are fairly common and can be easily seen in weather satellite pictures (Gruber, 1971, private communication). In such cases the atmosphere, even though initially dynamically and statically stable, may become unstable, either to shear instabilities or even to convective ones, when modified by the wave. The interaction then is more varied and of more direct application to real situations of wave-induced or wave-intensified storms (Bosart, 1973; Uccellini, 1973), squall lines (Tepper, 1955), dry lines (Palmén and Newton, 1969), fair weather trade wind conditions (Pennell and LeMone, 1974) and other phenomena.

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Substantial difficulties arise in trying to describe the propagation of gravity waves and their stability for the almost saturated case, since the wave background interactions are essentially nonlinear. A beginning can be made, at least for the wave-induced convection, by following the parcel or slice method, through the use of tephigrams on which the wet-bulb temperature and the usual dry temperature are plotted (see Normand, 1938; Hewson and Longley, 1951). This type of analysis gives an estimate of the minimum vertical displacement required for a parcel of fluid to find itself in an unstable environment and thus to ascend to greater heights; it does not determine, however, any stability parameter, growth rate or structure of the waves, nor does it properly take into consideration the dynamical processes.

The latter are the object of this paper. We will specifically investigate the stability and the characteristics of a nearly periodic gravity wave which propagates in an atmosphere near saturation and which, over some height range, creates the appropriate thermodynamic conditions for condensation to occur over a fraction of the wave cycle. It will be shown that under certain simplifying hypotheses a linear stability analysis, analogous to the one used by Howard (1961), Chimonas (1970) and Lalas and Einaudi (1973, 1974), can be used, based on a new amplitude-dependent Brunt-Väisälä frequency n_{av}. The background is taken to be statically stable so that its undisturbed Brunt-Väisälä frequency n_u is real and positive; the square of the new Brunt-Väisälä frequency is shown to be smaller than n_u^2 and in some cases even negative.

When the square of the new Brunt-Väisälä frequency is negative, convective activity will be triggered, and the present analysis is in question. It certainly does not apply when strong convective modes are excited as in the case of disturbances in a conditionally unstable tropical atmosphere at times when constructive interaction between deep cumulus convection and cyclone-scale disturbances is essential for the intensification of the disturbances themselves, as discussed by Charney and Eliassen (1964), Rosenthal and Koss (1968), Rosenthal (1970), Charney (1973), Lindzen (1974) and others. On the other hand, when convection is weak, it is possible that the wave solution can describe the initial stages of the process, i.e., for sufficiently short times, before other instabilities, triggered by the disturbance, invalidate the wave solution itself. This point is presently under investigation by the authors.

The substantial reduction of the Brunt-Väisälä frequency, due to wave-induced condensation that we have demonstrated, gives rise to enhanced growth rates of the waves, or, indeed, to excitation of unstable modes in an otherwise stable system, and, we believe, constitutes added evidence for the strong role gravity waves play in the dynamics of mesoscale systems, while at the same time points clearly at the need for further experimental examination of such wave-background interactions.

In Section 2 the model adopted is discussed and the governing equations are derived; in Section 3 the stability analysis of the system is carried out; and in Section 4 a numerical example is given to illustrate the quantitative effect of a gravity wave driving the atmosphere convectively unstable.

2. The model and the governing equations

We assume a horizontally homogeneous system in which the background temperature T_0 and wind velocity u_0 are functions of the vertical coordinate z only. If superscripts 1 and 2 indicate dry air and water vapor, respectively, the total background pressure p_{M0} and density ρ_{M0} are related by the hydrostatic equation

$$\frac{d}{dz}p_{M0} = -g\rho_{M0}, \tag{2.1}$$

where

$$p_{M0}=p_0^{(1)}+p_0^{(2)},\quad \rho_{M0}=\rho_0^{(1)}+\rho_0^{(2)}, \tag{2.2}$$

and g is the gravitational acceleration acting in the negative z direction. In addition, partial background densities and pressures, and temperature satisfy the equation of state for perfect gases:

$$p_0^{(i)}=R^{(i)}T_0\rho_0^{(i)},\quad i=1,2, \tag{2.3}$$

where $R^{(1)}$ and $R^{(2)}$ are the gas constants for dry air and water vapor respectively. Since the system is not saturated at any height, $p_0^{(2)}$ and $\rho_0^{(2)}$ are not related to T_0 via the Clausius-Clapeyron equation; but we further assume that $p_0^{(2)}$, $\rho_0^{(2)}$ and T_0 are such that the background water vapor is close to saturation within the height interval $z_1 \leqslant z \leqslant z_2$. This means that the point $Q_0(z,p_0^{(2)},V_0,T_0)$ which represents the state of the liquid water-vapor system on the thermodynamic diagram in Fig. 1a is rather close to the bell-shaped curve ℓ separating saturated from unsaturated regimes. V is the total volume of the liquid and vapor parts of the mixture.

An internal gravity wave, assumed to be essentially periodic in time, propagating in such a medium, changes the values of the thermodynamic variables at each point in space. Schematically, if T, $p^{(2)}$ and $\rho^{(2)}$ are the actual temperature, partial pressure and density for water vapor and $\rho^{(3)}$ is the partial density of water droplets, each of them being given by the sum of their background and perturbation values, then the point $Q(z,p^{(2)},V,T)$ will execute in the plane of Fig. 1 an essentially closed trajectory L around $Q_0(z,p_0^{(2)},V_0,T_0)$. If Q_0 is sufficiently close to the saturation curve ℓ, the trajectory L will intersect ℓ in two points A and B. For the portion of L inside ℓ, the system is saturated and subsequent changes will result in phase change with its associated latent heat exchange. For a wave

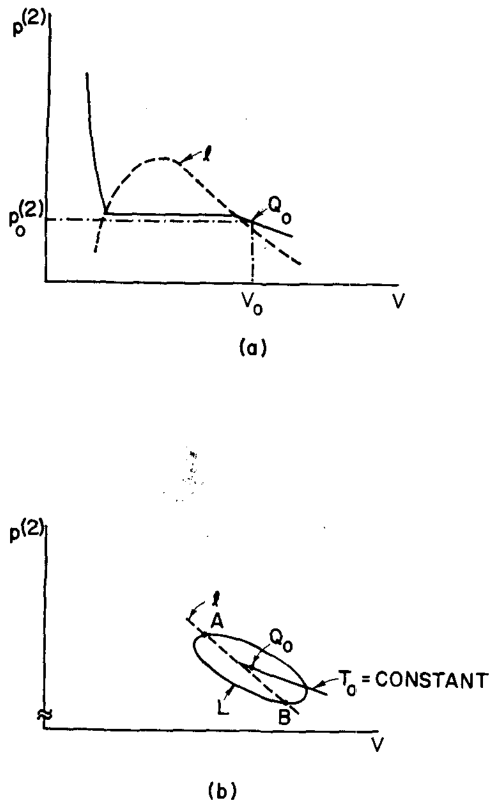

FIG. 1a. The thermodynamic diagram for water vapor, plotting partial pressure for water vapor $p^{(2)}$ vs the total volume V of the liquid and vapor parts of the mixture. The saturation curve ℓ is indicated by a dashed line. The initial state of the atmosphere before it is perturbed by the wave is denoted by $Q_0[p_0^{(2)}, V_0=1/\rho_0^{(2)}]$, and the isotherm through that point is indicated by a solid line.

