Analytical Solution to a Simple Climate Model with Diffusive Heat Transport

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ABSTRACT

A simple radiative balance climate model is presented which includes the ice feedback mechanism, zonal averaging, constant homogeneous cloudiness, and ordinary diffusive thermal heat transfer. The simplest version of the model with only one free parameter is solved explicitly in terms of hypergeometric functions and is used to study ice sheet latitude as a function of solar constant. A multiple branch structure of this function is found and discussed along with comparison to earlier results. A stability analysis about the equilibrium solutions shows that the present climate as well as an ice-covered earth are stable while an intermediate solution is unstable for small perturbations away from equilibrium.

1. Introduction

Mathematical modeling of climate has received renewed interest because of the recent concern about the planet's climatic response to man's activities (*Study of Man's Impact on Climate*, 1971). Schneider and Dickinson (1974) have reviewed a hierarchy of models that may be used to study the response to various external and internal perturbations to the present climate. It appears that little progress will be made in understanding the more complex models until the simpler members of the hierarchy are thoroughly analyzed. The present study focuses on a class of analytically soluble models, whose only feedback mechanism is the polar ice sheet.

Budyko (1969) and Sellers (1969) have recently proposed models for the study of climate change as a function of solar constant. In each case zonal averaging is performed, cloudiness (and infrared atmospheric opacity) is held constant and homogeneous, and the latitudinal extent of the polar ice sheet is allowed to vary with its accompanying albedo changes. Though the assumptions, parameterizations, and method of solution of these models seem to differ appreciably, the results are remarkably invariant. The most striking feature is that only a few percent drop in the solar constant is required for the earth to ice over completely. In fact, Budyko's model shows more than one equilibrium solution for the same value of solar constant (Budyko, 1972; Chýlek and Coakley, 1975). One solution for the present solar constant is the present climate with its ice line at about 72° latitude, another is a completely ice-covered earth, while a third is intermediate to these two.

In addition to this evidence some numerical experiments of time-dependent models closely resembling that of Sellers have been carried out recently by Schneider and Gal-Chen (1973). These experiments indicate that small perturbations in planetary temperature lead back to the present climate while large negative perturbations lead to an ice-covered earth.

Several questions arise from the preceding discussion: Are the Sellers and Budyko models nearly equivalent? Why didn't Schneider and Gal-Chen find the intermediate solution with earth about half-covered with ice? What are the stability properties of the solutions? Is there an explanation of the ice ages hidden in these models?

The model studied in this paper is intermediate to the Budyko and Sellers models. It has the advantage of being tractable analytically and it has only one empirical parameter—the phenomenological thermal diffusion coefficient. Once this constant is determined by fitting the present climate, it is possible to examine the consequences of changes in the solar constant due to the so called "ice feedback" mechanism.

2. Model

The model to be studied in this paper is very similar to that of Budyko except for the form of the heat transport term. The variable chosen to describe latitude is x, the sine of the latitude. This is the natural variable to use since it weights the area of each latitude circle. Therefore area averages over the hemisphere are simply integrals from 0 to 1 with respect to x. We shall consider the energy balance of the earth-atmosphere system at

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the latitude designated by x. The incoming absorbed radiation at latitude circle x is not exactly balanced by infrared outgoing radiation because of the meridional transport of heat parallel to the earth's surface.

Consider a latitude circle x of width dx. The area of this strip is proportional to the element dx. The amount of heat incident on the earth-atmosphere system per unit time on this area is the solar constant divided by 4 times a function $S(x)$ which represents the mean annual distribution of radiation at each latitude. The function $S(x)$ may be derived from geometry by using the tilt of the earth's orbit with respect to the ecliptic plane, etc. It is a complicated function of x, but it has been tabulated, for example, by Chýlek and Coakley (1975). If $S(x)$ is normalized so that its integral from 0 to 1 is unity, the amount of heat energy incident upon the Northern Hemisphere is the solar constant divided by 4 (the ratio of areas of a sphere to a disk).

