Analytical Analysis of a Budyko-Type Climate Model

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ABSTRACT

For a given value of the solar constant we have found that the Budyko-type climate model gives two positions for the edge of the polar ice sheet. The results of the Budyko model suggest that with a decrease in the solar constant of about 1.6%, the ice cap reaches a latitude of about 50°. At this point the model breaks down and the solution for the position of the edge of the polar ice sheet becomes complex. We have based our conclusions on analytic investigations. Numerical calculations were used only to support our analytic results.

1. Introduction

There exist a few simple climate models that allow one to investigate the possible response of the present global climate to changes in various factors which influence the climate such as the solar constant and the transmissivity of the atmosphere among others. The two well-known models of this type are the Budyko model (Budyko, 1969) and the Sellers model (Sellers, 1969). Although the parameterization of the terrestrial infrared radiation and of the horizontal heat transport are different in these models, it is believed that both models predict the same consequence: a small decrease in the solar constant—namely, a 1.6% decrease in the Budyko model and a 2% decrease in the Sellers model—were found to be sufficient for complete glaciation of the earth. The close coincidence of these results is considered to indicate the high instability of the present climate (Budyko, 1970). The result concerning the earth completely covered by ice was recently reproduced using a slightly modified Budyko model (Gordon and Davies, 1974).

The purpose of this paper is to point out the multiple-branch structure of the solution to the Budyko model.

2. Budyko-type climate model

The climate model of Budyko (1969) is based on three simple relations. The first of these is an empirical formula giving the zonal average outgoing terrestrial infrared radiation flux $I(\theta)$ as a function of the zonal average surface temperature $T(\theta)$ at any latitude θ:

$$I(\theta)=(a-a_1n)+(b-b_1n)T(\theta).$$

Here n is a cloudiness fraction which Budyko takes as constant ($=0.5$), independent of θ. Under this assumption we may rewrite the relation as

$$I(\theta)=A+BT(\theta), \tag{1}$$

where A and B are empirical coefficients. If $I(\theta)$ is in cal cm^{-2} min^{-1} and $T(\theta)$ in degrees Celsius, then Budyko's choice of coefficients a, a_1, b, b_1 and n leads to the values $A=0.289$ and $B=0.00208$.

The second relation describes the radiation balance at each latitude. The incoming annual average solar radiation at the top of the atmosphere is given by $Q(\theta)$. The albedo $\alpha(\theta)$ gives the fraction reflected, and thus the absorbed part is $Q(\theta)[1-\alpha(\theta)]$. Radiation balance requires that

$$Q(\theta)[1-\alpha(\theta)]-I(\theta)=D(\theta), \tag{2}$$

where $D(\theta)$ is the divergence of heat as a result of horizontal redistribution by the circulation of the atmosphere and ocean.

The system is closed by the third relation that gives an empirical dependence of $D(\theta)$ on the difference between $T(\theta)$ and the hemispherical mean surface temperature

$$\bar{T}=\int_0^{\pi/2} T(\theta)d(\sin\theta).$$

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Budyko takes this relation as linear, i.e.,

$$D(\theta)=\beta'[T(\theta)-\bar{T}], \quad (3)$$

with $\beta'=0.00544$ in the present units.

The two empirical relations (1) and (3) have coefficients chosen in such a way as to match present climate observations. These two linear relations may be combined into a single empirical relation

$$D(\theta)=\beta[I(\theta)-\bar{I}], \quad (4)$$

with a single empirical coefficient $\beta=\beta'/B=2.61$. Eqs. (2) and (4) form a closed system that permits a solution for $I(\theta)$ for any specified $Q(\theta)$ and $\alpha(\theta)$. The surface temperature $T(\theta)$, if desired, may then be obtained by use of (1).