FIG. 1b. The trajectory L that the thermodynamic variables $p^{(2)}$, V of the moist atmosphere trace under the influence of the wave perturbation. The states at which the atmosphere is just at saturation with infinitesimal liquid content present are marked A and B, and are specified by Eq. (2.56).

with small growth rate (small in some sense to be specified later) the situation is nearly periodic in time and, of course, in x. Thus, to the extent that no changes of the background quantities are allowed, the situation described above repeats itself for every subsequent passage of the internal wave at the height in question.

The end result of this process, therefore, can be thought of as being represented by a source (or sink) which feeds heat to (or subtracts it from) the wave with some phase difference from the wave itself. The determination of a suitable expression for this source term and its phase is the main difficulty of the problem.

a. The general form of the equations

In the absence of saturation, the system of dry air and water vapor can be treated as a mixture of two ideal gases. When saturation occurs, the atmosphere can be considered a mixture of three fluids—dry air, water vapor and water droplets—provided that the droplets are small in comparison with the characteristics scales of the system and numerous enough to be treated as a continuum. If, in addition, the disturbances to be discussed are slow enough so that the droplets are perfectly coupled with the gas as far as momentum and energy exchange are concerned, the linearized form of the equations, in the presence of a horizontal, height-dependent background wind $u_0(z)$, is as follows:

CONSERVATION OF MASS

$$\frac{d}{dt}\rho_s^{(i)}+\rho_0^{(i)}\nabla\cdot\mathbf{v}_s+v_{sz}\frac{d}{dz}\rho_0^{(i)}=S_s^{(i)},\quad i=1,2,3,\qquad(2.4)$$

$$\frac{d}{dt}\rho_{Ms}+\rho_{M0}\nabla\cdot\mathbf{v}_s+v_{sz}\frac{d}{dz}\rho_{M0}=0.\qquad(2.5)$$

CONSERVATION OF MOMENTUM

$$\rho_{M0}\frac{d}{dt}v_{sx}+\frac{\partial}{\partial x}p_{Ms}+\rho_{M0}v_{sz}\frac{du_0}{dz}=0,\qquad(2.6)$$

$$\rho_{M0}\frac{d}{dt}v_{sz}+\frac{\partial}{\partial z}p_{Ms}+\rho_{Ms}g=0.\qquad(2.7)$$

CONSERVATION OF ENERGY

$$[\rho_0^{(1)}c_p^{(1)}+\rho_0^{(2)}c_p^{(2)}]\left[\frac{d}{dt}T_s+v_{sz}\frac{d}{dz}T_0\right]-\left[\frac{d}{dt}p_{Ms}+v_{sz}\frac{d}{dz}p_{M0}\right]=-L_vS_s.\qquad(2.8)$$

EQUATIONS OF STATE

$$p_s^{(i)}=R^{(i)}\left[T_0\rho_s^{(i)}+\rho_0^{(i)}T_s\right],\ i=1,2.\qquad(2.9)$$

We refer to Einaudi and Lalas (1973) for a detailed derivation and justification of the above equations. The superscript 3 refers to the water-droplets continuum while the suffix s indicates perturbation quantities so that, for example, the total temperature T will be given by $T=T_0+T_s$; the background density of water droplets $\rho_0^{(3)}$ is assumed negligible throughout since the undisturbed system is unsaturated; v_{sx} and v_{sz} denote horizontal and vertical velocity perturbations, respectively; $c_p^{(i)}$ is the specific heat at constant pressure; L_v is the latent heat of evaporation; $S_s^{(1)}=0$; $S_s^{(2)}=-S_s^{(3)}=S_s$ is the production of water vapor per unit volume and unit time; and the total pressure and density, and the operator d/dt, are defined by

$$\rho_{Ms}=\sum_{i=1}^{3}\rho_s^{(i)},\quad p_{Ms}=\sum_{i=1}^{2}p_s^{(i)},\qquad(2.10)$$

$$\frac{d}{dt}=\frac{\partial}{\partial t}+u_0\frac{\partial}{\partial x}.\qquad(2.11)$$

Eq. (2.5) results from the addition of Eqs. (2.4).

If there were no source term, the above set of equations would be the same as the one governing the propagation of acoustic-gravity waves in a dry atmosphere (Hines, 1960).

While we defer a detailed discussion of the source term S_s, we would like to discuss here some of its qualitative characteristics. Because of the near periodicity of the internal gravity wave, of period τ, the source term $S_s(x,z,t)$ will have, for given x and z, the qualitative behavior, as a function of time, indicated by the solid line in Fig. 2. We then assume that such a function can be expanded in a Fourier series as

$$S_s(x,z,t)=\sum_{n=1}^{\infty}\left[S_n(z)e^{in(\omega t-k_x x)}+\text{c.c.}\right], \qquad (2.12)$$

where c.c. stands for the complex conjugate, k_x is the horizontal wavenumber, and ω is the fundamental frequency of the system. The frequency $\omega=\omega_r+i\omega_i$ will, in general, be complex, with $\omega_r=2\pi/\tau$. In order to be able to treat the wave as almost periodic, we must assume that

$$|\omega_i t|\ll 1, \qquad (2.13)$$

which imposes an upper limit to the values of time beyond which the analysis is no longer valid. The presence of this source term will cause harmonic distortion in all the thermodynamic variables of the system. All the variables in Eqs. (2.4)–(2.9) will have to be represented then by a series of the form

$$A_s(x,z,t)=\sum_{n=1}^{\infty}\left[A_n(z)e^{in(\omega t-k_x x)}+\text{c.c}\right], \qquad (2.14)$$

with different relative harmonic content for each individual variable.

The appearance of higher harmonics is common in processes of this nature as recognized by several authors (Phillips, 1966; Hodges, 1967; Orlanski and Bryan, 1969; Chimonas, 1972; Lindzen, 1974); higher wavenumbers appear in wave disturbances which act within otherwise stable systems and which cause the system to become unstable. It should be pointed out, though, that these higher harmonics are quite distinct from those which would appear if the nonlinear terms of the original governing equations were retained. Since the objective of the linear stability analysis is to solve an eigenvalue problem giving ω in terms of an assigned k_x, it will suffice to retain the fundamental harmonic only. Hence all quantities will reduce to the form

$$A_s(x,z,t)\approx A_1(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.15)$$

$$\varphi=\omega t-k_x x. \qquad (2.16)$$

In the following, we often use the notation $A_1=A_1(z)$.

We now choose the following expression for the vertical component of velocity:

$$v_{sz}(x,z,t)\approx v_{1z}(x,z,t)=v_{1z}(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.17)$$

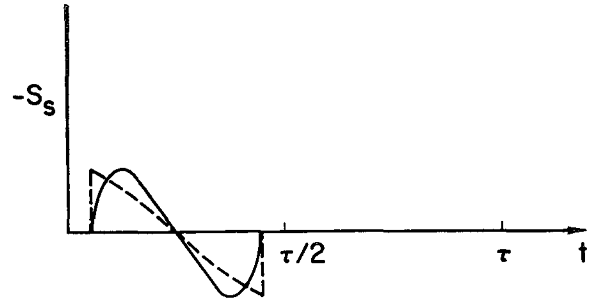

FIG. 2. The behavior of the rate of condensation over a period of the wave. Solid line indicates the "true" behavior while dashed line indicates the first approximation given by Eq. (2.41).

and we relate all the other dependent variables to v_{1z} in order to obtain a second-order differential equation in v_{1z}. The resulting polarization coefficients relating v_{1x}, ρ_{M1}, p_{M1}, T_1, $\rho_1^{(2)}$, etc., to v_{1z} and its derivative with respect to z depend on whether or not saturation occurs.