The amount of heat absorbed by the earth-atmosphere system is the incident radiation flux times one minus the albedo of the earth-atmosphere system. This difference is the absorption function which will be designated $a(x,x_s)$, where x_s is the sine of the latitude of the ice-sheet edge. Since the ice or snow cover has a much larger albedo than water or land, the absorption function will be strongly dependent upon the meridional extent of the ice. We shall take the step function form suggested by Budyko (1969):

$$a(x,x_s)=1-\alpha(x)=\begin{cases} a_1=0.38, & x>x_s \\ a_0=0.68, & x<x_s \end{cases}$$

Here $\alpha(x)$ is the albedo. This form differs from that of Sellers (1969), since Sellers allows a temperature dependence of the snow albedo and a latitude dependence of the albedo of ice free areas. Though Sellers' absorption seems to fit the recent satellite data on absorption better, we take Budyko's form for simplicity and note that a slight latitude (zenith angle) dependence may be induced by distorting the form of $S(x)$ toward that of an equinox distribution.

The form of the outgoing infrared radiation is also adopted from Budyko (1969). The infrared flux is given by the empirical formula

$$I(x)=A+BT(x), \tag{1}$$

where for 50% cloudiness, $A=201.4$ W m^{-2}, and $B=1.45$ W m^{-2} °C^{-1}, and $T(x)$ is the surface temperature (°C). An important assumption is incorporated in (1), namely that the outgoing radiation may be determined by prescribing the surface temperature. Clearly the radiation is actually determined by a temperature much higher in the atmosphere, but assuming a constant lapse rate the two temperatures are linearly related and (1) is approximately valid. Sellers' radiation form appears to be very different from (1), but when expressed in degrees Celsius and linearized the two forms agree to within about 5%. Linearizing is not serious here, since the expansion is in powers of $[T(x)\ (°\text{C})/273]$, and for all the temperatures of climatic interest the error from linearizing is less than 1%.

To close the energy balance equation we must express the meridional divergence of heat flux parallel to the earth's surface. While Budyko has chosen an empirical transport term of the form $\beta[T(x)-T_p]$, where β is an empirical constant and T_p the hemispheric average temperature, Sellers has chosen a much more complicated form with many empirical coefficients, which model the individual transport agents such as atmospheric sensible and latent heat transport as well as oceanic heat transport. The transport term adopted in this paper is the thermal diffusion form $D\nabla^2 T$, where D is an empirical coefficient to be determined by fitting (if possible) the present climate. All of the transport agents are presumably modeled by the single coefficient D. This form is similar to an eddy diffusion approach to dispersion by macroturbulence in the entire geofluid system. Its effectiveness in phenomenologically describing the transport of heat can only be tested by fitting the present climate. Taking the form of ∇^2 which admits no zonal or radial dependence, we may write the energy balance equation as

$$D\frac{d}{dx}(1-x^2)\frac{d}{dx}T(x)=I(x)-QS(x)a(x,x_s), \tag{2}$$

where Q is the solar constant divided by 4, and the radius of the earth squared from ∇^2 has been absorbed into D. In what follows, it will be more convenient to eliminate $T(x)$ and use $I(x)$ as the dependent variable by use of (1). Because of the linear relationship (1), I and T may be thought of interchangeably in the interpretation of results. Eq. (2) may then be rearranged slightly to give

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}I(x)-\frac{1}{D}I(x)=-\frac{QS(x)}{D}a(x,x_s), \tag{3}$$

where the coefficient B has been absorbed into D without loss of generality, since the coefficient D is to be determined empirically.

The model is not complete without specifying how the ice-sheet edge x_s is determined. We adopt the prescription used by both Sellers and Budyko: if $T(x)$ is less than −10°C ice will be present, while if $T(x)$ is greater than −10°C there will be no ice. In terms of the radiation function −10°C translates to

$$I(x_s)=I_s=186.8 \text{ W m}^{-2}.$$

When this condition is coupled with (3), the system becomes highly nonlinear. This nonlinearity is known as the ice-albedo feedback mechanism.

In order to solve (3) we must constrain the solution by boundary conditions. The solution procedure will be to solve (3) for $x>x_s$, $I_1(x)$, and for $x<x_s$, $I_0(x)$, and match these at the ice edge. The complete boundary conditions may be stated as follows.

(i) No heat transport at the pole [heat flux is proportional to the gradient of $I(x)$, $T(x)$, and must vanish at $x=1$]:

$$(1-x^2)^{\frac{1}{2}}\frac{d}{dx}I_1(x)\,|_{x=1}=0$$

where the differential operator on the left is the appropriate component of the gradient expressed in terms of x.