The latitudinal distribution of incoming solar radiation $Q(\theta)$ can be obtained from astronomical considerations, and its magnitude is related to the solar constant σ by

$$\int_0^{\pi/2} Q(\theta)\cos\theta d\theta=\bar{Q}\equiv\sigma/4.$$

It is convenient to introduce a new variable $x=\sin\theta$ and to write $Q(x)$ in the form

$$Q(x)=\bar{Q}S(x),$$

with $S(x)$ normalized to unity:

$$\int_0^1 S(x)dx=1.$$

Based on Budyko's paper, we assume that the local albedo $\alpha(x)$ has the form of a step function with a discontinuity at the edge of the ice cap and we will write

$$\alpha(x,x_s)=\begin{cases} a, & \text{for } x<x_s \\ \alpha_s, & \text{for } x=x_s \\ b, & \text{for } x>x_s \end{cases} \quad (5)$$

Assuming radiation balance is satisfied for the total hemisphere, we also have

$$\bar{I}=\bar{Q}[1-\alpha_P(x_s)], \quad (6)$$

where $\bar{I}$ is a hemispherical mean of $I(x)$ and the planetary albedo $\alpha_P(x_s)$ is given by

$$\alpha_P(x_s)=\bar{Q}^{-1}\int_0^1 \alpha(x,x_s)Q(x)dx=\int_0^1 \alpha(x,x_s)S(x)dx. \quad (7)$$

This completes the description of a Budyko-type climate model. Instead of solving the equations of the model numerically, as has usually been done, we will derive in the next section an analytical relation between the latitude of the edge of the ice cap and the magnitude of the solar constant $\sigma=4\bar{Q}$.

3. Analytical solution

According to Budyko the edge of the ice cap is defined by a definite temperature T_s. From (1) it follows that the edge of the ice cap can be alternatively characterized by an outgoing flux of infrared radiation $I_s=A+BT_s$.

Substituting (4) and (6) into (2) and taking the resulting equation at $x=x_s$, we obtain

$$1-\alpha_P(x_s)=\frac{I_s}{\bar{Q}}\frac{1+\beta}{\beta}-\frac{1-\alpha_s}{\beta}S(x_s). \quad (8)$$

To solve this equation explicitly for the sine of the latitude for the edge of the ice sheet, x_s, as a function of solar constant $4\bar{Q}$, we have to know the function $S(x)$ characterizing the relative distribution of the solar radiation with respect to latitude. As we have already mentioned, the value of this function can be calculated exactly for any latitude, however, its explicit form is rather complicated and unsuitable for an analytical analysis. To overcome this difficulty, we have divided the interval (0,1) into N subintervals by taking a sequence of increasing numbers $x_0,x_1,x_2,\ldots,x_i,x_{i+1},\ldots,x_{N-1},x_N$ such that $x_0=0$ and $x_N=1$. Inside each subinterval (x_i,x_{i+1}) we have approximated the function $S(x)$ by a linear function. By increasing the number of subintervals N we can achieve the final solution to any desired accuracy. Thus we have

$$S_i(x)=P_i-R_ix, \quad \text{for } x\in(x_i,x_{i+1}),\ i=0,1,2,\ldots,N-1. \quad (9)$$

Using (9) for the weighting function $S(x)$, we can calculate a planetary albedo $\alpha_p(x_s)$ using (7). The result can be written in the form

$$\alpha_P(x_s)=a+(b-a)(\tfrac{1}{2}R_ix_s^2-P_ix_s+K_i), \quad \text{for } x_s\in(x_i,x_{i+1}), \quad (10)$$

where the coefficients K_i are given by

$$K_i=P_ix_{i+1}-\tfrac{1}{2}R_ix_{i+1}^2+\sum_{j=i+1}^{N-1}[P_j(x_{j+1}-x_j)-\tfrac{1}{2}R_j(x_{j+1}^2-x_j^2)]. \quad (11)$$

Finally, substituting (9) and (10) into (8) we get a quadratic equation for x_s, the solutions of which are

$$x_s=A_i\pm(B_i-C_i/\bar{Q})^{\frac{1}{2}}, \quad x_s\in(x_i,x_{i+1}), \quad (12)$$

with the coefficients A_i, B_i and C_i given by

$$A_i=\frac{P_i}{R_i}-\frac{1-\alpha_s}{\beta(b-a)}, \quad (13a)$$

$$B_i=A_i^2+\frac{2(1-a)}{R_i(b-a)}+\frac{2P_i(1-\alpha_s)}{R_i\beta(b-a)}-\frac{2K_i}{R_i}, \quad (13b)$$