Let us define, for a given z, the two values of φ, $\varphi_1(z)$ and $\varphi_2(z)$, with $0\leqslant\varphi_1<\varphi_2\leqslant 2\pi$, such that where $\varphi_1\leqslant\varphi\leqslant\varphi_2$ the system is saturated. The method of determination of φ_1 and φ_2 will be described later. We now write:

$$v_{sx}^{(j)}(x,z,t)\approx v_{1x}^{(j)}(x,z,t)=v_{1x}^{(j)}(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.18)$$

$$p_{Ms}^{(j)}(x,z,t)\approx p_{M1}^{(j)}(x,z,t)=p_{M1}^{(j)}(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.19)$$

$$\rho_{Ms}^{(j)}(x,z,t)\approx \rho_{M1}^{(j)}(x,z,t)=\rho_{M1}^{(j)}(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.20)$$

$$T_s^{(j)}(x,z,t)\approx T_1^{(j)}(x,z,t)=T_1^{(j)}(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.21)$$

$$\rho_s^{(2,j)}(x,z,t)\approx \rho_1^{(2,j)}(x,z,t)=\rho_1^{(2,j)}(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.22)$$

$$p_s^{(2,j)}(x,z,t)\approx p_1^{(2,j)}(x,z,t)=p_1^{(2,j)}(z)\exp(i\varphi)+\text{c.c.}, \qquad (2.23)$$

where

$$j=\left\{\begin{matrix} u, & \text{for} & 0\leqslant\varphi<\varphi_1,\ \varphi_2<\varphi\leqslant 2\pi \\ c, & \text{for} & \varphi_1\leqslant\varphi\leqslant\varphi_2 \end{matrix}\right\}.$$

When the superscript j does not appear on a variable, as in (2.17), then the expression for such a variable is valid for any φ. Each composite variable valid for all φ is obtained as follows: Defining

$$A_s(x,z,t)\approx A_1(z)\exp(i\varphi)+\text{c.c.}, \quad \text{for any value of } \varphi,$$

$$A_s(x,z,t)\approx A_1^{(u)}(z)\exp(i\varphi)+\text{c.c.}, \quad \text{for } 0\leqslant\varphi<\varphi_1,\ \varphi_2<\varphi\leqslant 2\pi,$$

$$A_s(x,z,t)\approx A_1^{(c)}(z)\exp(i\varphi)+\text{c.c.}, \quad \text{for } \varphi_1\leqslant\varphi\leqslant\varphi_2,$$

we expand $A_s(x,z,t)$ in a Fourier series and retain the first term only:

$$A_s(x,z,t)=\sum_{n=1}^{\infty}\left[A_n(z)e^{in\varphi}+\text{c.c.}\right]\approx A_1(z)e^{i\varphi}+\text{c.c.},$$

so that $A_1(z)$ is by definition

$$A_1(z)=\frac{1}{2\pi}\int_0^{2\pi} A_s(x,z,t)e^{-i\varphi}d\varphi$$

$$\approx\frac{1}{2\pi}\left\{\int_0^{\varphi_1}[A_1^{(u)}(z)e^{i\varphi}+\text{c.c.}]e^{-i\varphi}d\varphi\right.$$

$$+\int_{\varphi_1}^{\varphi_2}[A_1^{(c)}e^{i\varphi}+\text{c.c.}]e^{-i\varphi}d\varphi$$

$$\left.+\int_{\varphi_2}^{2\pi}[A_1^{(u)}e^{i\varphi}+\text{c.c.}]e^{-i\varphi}d\varphi\right\}$$

$$=\frac{1}{2\pi}\left\{A_1^{(u)}[\varphi_1-\varphi_2+2\pi]+(\varphi_2-\varphi_1)A_1^{(c)}\right.$$

$$\left.-\frac{1}{2i}[e^{-2i\varphi_1}-e^{-2i\varphi_2}][A_1^{(u)*}-A_1^{(c)*}]\right\}. \quad (2.24)$$

Complex conjugates are also indicated by an asterisk. The above formulas are valid for only real φ, i.e., only if relation (2.13) is satisfied. Here let us again emphasize that (2.13) restricts our analysis to the earliest stage of growth of the wave.

If the unperturbed atmosphere is so close to saturation that

$$\varphi_2-\varphi_1=\pi+O(\delta), \quad (2.25)$$

where δ is a smallness parameter to be specified later, then (2.24) reduces to

$$A_1(z)\approx\tfrac{1}{2}[A_1^{(u)}+A_1^{(c)}]. \quad (2.26)$$

The parameter δ is defined by (2.57). The condition $\delta\ll 1$ is a measure of how close the background system has to be to saturation for a gravity wave of small but finite amplitude to bring the system to saturation over almost half of the wave cycle. We shall restrict ourselves in this paper to the case $\varphi_2-\varphi_1\approx\pi$, in part because the calculations are much simpler, in part because if one were to include the case $\varphi_2-\varphi_1$ small compared to π, then a careful analysis of the order of magnitude of the source term compared to the neglected nonlinear terms would be required.

We now proceed to relate $v_{1x}^{(j)}$, $p_{M1}^{(j)}$, etc., to v_{1z} in order to obtain the polarization coefficients.

b. Calculations for $0\leqslant\varphi<\varphi_1$ *and* $\varphi_2<\varphi\leqslant 2\pi$

Since, in this range of values of φ, the source term is zero, the perturbed water vapor density can readily be expressed in terms of v_{1z}, $\rho_{M1}^{(u)}$ and $p_{M1}^{(u)}$ using the continuity Eq. (2.5):

$$\rho_1^{(2,u)}=\frac{\rho_0^{(2)}}{\rho_{M0}}\rho_{M1}^{(u)}+\frac{v_{1z}}{i\Omega\rho_{M0}}\left[\rho_0^{(2)}\frac{d}{dz}\rho_{M0}-\rho_{M0}\frac{d}{dz}\rho_0^{(2)}\right], \quad (2.27)$$

where

$$\Omega=\omega-k_x u_0.$$

From the equation of state one then obtains the following expression for $T_1^{(u)}$:

$$T_1^{(u)}=\frac{1}{[R^{(1)}\rho_0^{(1)}+R^{(2)}\rho_0^{(2)}]}\times\left\{p_{M1}^{(u)}-\frac{T_0}{p_{M0}}[R^{(1)}\rho_0^{(1)}+R^{(2)}\rho_0^{(2)}]\rho_{M1}^{(u)}\right.$$
$$\left.-\frac{T_0[R^{(2)}-R^{(1)}]}{i\Omega\rho_{M0}}\left[\rho_0^{(2)}\frac{d}{dz}\rho_{M0}-\rho_{M0}\frac{d}{dz}\rho_0^{(2)}\right]v_{1z}\right\}. \quad (2.28)$$