(ii) No heat transport across the equator:

$$(1-x^2)^{\frac{1}{2}}\frac{d}{dx}I_0(x)\,|_{x=0}=0$$

(iii) The solution $I_1(x)$ at the ice boundary must attain the critical value (−10°C):

$$I_1(x_s)=I_s$$

(iv) The same must be true of the ice-free solution:

$$I_0(x_s)=I_s$$

(v) The gradient of $I(x)$ must be continuous at the ice edge since otherwise heat would build up at this latitude circle:

$$\frac{d}{dx}I_1(x)\,|_{x=x_s}=\frac{d}{dx}I_0(x_s)\,|_{x=x_s}.$$

The problem is now a well-posed boundary value problem which may be attacked by computer or by analytical methods. In the next section the solution will be obtained analytically by use of hypergeometric functions.

It is well to bear in mind that our objective will be to find x_s as a function of Q. As has been shown by Budyko and Sellers, the functional relationship is a peculiar one. In fact, Budyko (1972) and Chýlek and Coakley (1975) have seen that more than one value of x_s obtains for a given value of Q. This multiple-valued structure is most easily revealed by reversing the question and obtaining Q as a function of x_s; this may seem a peculiar way of viewing the relationship, but it simplifies the mathematics considerably. The reader uninterested in mathematical detail may skip directly to the discussion of equilibrium results.

3. Solution

The general solution of Eq. (3) consists of the homogeneous solution plus a particular solution. We first consider the particular solution. Since the function $S(x)$ may be determined from astronomical calculations, it may be developed in a Fourier-Legendre series with known coefficients, i.e.,

$$S(x)=\sum_{n=0,2,4,\cdots} S_nP_n(x). \tag{4}$$

Odd terms have been omitted, since $S(x)$ is an even function of x. Because of the integral constraint on $S(x)$ we also have $S_0=1$. It can be shown that a particular solution to (3) is

$$QI_{pi}(x)=\frac{Q}{D}a_i\sum_{n\text{ even}}\frac{S_nP_n(x)}{n(n+1)+1/D},\quad i=0,1. \tag{5}$$

It is sufficient to retain only the first two terms of Eq. (4) to obtain 2% accuracy to the mean annual heating function, given in tabular form by Chýlek and Coakley (1975). The coefficient S_2 obtained in this fit is −0.482. The particular solution may then be written

$$I_{pi}(x)=a_i\left[1+\frac{S_2P_2(x)}{6D+1}\right],\quad i=0,1. \tag{6}$$

The homogeneous solution to (3) is given by the Legendre functions (Kamke, 1959):

$$I_{hi}(x)=A_iP_\nu(x)+B_iQ_\nu(x), \tag{7}$$

where

$$-\frac{1}{D}=\nu(\nu+1), \tag{8}$$

or

$$\nu=-\tfrac{1}{2}+\tfrac{1}{2}(1-4/D)^{\frac{1}{2}}, \tag{9}$$

and A_i, B_i are to be determined by boundary conditions. The second root of (8) is ignored since it may be shown that it does not lead to a new linearly independent solution. In order to satisfy boundary condition (i), B_1 must vanish since $Q_\nu(1)$ diverges even for ν complex. For $x>x_s$ the solution then reduces to

$$I_1(x)=A_1P_\nu(x)+QI_{p1}(x). \tag{10}$$

For $x<x_s$, both P_ν and Q_ν must be retained. For computational purposes it is more convenient to choose a slightly different form for the linearly independent solutions (Kamke, 1959), i.e.,

$$I_0(x)=A_0f_{1\nu}(x)+B_0f_{2\nu}(x)+QI_{p0}(x); \tag{11}$$

the functions $P_\nu(x)$, $f_{1\nu}(x)$, $f_{2\nu}(x)$ may all be related to hypergeometric functions (Kamke, 1959; Erdélyi, 1953):

$$P_\nu(x)=F(\tfrac{1}{2}+\tfrac{1}{2}\nu,\,-\tfrac{1}{2}\nu,\,1,\,1-x^2) \tag{12}$$

$$f_{1\nu}(x)=F(-\tfrac{1}{2}\nu,\,\tfrac{1}{2}+\tfrac{1}{2}\nu,\,\tfrac{1}{2},\,x^2) \tag{13}$$

$$f_{2\nu}(x)=xF(\tfrac{1}{2}-\tfrac{1}{2}\nu,\,1+\tfrac{1}{2}\nu,\,\tfrac{3}{2},\,x^2). \tag{14}$$

Power series expansions for these hypergeometric functions all converge well in the domains of interest. One further complication is that for $D<4$ the index ν becomes complex, and so do the functions defined by Eqs. (12)–(14). However, since Eq. (3) has real coefficients and a real inhomogeneous term, it will be sufficient to take for P_ν, $f_{1\nu}$, $f_{2\nu}$ their real parts. This last is true since if any pair of these functions are linearly independent, their respective real parts will also be linearly independent. In what follows this will be done without change of notation.