$$C_i=\frac{2I_s(1+\beta)}{R_i\beta(b-a)}. \quad (13c)$$

Thus, for the region of x and $\bar{Q}$ in which $B_i - C_i/\bar{Q} > 0$ we obtain two possible solutions for x_s, provided that both roots of (12) lie inside the interval (x_i, x_{i+1}). If $A_i \in (x_i, x_{i+1})$ then there exists a subinterval of (x_i, x_{i+1}) which contains both roots of Eq. (12). We have calculated numerically the values of A_i for latitudinal belts of 5°. We have found that in the latitudinal belt between 55° and 50° the value of A_i lies inside the corresponding interval $(x_i, x_{i+1}) = (0.819, 0.766)$. Therefore somewhere between 55° and 50° there exists an interval inside which both roots of (12) lie inside (x_i, x_{i+1}). This means that for a given value of the solar constant we have two possible positions for the edge of the ice cap provided the condition $B_i - C_i/\bar{Q} > 0$ is satisfied. We expect that the extrapolation of the solution which includes our present climate is characterized by the property that a decrease of the solar constant will cause a shift of the ice cap toward the equator. This requirement can be

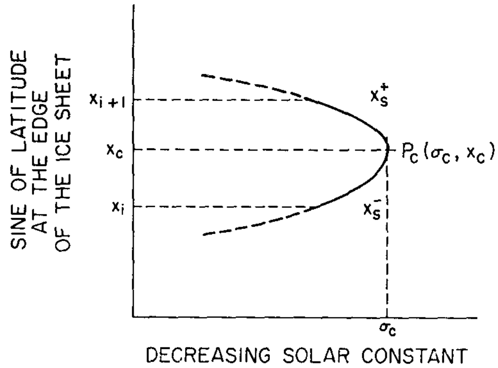

Fig. 1. Schematic behavior of the solutions x_s^+ and x_s^- of the Budyko climate model. The present climatic state moves down along the x_s^+ curve with decreasing solar constant until it reaches the critical point P_c. At this point both solutions disappear into the complex x_s plane.