Using (2.28), we can cast the energy equation in the form

$$i\Omega p_{M1}^{(u)}=c_u^2\left[i\Omega\rho_{M1}^{(u)}-\frac{\rho_{M0}}{g}n_u^2 v_{1z}\right], \quad (2.29)$$

where

$$c_u^2=\frac{T_0}{\rho_{M0}}\frac{[\rho_0^{(1)}R^{(1)}+\rho_0^{(2)}R^{(2)}][\rho_0^{(1)}c_p^{(1)}+\rho_0^{(2)}c_p^{(2)}]}{\{\rho_0^{(1)}[c_p^{(1)}-R^{(1)}]+\rho_0^{(2)}[c_p^{(2)}-R^{(2)}]\}}, \quad (2.30)$$

$$n_u^2=-g\left(\frac{1}{\rho_{M0}}\frac{d\rho_{M0}}{dz}+\frac{g}{c_u^2}\right). \quad (2.31)$$

The quantities c_u and n_u are the speed of sound and the Brunt-Väisälä frequency, respectively, for a system which is not saturated.

c. Calculations for $\varphi_1\leqslant\varphi\leqslant\varphi_2$ *and the source term*

The atmosphere, at some height, is now saturated and droplets start forming. The microstructure around each droplet regulates all the detailed phenomena that affect the mass, momentum and energy exchanges between the constituents. In the region surrounding the droplets, all assumed the same, the local average values of $p_0^{(2)}+p_1^{(2,c)}$, $\rho_0^{(2)}+\rho_1^{(2,c)}$ and $T_0+T_1^{(c)}$ differ, in general, from the vapor pressure, density, and temperature at the droplet surface (p_ω, ρ_ω, T_ω). The latter are related to each other through the Clausius-Clapeyron equation

$$p_\omega=p_{\omega g}\exp\left[\frac{L_v}{R^{(2)}T_g}\left(1-\frac{T_g}{T_\omega}\right)\right], \quad (2.32)$$

as well as the usual equation of state.

Einaudi and Lalas (1973) have shown that for the problem at hand we can set $T_\omega=T_0+T_1^{(c)}$. Furthermore, if one assumes that the temperature perturbation is sufficiently small that

$$\frac{L_v}{R^{(2)}T_0}\frac{|T_1^{(c)}|}{T_0}\ll 1, \quad (2.33)$$

then we can linearize Eq. (2.32) and obtain

$$p_\omega = p_{\omega 0} + p_{\omega 1} = p_{\omega 0}\left[1 + \frac{L_v}{R^{(2)}T_0}\frac{T_1^{(c)}}{T_0}\right], \quad (2.34)$$

where

$$p_{\omega 0} = p_{\omega g}\exp\left[\frac{L_v}{R^{(2)}T_g}\left(1-\frac{T_g}{T_0}\right)\right]. \quad (2.35)$$

$p_{\omega 0}$ is the value that the background water vapor pressure should have for the system to be saturated at the background temperature T_0 and $p_{\omega g}$ is such a value at some reference height z_g where the background temperature is T_g.

Similarly we can write

$$\rho_\omega = \rho_{\omega 0}\left\{1 + \left[\frac{L_v}{R^{(2)}T_0} - 1\right]\frac{T_1^{(c)}}{T_0}\right\}, \quad (2.36)$$

where

$$\rho_{\omega 0} = p_{\omega 0}/[R^{(2)}T_0]. \quad (2.37)$$

Turning our attention to the source term now, we see that if diffusion is taken to be the controlling mechanism of mass exchange between each droplet and the ambient water vapor, then we can write the source term as

$$S_s = [\rho_{\omega s} - \rho_s^{(2)}]/\tau_m, \quad \frac{1}{\tau_m} = 4\pi n_p a_0 D \quad (2.38)$$

(Byers, 1965), where n_p is the number of droplets, all assumed the same, per unit volume of the mixture; D is the diffusivity of water vapor, a_0 the average radius of the droplets, and τ_m the characteristic time for mass transfer (see also Einaudi and Lalas, 1973).

We can express $\rho_{\omega s}$ as

$$\rho_{\omega s}(x,z,t) \approx \rho_{\omega 1}(x,z,t) = \rho_{\omega 1}(z)\exp(i\varphi) + \text{c.c.}, \quad (2.39)$$

with

$$\rho_{\omega 1}(z) = \rho_{\omega 0}\left[\frac{L_v}{R^{(2)}T_0} - 1\right]\frac{T_1^{(c)}}{T_0}. \quad (2.40)$$

The above relation, valid for $\varphi_1 \leqslant \varphi \leqslant \varphi_2$, can be obtained from (2.36). Using (2.22) and (2.40), we can write

$$\left.\begin{aligned} S_1^{(c)} &= \frac{\rho_{\omega 1}(z) - \rho_1^{(2,c)}(z)}{\tau_m}\exp(i\varphi) + \text{c.c.}, \\ &\qquad \text{for } \varphi_1 \leqslant \varphi \leqslant \varphi_2 \\ S_1^{(u)} &= 0, \qquad \text{for } 0 \leqslant \varphi < \varphi_1 \text{ and } \varphi_2 < \varphi \leqslant 2\pi \end{aligned}\right\}. \quad (2.41)$$

The source term, therefore, has the behavior schematically indicated by the dotted line of Fig. 2, with discontinuities at φ_1 and φ_2. The jumps at $\varphi = \varphi_1$ and $\varphi = \varphi_2$, present in the source term, will also manifest themselves in all dependent variables of the system, except v_{1z}, so that for example, $p_{M1}^{(u)}(\varphi_j) \neq p_{M1}^{(c)}(\varphi_j)$, for $j = 1, 2$, etc. Since only the first harmonic has any effect in the present stability analysis, we can write

$$S_s(x,z,t) \approx S_1(z)\exp(i\varphi) + \text{c.c.} \quad (2.42)$$

The true behavior of the source term, indicated by the solid line in Fig. 2, would require a substantial harmonic content for a true representation. These higher harmonics would tend to eliminate the discontinuities evident in the dotted line; but because their contribution is small in comparison with the fundamental one and at most of the order of the nonlinear terms in the original set of equations, all of which have already been neglected, they are ignored in the present analysis.