We derive from the equatorial boundary condition (ii) the relation

$$B_0+QI'_{p0}(0)=0, \tag{15}$$

where we have used the properties (Whittaker and Watson, 1963):

$$\frac{d}{dz}F(a,b,c,z)=\frac{ab}{c}F(a+1,b+1,c+1,z), \tag{16}$$

$$F(a,b,c,0)=1. \tag{17}$$

Since $I_{p0}(x)$ is an even function of x, it follows that $I'_{p0}(0)$ is zero and, therefore, B_0 vanishes.

The remaining three boundary conditions may be expressed as

$$A_0f_{1\nu}(x_s)+QI_{p0}(x_s)=I_s, \tag{18}$$

$$A_1P_\nu(x_s)+QI_{p1}(x_s)=I_s, \tag{19}$$

$$A_1P_\nu'(x_s)+Q[I'_{p1}(x_s)-I'_{p0}(x_s)]=A_0f'_{1\nu}(x_s). \tag{20}$$

The unknowns to be determined by these equations are A_0, A_1 and x_s; however, it is much more convenient to consider x_s as given and determine Q instead, since the equations are linear in Q. Eqs. (18)–(20) are easily solved for Q:

$$Q=I_s(Pf'-fP')/[I_{p0}Pf'-I_{p1}P'f+(I'_{p1}-I'_{p0})fP], \tag{21}$$

where $P=P_\nu(x_s)$, etc., all functions being evaluated at x_s. The remaining coefficients are given by

$$A_0=(I_s-QI_{p0})/f, \tag{22}$$

$$A_1=(I_s-QI_{p1})/P. \tag{23}$$

The terms in (21) are easily evaluated by the power series representations of the hypergeometric functions (12), (13), (14) together with (16).

Before presenting graphical results for $0<x_s<1$, let us inquire about the special cases $x_s=0$, 1. For $x_s=0$ (ice-covered earth) we have the inequality

$$I_1(x)\leqslant I_s,\quad 0\leqslant x\leqslant 1. \tag{24}$$

In particular,

$$A_1P_\nu(0)+QI_{p1}(0)\leqslant I_s. \tag{25}$$

In this case the condition (ii) is trivially satisfied since $P_\nu'(0)=I'_{p1}(0)=0$. In this case it follows also from (3) that

$$\int_0^1 I_1(x)dx=a_1Q. \tag{26}$$

Substituting for $I_1(x)$ and noting that

$$\int_0^1 I_{p1}(x)dx=a_1,$$

it follows that $A_1=0$. Inequality (25) now reduces to

$$Q\leqslant I_s/I_{p1}(0),\quad x_s=0, \tag{27}$$

which states that for $x_s=0$, a continuous range of Q below $I_s/I_{p1}(0)$ is acceptable.

For the special case $x_s=1$, a similar result obtains:

$$Q\geqslant I_s/I_{p0}(1),\quad x_s=1. \tag{28}$$

One procedes now to vary D until the present climate is approximately fitted: $x_s=0.95$, $4Q=1338$ W m^{-2}, and an acceptable fit to the temperature vs x curve. It is remarkable that a single free parameter can be adjusted to meet these two independent conditions.

The model remains soluble analytically under several generalizations:

1) D may take different values on and off the ice, the solution procedure being the same as above only now there are two free parameters.

2) D may take different values in different finite width zones, the number of matching conditions increasing correspondingly.

3) The absorption functions $a_1(x)$, $a_0(x)$ may take on any smooth forms, since only the particular solutions $I_{p1}(x)$, $I_{p0}(x)$ are modified by this alteration. The procedure in this case would be to develop $a_i(x)S(x)$ in a Fourier-Legendre series instead of expanding only the $S(x)$ factor.