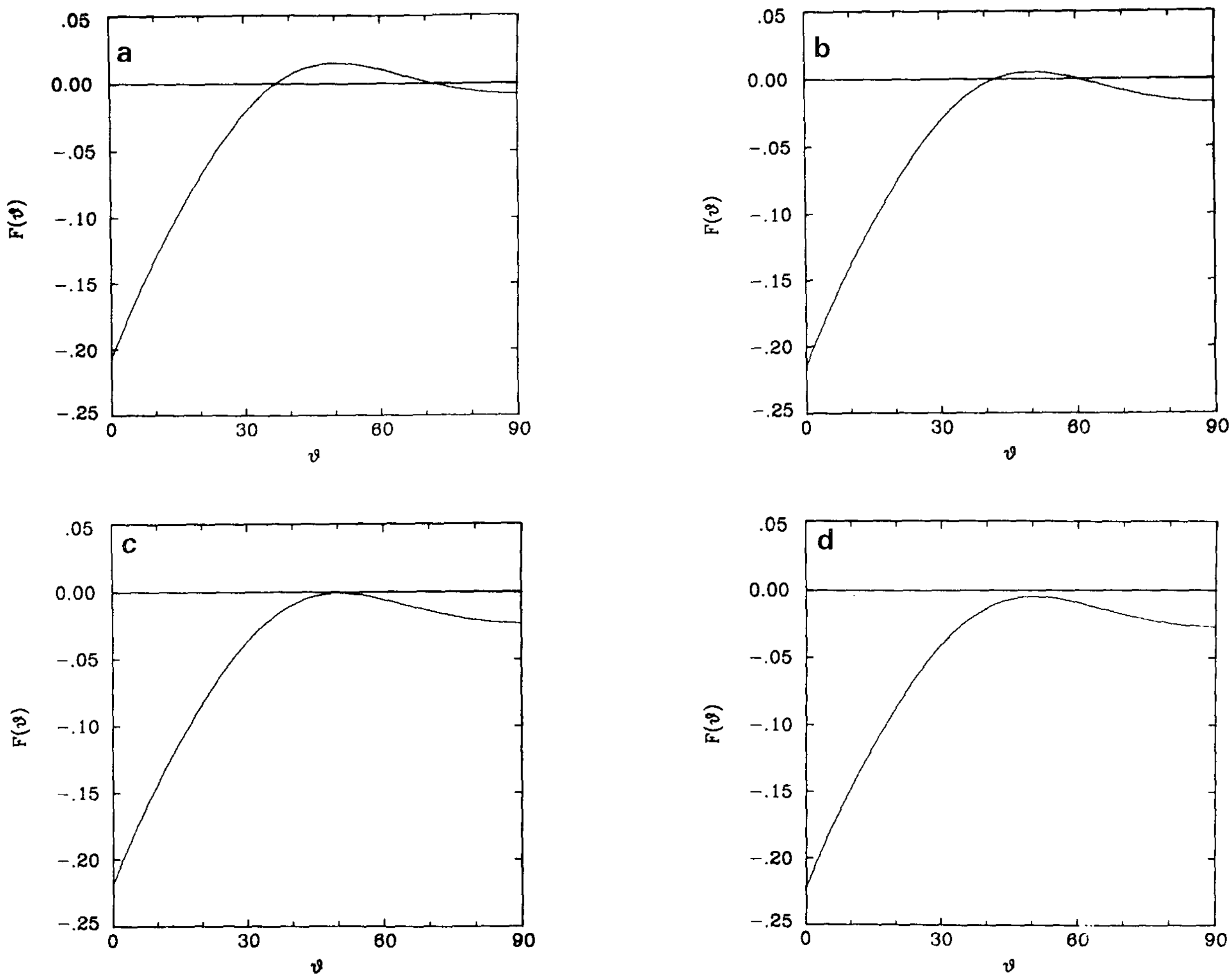

Fig. 2. Numerical calculations used to support our theoretical conclusions concerning the behavior of the solutions of the Budyko model. (a) At the present value of the solar constant there are two possible solutions for the latitude of the edge of the ice sheet. For the present climate the edge is at $\theta = 72°$. (b) With the solar constant decreased by 1% the two solutions approach each other. (c) With the solar constant decreased by 1.6%, the two solutions meet each other and the edge of the ice cap is at about 50°. (d) If the solar constant is decreased by 2% (or by any amount larger than 1.6%), there is no real solution to the problem.

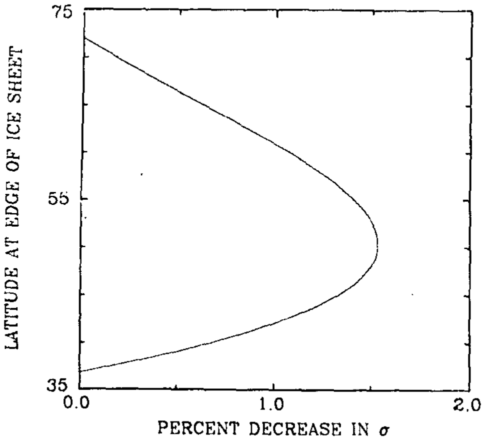

FIG. 3. Numerical solution of the Budyko-model for the latitude at the edge of the ice sheet as a function of the percent decrease in the solar constant σ.

expressed mathematically by

$$\frac{\partial x_s}{\partial \bar{Q}} > 0.$$

From (12) it follows that

$$\frac{\partial x_s}{\partial \bar{Q}} = \pm \frac{C_i/\bar{Q}}{(B_i - C_i/\bar{Q})^{\frac{1}{2}}}.$$

Since $C_i > 0$ the sign of $\partial x_s/\partial \bar{Q}$ is determined by the sign we take before the square root in (12). Therefore, the present climate is characterized by the solution of the form (12) with a positive sign before the square root.

TABLE 1. Normalized, annual mean solar radiation $S(x)$ at the top of the atmosphere. The latitude $\theta = \arcsin x$.

θ(deg)	x	$S(x)$
0	0.000	1.224
5	0.087	1.219
10	0.174	1.214
15	0.259	1.189
20	0.342	1.160
25	0.423	1.120
30	0.500	1.075
35	0.574	1.021
40	0.643	0.961
45	0.707	0.892
50	0.766	0.834
55	0.819	0.770
60	0.866	0.694
65	0.906	0.624
70	0.940	0.565
75	0.966	0.531
80	0.985	0.510
85	0.996	0.500
90	1.000	0.496

by using a least-squares method. The fit is shown in Fig. 4.