If we now return to (2.4) with the help of (2.41) and (2.5), we can write for $\varphi_1 \leqslant \varphi \leqslant \varphi_2$

$$\rho_1^{(2,c)} = \frac{1}{1+i\Omega\tau_m}\left\{\rho_{\omega 1} + i\Omega\tau_m\frac{\rho_0^{(2)}}{\rho_{M0}}\rho_{M1} - \tau_m\frac{v_{1z}}{\rho_{M0}}\left[\rho_0^{(1)}\frac{d}{dz}\rho_0^{(2)} - \rho_0^{(2)}\frac{d}{dz}\rho_0^{(1)}\right]\right\}. \quad (2.43)$$

In addition, summing Eqs. (2.4), for $i = 2$ and 3, and using Eq. (2.5), we can write

$$\rho_1^{(2,c)} + \rho_1^{(3)} = \frac{1}{i\Omega}\left\{\frac{\rho_0^{(2)}}{\rho_{M0}}\left[i\Omega\rho_{M1} + v_{1z}\frac{d}{dz}\rho_{M0}\right] - v_{1z}\frac{d}{dz}\rho_0^{(2)}\right\}. \quad (2.44)$$

The equation of state, then, with the use of (2.43) and (2.44) yields the following relation between $T_1^{(c)}$ and $p_{M1}^{(c)}$, $\rho_{M1}^{(c)}$ and v_{1z}:

$$\begin{aligned} p_{M1}^{(c)} = \frac{1}{1+i\Omega\tau_m}\Bigg\{ & T_1^{(c)}\left[(R^{(1)}\rho_0^{(1)} + R^{(2)}\rho_0^{(2)})(1+i\Omega\tau_m) + R^{(2)}\rho_{\omega 0}\left(\frac{L_v}{R^{(2)}T_0} - 1\right)\right] \\ & + \frac{\rho_{M1}^{(c)}}{\rho_{M0}}[(1+i\Omega\tau_m)T_0R^{(1)}\rho_0^{(1)} + i\Omega\tau_m T_0 R^{(2)}\rho_0^{(2)}] \\ & + \frac{v_{1z}}{i\Omega\rho_{M0}}\left[\rho_0^{(2)}\frac{d}{dz}\rho_0^{(1)} - \rho_0^{(1)}\frac{d}{dz}\rho_0^{(2)}\right] \\ & \times [i\Omega\tau_m T_0 R^{(2)} - T_0 R^{(1)}(1+i\Omega\tau_m)]\Bigg\}. \end{aligned} \quad (2.45)$$

But for the typical values of $a_0 \approx 10^{-3}$ cm, $D \approx 2.58 \times 10^{-1}$ cm^2 s^{-1} and $n_p \approx 300$ particles cm^{-3}, $\tau_m \approx 1$ s and, if we restrict ourselves to frequencies in the internal gravity wave range with $u_0 \approx 10$–100 m s^{-1}, and horizontal wavelengths λ_x of at least 1 km, we see that

$$|\Omega\tau_m|\ll 1. \tag{2.46}$$

In this limit Eq. (2.45) reduces to

$$T_1^{(c)}=\frac{1}{\psi_1}\left\{p_{M1}^{(c)}-\frac{T_0R^{(1)}\rho_0^{(1)}}{\rho_{M0}}\rho_{M1}\right.$$
$$\left.+\frac{T_0R^{(1)}}{\rho_{M0}}\frac{v_{1z}}{i\Omega}\left[\rho_0^{(2)}\frac{d}{dz}\rho_{M0}-\rho_{M0}\frac{d}{dz}\rho_0^{(2)}\right]\right\}, \tag{2.47}$$

where

$$\psi_1=R^{(1)}\rho_0^{(1)}+R^{(2)}\rho_0^{(2)}+R^{(2)}\rho_{\omega0}\left[\frac{L_v}{R^{(2)}T_0}-1\right]. \tag{2.48}$$

Using (2.4) and (2.47), we can write the energy equation as

$$i\Omega p_{M1}^{(c)}=c_c^2\left[i\Omega\rho_{M1}^{(c)}-\frac{\rho_{M0}}{g}n_c^2v_{1z}\right], \tag{2.49}$$

where

$$c_c^2=\frac{T_0}{\rho_{M0}}\frac{\rho_0^{(1)}R^{(1)}\psi_2+\rho_0^{(2)}R^{(2)}\tilde{L}_v}{(\psi_2-1)}, \tag{2.50}$$

$$n_c^2=-g\left(\frac{1}{\rho_{M0}}\frac{d\rho_{M0}}{dz}+\frac{g}{c_c^2}\right)-R, \tag{2.51}$$

$$R=\frac{gR^{(2)}[(\rho_0^{(1)}R^{(1)}+\rho_0^{(2)}R^{(2)})\tilde{L}_v-(\rho_0^{(1)}c_p^{(1)}+\rho_0^{(2)}c_p^{(2)})]\left[\dfrac{d\rho_{\omega0}}{dz}-\dfrac{d\rho_0^{(2)}}{dz}\right]}{\{(\rho_0^{(1)}c_p^{(1)}+\rho_0^{(2)}c_p^{(2)})\rho_0^{(1)}R^{(1)}+(\rho_0^{(1)}R^{(1)}+\rho_0^{(2)}R^{(2)})\tilde{L}_v[\rho_0^{(2)}R^{(2)}+\rho_{\omega0}R^{(2)}(\tilde{L}_v-1)]\}}, \tag{2.52}$$

$$\psi_2=\frac{[\rho_0^{(1)}c_p^{(1)}+\rho_0^{(2)}c_p^{(2)}+\rho_{\omega0}R^{(2)}\tilde{L}_v(\tilde{L}_v-1)]}{[\rho_0^{(1)}R^{(1)}+\rho_0^{(2)}R^{(2)}+\rho_{\omega0}R^{(2)}(\tilde{L}_v-1)]},\quad \tilde{L}_v=\frac{L_v}{R^{(2)}T_0}. \tag{2.53}$$

When the atmosphere is fully saturated, $\rho_{\omega0}=\rho_0^{(2)}$ by definition, and c_c^2 and n_c^2 reduce to the usual form appropriate to a fully saturated atmosphere. This form, and its relation to the wet adiabatic lapse rate, is discussed in detail in Lalas and Einaudi (1974).

We defer to Section 3 the stability analysis of the system, and we turn our attention now to the determination of φ_1 and φ_2.

d. *Determination of φ_1 and φ_2*

Let $\rho_1^{(2)}(x,z,t)$ and $T_1(x,z,t)$ be the expressions for the partial water vapor density and temperature valid for all φ's. By definition, the values of $\varphi_1(z)$ and $\varphi_2(z)$ can be obtained by demanding that the partial density for water vapor and the temperature are related via the Clausius-Clapeyron equation. This is equivalent to demanding that the partial vapor density derived at the unsaturated part of the cycle be equal to the partial vapor density derived at the saturated part, i.e.,

$$\rho_0^{(2)}(z)+\rho_1^{(2)}(z)\exp(i\varphi_j)+\text{c.c.}$$
$$=\rho_{\omega0}(z)+\rho_{\omega0}(z)\left[\frac{L_v}{R^{(2)}T_0}-1\right]\frac{T_1(z)}{T_0}\exp(i\varphi_j)+\text{c.c.},$$
$$j=1,2. \tag{2.54}$$

If we define the real quantities $B(z)\geqslant 0$ and $\alpha(z)$ so that

$$B(z)\exp(i\alpha)$$
$$=\frac{2}{\rho_{\omega0}}\left\{\rho_1^{(2)}(z)-\rho_{\omega0}(z)\left[\frac{L_v}{R^{(2)}T_0}-1\right]\frac{T_1(z)}{T_0}\right\}, \tag{2.55}$$

then φ_1 and φ_2 are the solutions of the equation

$$\cos(\alpha+\varphi_j)=\frac{1}{B(z)}[1-\rho_0^{(2)}/\rho_{\omega0}],\quad j=1,2. \tag{2.56}$$

We are now in a position to define the smallness parameter δ, introduced in Eq. (2.25). If

$$\delta=[1-\rho_0^{(2)}/\rho_{\omega0}]/B(z)\ll 1,\quad B(z)\neq 0, \tag{2.57}$$

then indeed

$$\varphi_2(z)-\varphi_1(z)=\pi+O(\delta). \tag{2.58}$$

It should be emphasized that the above derivation assumes φ_j to be essentially real, consistent with our hypothesis (2.13). The amount that $\rho_0^{(2)}$ can differ from $\rho_{\omega0}$ with (2.58) still valid depends on $B(z)$ and, in turn, on the amplitude of the internal wave impinging on the system. The point to be noted here is that (2.56) will certainly have solutions, provided that $\rho_{\omega0}$ is sufficiently close to $\rho_0^{(2)}$, i.e., that the background system is sufficiently close to saturation. For those heights at which $B(z)=0$, of course, saturation will not take place unless $\rho_0^{(2)}$ is indeed equal to $\rho\omega_0$.