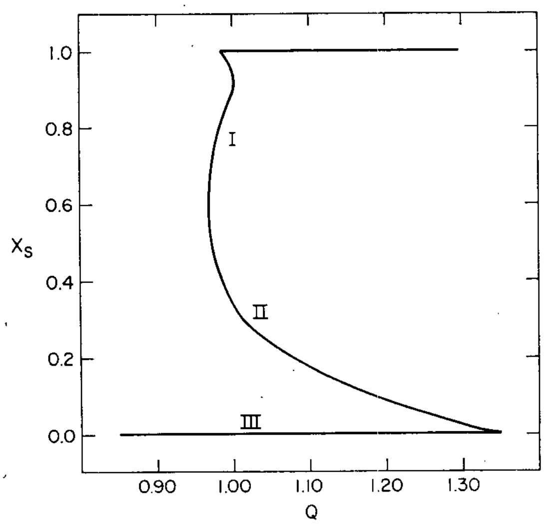

FIG. 1. The ice-sheet edge x_s (sine of latitude) as a function of the solar constant in units of the present solar constant (1337.6 W m^{-2}). The solar heating in this case is the mean annual heating function, and the fitted thermal diffusion coefficient is $D=0.310$. See text for meaning of branches I, II, III.

4. Exact equilibrium results

Fig. 1 shows the graph of x_s vs Q for the astronomically determined mean annual solar heating function

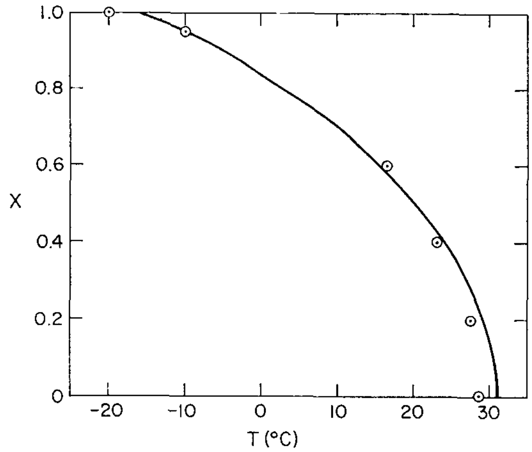

FIG. 2. Zonally averaged temperature as a function of x (sine of latitude) for observed (points) and model values (curve). The conditions are the same as in Fig. 1, with $x_s=0.95$ (the present climate).

given in tabular form by Chýlek and Coakley (1975). The multiple-branch nature of the solution was first discovered by Budyko (1972) and later discussed by Chýlek and Coakley (1975) in the simpler Budyko model (Budyko, 1969); the latter authors, however, did not consider the inequality (24) which leads to branch III of Fig. 1 (ice-covered earth). The fit in this case is for a value of $D=0.310$. Clearly, as Q is lowered quasi-statically below $0.97Q$, the solution must jump discontinuously from branch I to branch III. Fig. 2 shows the comparison of model temperatures with the observed mean annual temperatures taken from Sellers (1969).

Fig. 3 shows the x_s vs Q curve for $S(x)=\frac{3}{2}(1-x^2)$, $(S_2=-1)$, the heating function which would obtain at equinox conditions. This heating function goes to zero at the pole and in general has a larger gradient from equator to pole. In this case a value of $D=0.65$ gives a reasonable fit [$T(0)=33.5°C$]. The curve is similar to that in Fig. 1 except near $x_s=1$. The latter shape near $x_s=1$ seems intuitively more reasonable, since we expect the ice cap to gradually recede with increasing Q. In addition, the $S_2=-1$ solution may more faithfully represent the strong zenith angle dependence of the albedo near the pole. In this case a reduction of Q by 7% is required for ice age instability. The best solution probably lies between these two extremes, though it seems premature to introduce more assumptions in order to pin it down in such a crude model.

5. Stability analysis

Since equilibrium solutions with more than one value of x_s have been found for the same (present) value of Q, it is important to examine the stability of these solutions. The problem may be posed as follows: if a small perturbation away from equilibrium is allowed, does it grow exponentially in time or does it relax back to the equilibrium solution?

The time-dependent equation to be studied is

$$\frac{\partial}{\partial t}I(x,t)=D\frac{\partial}{\partial x}(1-x^2)\frac{\partial}{\partial x}I(x,t)+QS(x)a(x,x_s)-I(x,t). \quad (29)$$

The thermal inertia coefficient on the left hand side is taken to be unity since it merely scales the time.