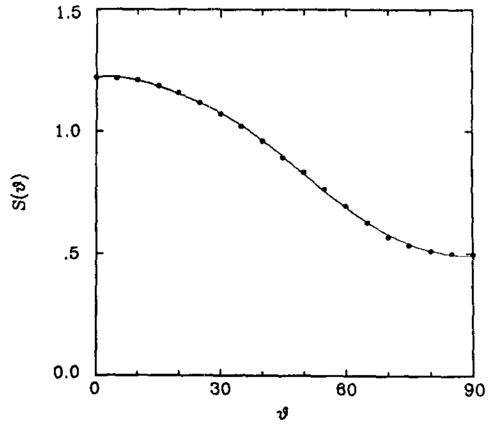

FIG. 4. Fifth-order polynomial fit (solid line) to the points of $S(\theta)$ as given in Table 1.

The behavior of both solutions is schematically shown in Fig. 1. The solution which includes the present climate regime is labeled by x_s^+, the second possible solution by x_s^-. With decreasing solar constant the two possible solutions x_s^+ and x_s^- approach each other and finally they meet at the critical point P_c, when the solar constant reaches its critical value σ_c and the ice sheet reaches the critical latitude $\theta_c = \arcsin x_c$. At this point we have $B_i - C_i/\bar{Q} = 0$. If the solar constant is now decreased below its critical value σ_c the expression $B_i - C_i/\bar{Q}$ becomes negative and there is no real solution to the problem.

After substitution of the appropriate expressions for $\alpha_P(x_s)$ and $S(x_s)$, Eq. (8) becomes a quadratic equation for x_s. It has two distinct real roots x_s^+ and x_s^- for $\sigma > \sigma_c$, one double root x_c for $\sigma = \sigma_c$, and a pair of complex conjugate roots for $\sigma < \sigma_c$. This means that once the solar constant reaches its critical value σ_c and the ice sheet reaches the critical latitude θ_c, the Budyko model breaks down. The real solution x_s^+ which exists for $\sigma > \sigma_c$ goes through a critical point $P_c(\sigma_c, x_c)$ and disappears into the complex x_s plane. Provided the edge of the polar ice sheet is defined by a given temperature T_s (approximately equal to the present average temperature at 72°N), x_s^+ and x_s^- are the only possible solutions of the model. However, if the temperature is allowed to drop below the value of T_s everywhere, then there exists a third solution corresponding to the completely ice-covered earth[3].

To support our theoretical analysis we have calculated numerically the latitude of the ice cap as a function of the solar constant σ. To insure that we would find all of the possible solutions, we wrote (8) in the form

$$F(x_s) = \frac{I_s}{\bar{Q}} \frac{1+\beta}{\beta} - \frac{1-\alpha_s}{\beta} S(x_s) - 1 + \alpha_P(x_s). \quad (14)$$

[3] We are grateful to both M. I. Budyko and G. R. North for bringing this fact to our attention.

$F(x_s)$ was evaluated as a function of x_s for each value of the solar constant $\sigma = 4\bar{Q}$. The values of x_s for which $F(x_s) = 0$ are the solutions of the problem. The numerical results are shown in Fig. 2 and 3, where the latitude θ is used as the independent variable. We found two solutions which approach each other and finally met at a point when the solar constant was decreased by 1.6% and the ice cap reached the latitude of about 50°. The values we used for the parameters were the same as those used by Budyko (1969)—namely, $a = 0.32$, $b = 0.62$, $\alpha_s = 0.50$, $A = 0.289$, $B = 0.00208$, $\beta = 2.61$ with the present position of the ice cap extending to the latitude $\theta = 72°$ and $\sigma = 1.92$ cal cm^{-2} min^{-1}. Our numerical result for the critical decrease of the solar constant is 1.6% which is in agreement with the result given by Budyko (1969). Budyko (1969) does not specify what values he used for the weighting function $S(x)$. We have used values obtained from a fifth-order polynomial in x which has been fitted to the points (based on theoretical calculations) given in Table 1,

4. Conclusions

We have found that the Budyko-type climate model predicts the spread of the polar ice cap to the critical latitude of 50° with a decrease of the solar constant by about 1.6%. At this point the two branches of the solution meet, and they become complex for smaller values of the solar constant.

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