3. The stability analysis

Eqs. (2.29) and (2.49) derived in the previous section along with Eqs. (2.5), (2.6) and (2.7) constitute a complete set. This set can be reduced to a second-order differential equation for one of the unknowns, say v_{1z}, with coefficients that depend on whether or not, at a particular height range, condensation caused by the wave is possible.

Let us first consider the height interval where condensation occurs and introduce the new variable $q(z)$ defined by

$$q=\frac{v_{1z}}{i\epsilon\Omega}, \tag{3.1}$$

where ϵ is defined by (3.10). From (2.5) and (2.6) one derives

$$\rho_{M1}^{(j)}=\frac{k_x^2}{\Omega^2}p_{M1}^{(j)}-\frac{d}{dz}(\epsilon\rho_{M0}q),\quad j=u,c, \tag{3.2}$$

while (2.29) and (2.49) now read

$$\rho_{M1}^{(j)}=\frac{1}{c_j^2}p_{M1}^{(j)}+\rho_{M0}\frac{n_j^2}{g}\epsilon q,\quad j=u,c. \tag{3.3}$$

Eliminating $\rho_{M1}^{(j)}$ between (3.2) and (3.3), we can write

$$p_{M1}^{(j)}=\Omega^2\left(k_x^2-\frac{\Omega^2}{c_j^2}\right)^{-1}\times\left[\epsilon\rho_{M0}\frac{n_j^2}{g}q+q\frac{d}{dz}(\epsilon\rho_{M0})+\epsilon\rho_{M0}\frac{dq}{dz}\right]. \tag{3.4}$$

In the internal gravity wave range, we can simplify (3.4) to

$$p_{M1}^{(j)}=\frac{\Omega^2}{k_x^2}\left[\epsilon\rho_{M0}\frac{n_j^2}{g}q+q\frac{d}{dz}(\epsilon\rho_{M0})+\epsilon\rho_{M0}\frac{dq}{dz}\right], \tag{3.5}$$

by assuming that the phase velocity of the waves is much smaller than the sound speed, i.e.,

$$|\Omega|^2\ll k_x^2c_j^2,\quad j=u,c. \tag{3.6}$$

As shown earlier, for the case in which $\varphi_2-\varphi_1\approx\pi$, the value of p_{M1} valid for all φ's is simply

$$p_{M1}=\tfrac{1}{2}[p_{M1}^{(c)}+p_{M1}^{(u)}], \tag{3.7}$$

which, from (3.5), becomes

$$p_{M1}=\frac{\Omega^2}{k_x^2}\left[\epsilon\rho_{M0}\frac{q}{g}n_{\mathrm{av}}^2+q\frac{d}{dz}(\epsilon\rho_{M0})+\epsilon\rho_{M0}\frac{d}{dz}q\right], \tag{3.8}$$

where

$$n_{\mathrm{av}}^2=\tfrac{1}{2}(n_u^2+n_c^2). \tag{3.9}$$

If we choose

$$r_{\mathrm{av}}=\epsilon\rho_{M0}=\exp\left(-\int\frac{n_{\mathrm{av}}^2}{g}dz\right), \tag{3.10}$$

(3.8) simplifies to

$$p_{M1}=\frac{\Omega^2}{k_x^2}r_{\mathrm{av}}\frac{dq}{dz}. \tag{3.11}$$

Using (3.2), (3.10) and (3.11), we can write ρ_{M1} as

$$\rho_{M1}=r_{\mathrm{av}}n_{\mathrm{av}}^2q/g, \tag{3.12}$$

and substitution of (3.11) and (3.12) into Eq. (2.7) yields the final equation for q:

$$\frac{d}{dz}\left(r_{\mathrm{av}}\frac{\Omega^2}{k_x^2}\frac{dq}{dz}\right)+r_{\mathrm{av}}(n_{\mathrm{av}}^2-\Omega^2)q=0. \tag{3.13}$$

The above equation applies to the height range over which condensation is induced by the wave. When conditions are such that the wave does not produce condensation, we introduce the transformation, analogous to (3.1),

$$q_d=\frac{v_{1z}}{i\epsilon_d\Omega}, \tag{3.14}$$

and, following a similar procedure, we obtain the equation

$$\frac{d}{dz}\left(r_d\frac{\Omega^2}{k_x^2}\frac{dq_d}{dz}\right)+r_d(n_u^2-\Omega^2)q_d=0, \tag{3.15}$$

where

$$r_d=\epsilon_d\rho_{M0}=\exp\left(-\int\frac{n_u^2}{g}dz\right). \tag{3.16}$$

Eqs. (3.13) and (3.15) can be solved numerically, with the appropriate boundary conditions, to calculate vertical velocities, evaporation and condensation rates, phase velocities or any other variable associated with the disturbance. An example of such a calculation will be given in Section 4.

In order to obtain from Eqs. (3.13) and (3.15) some analytical stability criteria, we have to introduce a new simplifying hypothesis. We assume that the total height range of our system is such that we can set

$$r_{\mathrm{av}}\approx r_d\approx 1. \tag{3.17}$$

Since n_{av}^2/g and n_u^2/g are both of order 10^{-5} m^{-1}, it follows that we have to limit ourselves to heights of about 10 km. In this case, it follows that

$$\epsilon\approx\epsilon_d,\quad q\approx q_d, \tag{3.18}$$

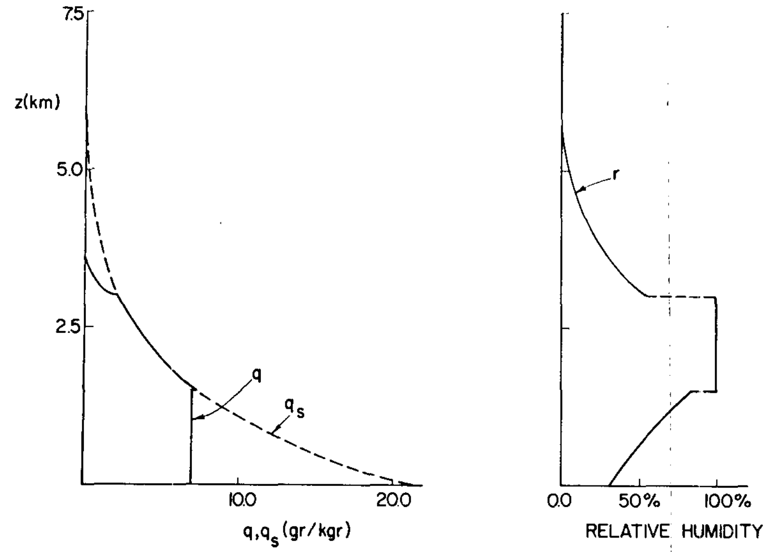

FIG. 3. Relative (r) and specific (q) humidity profiles for a hypothetical sounding. The specific humidity required for saturation is denoted by q_s.