If $I(x)$ is developed in a Fourier-Legendre series using observed mean annual values of $T(x)$ [Fig. 2], it can be shown that only the first two terms are required to obtain accuracy of a few percent. This suggests that a time-dependent analysis of (29) can be made by assuming a two-mode solution:

$$I(x,t)=I_0(t)+I_2(t)P_2(x), \quad (30)$$

where odd terms have been omitted since $I(x,t)$ is assumed to be an even function of x.

Substituting (30) into (29) and using standard projection methods, one obtains equations for the mode amplitudes:

$$\dot{I}_0(t)=-I_0(t)+Q\int_0^1 S(x)a(x,x_s)dx, \quad (31)$$

$$\dot{I}_2(t)=-(6D+1)I_2(t)+5Q\int_0^1 S(x)a(x,x_s)P_2(x)dx. \quad (32)$$

A more convenient form for (31) and (32) obtains if

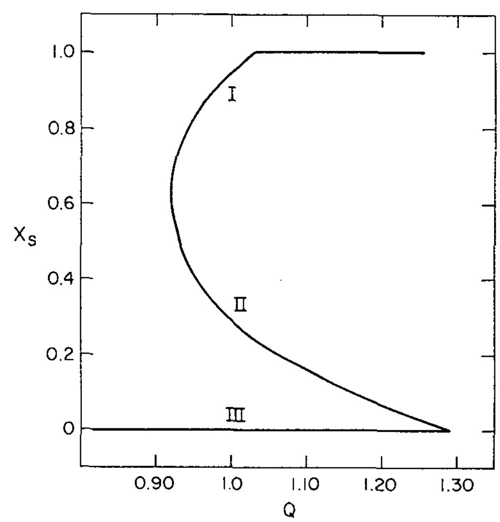

FIG. 3. The ice-sheet edge x_s as a function of the solar constant in units of the present solar constant. The heating function in this case is the equinox distribution $(s_2=-1)$, and the fitted $D=0.65$.

the explicit properties of $S(x)$ and $a(x,x_s)$ are used:

$$\dot{I}_0(t)=-I_0(t)+Q(a_0-a_1)\int_0^{x_s}S(x)dx+Qa_1, \quad (33)$$

$$\dot{I}_2(t)=-(6D+1)I_2(t)+5Q(a_0-a_1)\int_0^{x_s}S(x)P_2(x)dx+Qa_1S_2. \quad (34)$$

Eqs. (33) and (34) are highly nonlinear because of the definition of x_s, which in the two-mode approximation takes the form

$$I_s=I_0(t)+I_2(t)P_2(x_s). \quad (35)$$

In order to study the stability of steady-state solutions it is convenient to consider small departures from equilibrium:

$$I_0(t)=I_0^0+\delta_0(t), \quad (36)$$

$$I_2(t)=I_2^0+\delta_2(t), \quad (37)$$

$$x_s(t)=x_0+\epsilon(t), \quad (38)$$

where x_0 is a particular point on branch I, II or III and I_0^0, I_2^0 are the equilibrium values corresponding to the point x_0.

If (35) is time-differentiated, we obtain to first order in small quantities:

$$0=\dot{\delta}_0+\dot{\delta}_2P_2(x_0)+I_2^0 3x_0\dot{\epsilon}. \quad (39)$$

To the same order of approximation (33) and (34) reduce to

$$\dot{\delta}_0=-\delta_0+q\epsilon, \quad (40)$$

$$\dot{\delta}_2=-(6D+1)\delta_2+5qP_2(x_0)\epsilon, \quad (41)$$

where $q=Q(a_0-a_1)S(x_0)$. In the derivation of (40) and (41) several large terms cancel because the perturbation is about an equilibrium solution. Eq. (35) becomes

$$0=\delta_0+\delta_2P_2(x_0)+3x_0I_2^0\epsilon. \quad (42)$$

Eqs. (42) and (39) may now be used to eliminate δ_2 in (41); the result is

$$-\dot{\delta}_0-3x_0I_2^0\dot{\epsilon}=+(6D+1)\delta_0+3x_0I_2^0(6D+1)\epsilon+5qP_2(x_0)^2\epsilon. \quad (43)$$

Eqs. (40) and (43) now constitute a pair of simultaneous linear differential equations for $\delta_0(t)$, which may be interpreted as the planetary average temperature fluctuation, and $\epsilon(t)$, which is the fluctuation in the ice line.