and (3.13) and (3.15) reduce to

$$\frac{d}{dz}\left(\frac{\Omega^2}{k_x^2}\frac{dq}{dz}\right)+(n^2-\Omega^2)q=0. \tag{3.19}$$

where

$$n=\begin{cases} n_{\mathrm{av}}, & \text{in the range where condensation is induced} \\ n_u, & \text{in the range where condensation is not induced} \end{cases} \tag{3.20}$$

The stability analysis of the system is now reduced to an investigation of the conditions under which (3.19) demands that the eigenvalues ω be complex. It can be shown that, for n^2 real and positive throughout the flow, a sufficient condition for the stability of a system obeying (3.19) is that throughout the flow

$$n^2 \geqslant (du_0/dz)^2/4. \tag{3.21}$$

The proof that, for stability, the square of the Brunt-Väisälä frequency has to be larger than the square of the vertical wind shear divided by 4, was originally given, for incompressible flows, by Miles (1961) and also, via a very simple and elegant method, by Howard (1961). Chimonas (1970) followed Howard's approach and extended this result to compressible fluids, while Lalas and Einaudi (1973, 1974) have proven its validity for a fully saturated atmosphere. The present analysis extends the criterion to situations where condensation may occur, induced by the wave itself, in an otherwise unsaturated atmosphere, and provides the appropriate n^2 to be used. When no background wind is present, (3.21) reduces to

$$n^2 \geqslant 0. \tag{3.22}$$

On the other hand, when (3.21) is violated somewhere in the flow, i.e., when

$$n^2 < (du_0/dz)^2/4, \tag{3.23}$$

the system may be unstable, since (3.23) is only a necessary but not a sufficient condition for instability.

It is clear from the above considerations that n^2 is the basic parameter on which the stability of the system depends. To appreciate the effect of the wave-induced condensation, it is important to compare n_{av}^2 with the value n_u^2 corresponding to an amplitude of the disturbance so small that condensation does not occur. To this end, we note that in general

$$n_c^2 < n_u^2, \tag{3.24}$$

since the numerator of n_c^2 in (2.51), which is enhanced by terms of order $[L_v/R^{(2)}T_0] \approx 20$, increases less than the denominator, which is enhanced by terms of order $[L_v/R^{(2)}T_0]^2$. It follows then that

$$n_{\mathrm{av}}^2 < n_u^2. \tag{3.25}$$

This result has been verified by numerical calculations utilizing a variety of background temperature and humidity soundings.

The main result then is that the equivalent Brunt-Väisala frequency of the system is reduced and therefore the system is more unstable than it would be in the absence of wave-induced condensation. Two cases are particularly significant. In the first case, u_0, n_{av} and n_u are such that

$$n_u^2 \geqslant (du_0/dz)^2/4, \quad n_{\mathrm{av}}^2 < (du_0/dz)^2/4. \tag{3.26}$$

Then the system is dynamically stable for infinitesimal perturbations, while it may be unstable for small, but finite disturbances that can induce condensation. The latter will grow in time and may ultimately alter the

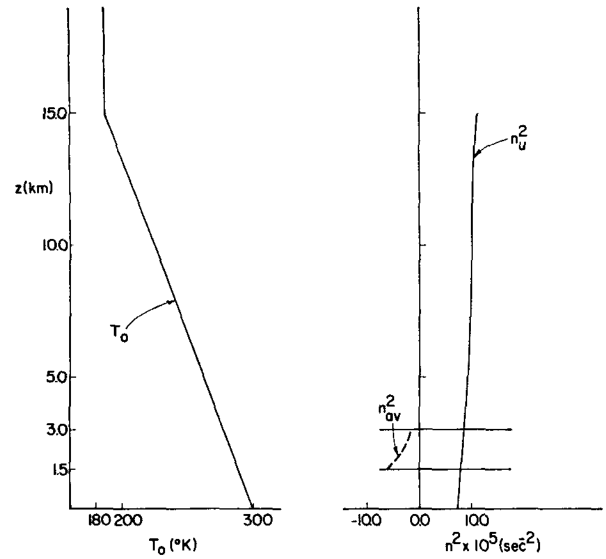

FIG. 4. The temperature and square of the Brunt-Väisälä frequency profiles utilizing the specific humidity shown in Fig. 3. The solid line, marked n_u^2, is the value for the unperturbed atmosphere. Over the height range $1.5<z<3.0$ km, where the wave can bring about condensation, the effective Brunt-Väisälä frequency squared should be calculated by Eq. (3.9) and is indicated by the dashed line marked n_{av}^2.

background quantities substantially. Such alteration unfortunately cannot be described by the present linear analysis.

The second case comprises a background with no wind and such that

$$n_u^2>0, \quad n_{av}^2<0 \tag{3.27}$$

hold. The system is then statically and dynamically stable for infinitesimal disturbances, whereas it becomes convectively unstable for small, but finite perturbations. Convective motions can be initiated leading to further dynamic interactions with the wave itself. The present analysis can only hope to describe the early stages of such processes.

Finally, we mention that the need for having two distinct Eqs. (3.13) and (3.15), each with its own range of applicability, stems from the earlier assumption that $\varphi_2-\varphi_1\approx\pi$ over the entire layer in which condensation occurs. If, instead, $\varphi_2-\varphi_1$ is allowed to be a continuous function of z, then the two equations will merge into one with coefficients given by continuous functions of z.

4. A numerical example of the influence of wave-induced condensation

Having discussed stability criteria in the previous section, we will illustrate here the effect of wave-induced condensation on the propagation and vertical structure of a neutral gravity wave. We will consider an atmosphere with the properties shown in Figs. 3 and 4. The atmosphere is stably stratified over all of its height and capped by an isothermal layer. The velocity is assumed constant with height. The water vapor content is such that the relative humidity is almost 100% over a certain height range. If no phase change takes place, the square of the Brunt-Väisala frequency is positive throughout, and the atmosphere is stable. If some wave induces condensation over the height range where the relative humidity is almost 100%, Eq. (3.9) gives the square of the effective Brunt-Väisälä frequency equal to n_{av}^2, which is now negative, and the atmosphere becomes convectively unstable over that height range. This vertical structure of the atmosphere is not unlike the Topeka, Kan., sounding of 18 May 1971, reported by Uccellini (1973) where the relative humidity was 100% from 770 to 680 mb in (Fig. 5).

If waves are generated by an arbitrary source at the ground, they will propagate vertically through the atmosphere, according to (3.13) and (3.15) with the appropriate n^2. In particular, we analyze the behavior of a monochromatic wave generated by an idealized source such as a corrugated surface of wavelength λ_x and oscillating with real frequency ω. The boundary conditions for the vertical velocity of the wave at such a surface are

$$w(z=z_g)=\rho_{M0}^{\frac{1}{2}}v_z(z=z_g)=A\cos\varphi+B\sin\varphi, \tag{4.1}$$

which can be synthesized by the superposition of the two terms

$$w^{(1)}(z=z_g)=A\cos\varphi, \tag{4.2}$$

$$w^{(2)}(z=z_g)=B\sin\varphi. \tag{4.3}$$

At any height then, the vertical velocity field is

$$\begin{aligned} w(z,\varphi)&=w^{(1)}(z,\varphi)+w^{(2)}(z,\varphi)\\ &=A[Z_c^{(1)}\cos\varphi+Z_s^{(1)}\sin\varphi]\\ &\qquad+B[Z_c^{(2)}\cos\varphi+Z_s^{(2)}\sin\varphi], \end{aligned}$$

where $Z_c^{(1)}$, $Z_c^{(2)}$, $Z_s^{(1)}$, $Z_s^{(2)}$ are functions of z only, and

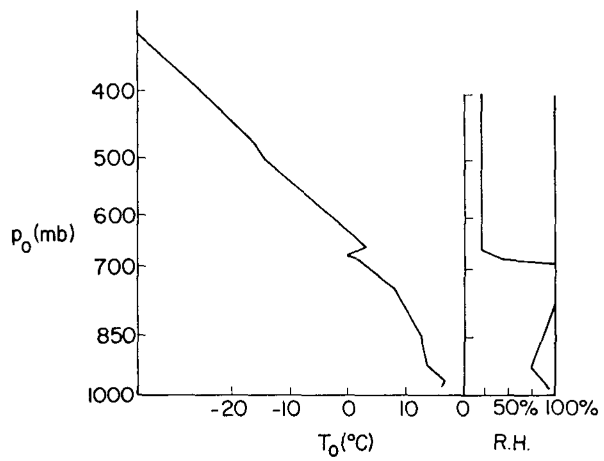

FIG. 5. The Topeka, Kan., sounding of 1200 GMT 18 May 1971, showing temperature and relative humidity profiles.

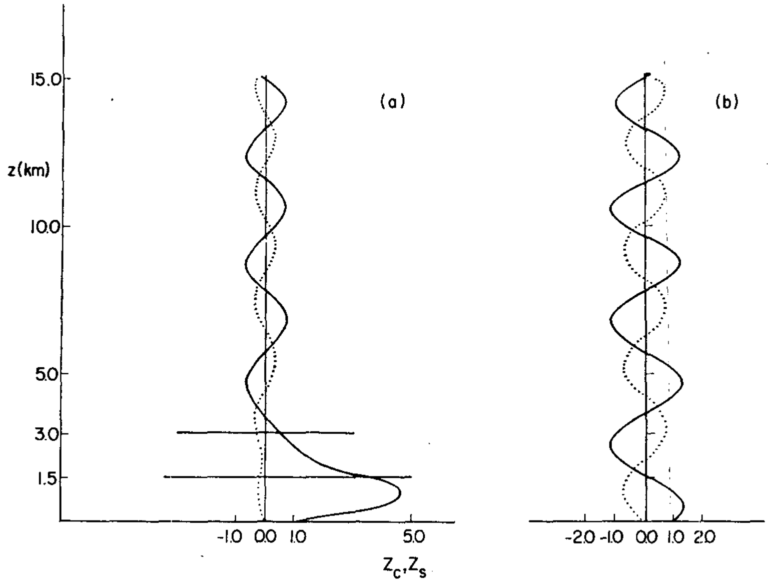

FIG. 6. The $Z_c^{(1)}$ and $Z_s^{(1)}$ components of the vertical velocity for a wave of 30 min period and 10 km horizontal wavelength that induces condensation (a) and for the same wave if wave-induced condensation is neglected (b). For both cases the background temperature and humidity are the ones of Figs. 3 and 4.

obey the following boundary conditions at the source:

$$\left.\begin{array}{ll} Z_c^{(1)}(z=z_g)=1, & Z_s^{(1)}(z=z_g)=0 \\ Z_c^{(2)}(z=z_g)=0, & Z_s^{(2)}(z=z_g)=1 \end{array}\right\}. \quad (4.4)$$

In the isothermal layer we demand that the wave be such that the radiation condition is satisfied at $z=\infty$; this requires, at $z=z_0$, where the isothermal layer begins, that the Z's satisfy the conditions

$$\left.\begin{array}{ll} \dfrac{dZ_c^{(i)}}{dz}(z=z_0)=k_z Z_s^{(i)}(z=z_0), & i=1,2 \\ \dfrac{dZ_s^{(i)}}{dz}(z=z_0)=-k_z Z_c^{(i)}(z=z_0), & i=1,2 \end{array}\right\}, \quad (4.5)$$

where k_z is the vertical wavenumber for the isothermal region. The results show that for all heights

$$\left.\begin{array}{l} Z_c^{(1)}=Z_s^{(2)} \\ Z_c^{(2)}=-Z_s^{(1)} \end{array}\right\}, \quad (4.6)$$

so that one need only plot $Z_c^{(1)}$ and $Z_s^{(1)}$. For a source with period 30 min and wavelength 10 km, the profiles $Z_c^{(1)}(z)$ and $Z_s^{(1)}(z)$ for the case of wave-induced condensation and for the case where this condensation is not taken into account are shown in Figs. 6a and 6b respectively.

The differences in the two profiles are substantial. First, the amplitude near the source for the condensation case is four times larger, while the amplitude above the almost saturated layer is smaller by a factor of 0.5. Second, the phase of the wave is altered, even though the vertical wavelength is almost the same. Clearly some resonance effect is introduced by the existence of the layer almost at saturation, which, in combination with the exponential growth in the convectively unstable regime, drastically alters the characteristics of the disturbance. The quantitative implications of this change on the feedback mechanism and on the intensification of the disturbance itself are currently being investigated in a model where the restriction of $\varphi_2-\varphi_1\approx\pi$ is relaxed, and the background velocity is a function of height, so that a wider variety of soundings amenable to wave-induced condensation can be analyzed.

Finally, we mention, as we have already done in the Introduction, that some questions exist about the validity of a wave solution itself in the height interval where n_{ov}^2 becomes negative. A full numerical analysis, which makes no assumption of a wavelike behavior, is presently under investigation and will be compared to the one presented here.

5. Conclusions

The possibility of gravity waves initiating convective activities and inducing condensation in an atmosphere near saturation over some height is investigated analytically and numerically. The appropriate quasi-linear equations governing wave propagation with condensation and subsequent heat release, induced by the wave over a fraction of its period, are derived.

For the particular case of the atmosphere just at saturation so that condensation or evaporation occurs over half of the period, the wave is shown to propagate in an atmosphere with a new effective Brunt-Väisälä frequency n_{av} smaller than the original frequency n_u of the unperturbed atmosphere. A linear stability analysis is then feasible, and shows that n_{av}^2 can become negative so that an atmosphere which is statically stable when no wave is present, becomes convectively unstable due to wave-induced condensation. Similarly, the Richardson number, characteristic of the dynamic stability of the atmosphere, will be lower when the new Brunt-Väisälä frequency is used, implying possible dynamic instability induced by the wave itself.

The propagation characteristics of the wave are altered, even when no convection is triggered, by ducting phenomena generated by the wave itself, through the induced changes of the vertical structure of the atmosphere.

Further numerical calculations, based on the quasi-linear analysis presented here and for specific cases of convective activities induced by the gravity waves, are presently under investigation and could provide further insight into the importance of gravity waves in mesoscale processes.

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