Following a standard method we introduce

$$\delta_0=D_0e^{i\omega t}, \quad \epsilon=Ee^{i\omega t}, \quad (44)$$

where D_0 and E are complex constants, and ω is a frequency to be determined. Substituting (44) into (40) and (43) one obtains

$$(\Omega-6D)D_0-qE=0, \quad (45)$$

$$\Omega D_0+(3x_0I_2^0\Omega+5qP_2(x_0)^2)E=0, \quad (46)$$

where $\Omega=i\omega+6D+1$. For a solution to exist the determinant of coefficients must vanish, i.e.,

$$\alpha\Omega^2+\beta\Omega+\gamma=0, \quad (47)$$

where

$$\left.\begin{aligned}\alpha&=3x_0I_2^0\\ \beta&=-18DI_2^0x_0+5qP_2(x_0)^2+q\\ \gamma&=-30DqP_2(x_0)^2\end{aligned}\right\}.$$

The condition for stability is $\mathrm{Im}\,\omega>0$. The coefficients α,β,γ are readily calculated from the steady-state solutions.

We first consider branch I of Fig. 3 ($S_2=-1$, equinox heating function) where $D=0.65$, $x_0=0.95$, $q=14.70$, $I_0^0=226.4$, $I_2^0=-46.54$, $P_2(x_0)=0.853$; the resulting roots are

$$\omega_{\mathrm{I}}=2.127i,\ 4.383i. \qquad \text{(Branch I)}$$

Therefore both roots lead to critically damped relaxation back to equilibrium.

For branch II of Fig. 3 [$x_0=0.29$, $q=137.8$, $I_0^0=160.2$, $I_2^0=-70.78$, $P_2(x_0)=-0.374$], the resulting roots are

$$\omega_{\mathrm{II}}=3.948i,\ -1.900i. \qquad \text{(Branch II)}$$

The second root in this case causes exponential growth away from branch II, and, therefore, branch II is unstable.

In the case of $x_0=0$, branch III, $\alpha=0$ so that only one root is possible:

$$\omega_{\mathrm{III}}=i[(8/3)D+1] \qquad \text{(Branch III)}$$

which is clearly stable and critically damped. Critical damping is important for self-consistency on branch III since $x_s<0$ is not defined.

6. Conclusion

A very simple climate model has been developed and solved using analytical methods. The only feedback mechanism in the model is the latitudinal extent of the polar ice sheet. The model is crude and probably should not be taken too seriously when it is used to extrapolate to conditions very far from those in present existence. Among the cruder assumptions are constant homogeneous cloudiness, all dynamics and outgoing infrared radiation are parameterized by the ground level temperature, and the meridional heat transport is characterized by a thermal diffusion (whose transport coefficient does not depend on the climate itself). The success of the model in fitting the present climate by adjusting only one parameter, suggests that it may have some limited validity.

The model shares the multi-climate property with the Budyko model. A linear stability analysis about the equilibrium solutions confirms the results of Budyko (1972) that the present climate and the ice-covered earth solutions are stable while the intermediate solution is unstable. These results explain why in time-dependent numerical experiments one does not find the intermediate solution. Time-dependent solutions will locate either branch I or branch III, but will never settle down on branch II. Branch II is therefore of no physical consequence and hence not a possibility for developing a theory of the ice ages, based on quantum jumping from branch I to branch II because of non-linear self-induced fluctuations.

If the assumptions of this model are taken seriously, the most recent advances of the ice sheet might be explained by lowering the effective solar constant by a slight amount to bring the ice sheet to mid-latitudes. According to the model, however, this last is dangerously close to the catastrophic plunge to complete ice-cover. There is no geological evidence that this state has ever existed, which is comforting since Fig. 1 suggests that the solar constant would then have to be boosted by about 35% to remove the ice, and cause a jump to an ice-free earth.

It seems improbable that the ice sheet could have advanced several times to mid-latitudes without making the catastrophic plunge. The explanation of this paradox may lie in the deficiencies of the models. The most obvious shortcoming is their failure to incorporate cloudiness as a feedback mechanism. Other feedbacks are also neglected such as nonlinear thermal diffusion, salinity advection in sea water, availability of moisture for producing snow, and time lag effects which can lead to build up of snow over long periods. Though, the path to understanding global climate is obviously complicated, it is clear that mathematical modeling is the best starting point to test assumptions against geological and present day evidence.

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